SVOLGIMENTO PROVA SCRITTA ANALISI 1 dell'11/2/2015

1A

COMPITO A

1) $\bar{z} \neq 3i \Rightarrow z \neq -3i$

$$\bar{z} - 3i = \frac{1}{i^{540}} = (-i)^{540} = (-i)^{4 \cdot 135} = 1^{135} = 1$$

$$\Rightarrow x - iy - 3i = 1 \Rightarrow \begin{cases} x = 1 \\ y = -3 \end{cases} \Rightarrow z = 1 - 3i$$

2) Serie a segno alterno

$$a_n = \frac{n+1}{2n^2-1} > 0$$

Criterio di Leibniz

i) $\lim_{n \to +\infty} a_n = 0$

ii) $a_n \geq a_{n+1} \Rightarrow \frac{n+1}{2n^2-1} \geq \frac{n+2}{2(n+1)^2-1}$

$$\left[2(n+1)^2 - 1\right](n+1) \geq (n+2)(2n^2-1)$$

$$2(n+1)^3 - (n+1) \geq 2n^3 - n + 4n^2 - 2$$

$$\cancel{2n^3} + 6n + 6n^2 + 2 \cancel{- n} - 1 \geq \cancel{2n^3} \cancel{- n} + 4n^2 - 2$$

$$2n^2 + 6n + 3 \geq 0$$

Studio $2x^2 + 6x + 3 = 0 \Rightarrow x_{1,2} = \frac{-3 \pm \sqrt{9-6}}{2}$

$\Rightarrow \quad n \geq \frac{-3+\sqrt{3}}{2} \Rightarrow \forall n \in \mathbb{N}$ (2A)

a_n decrescente

$\Rightarrow$ la serie converge.

3) $B(y) = \frac{1+e^y}{e^y} = 1+e^{-y} \in C^\infty(\mathbb{R})$

$A(x) = \frac{1}{1+x^2} \in C^\infty(\mathbb{R})$

$\Rightarrow \exists I(0)$ in cui $\exists !$ sol. $y(x) \in C^1(I)$

$B(y) \neq 0 \quad \forall y \Rightarrow \nexists$ soluzioni singolari.

$\Rightarrow \int \frac{e^y}{1+e^y} dy = \int \frac{1}{1+x^2} dx$

$\log(1+e^y) = \operatorname{arctg} x + C$

$y(0) = 0 \quad \Rightarrow \log 2 = C$

$\Rightarrow \log(1+e^y) = \operatorname{arctg} x + \log 2$

$1+e^y = \operatorname{arctg} x + \log 2$

$e^y = e^{\log 2 + \operatorname{arctg} x} - 1$

$= 2e^{\operatorname{arctg} x} - 1$

③

$$y(x) = \log\left[2e^{\operatorname{arctg} x} - 1\right].$$

4) $\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} = 1$

$\lim_{x \to 0} \frac{\sin(\alpha x)}{x} = \alpha$

$\Rightarrow \alpha = \beta = 1.$

$$f'(x) = \begin{cases} \frac{2xe^{x^2}x^2 - 2x(e^{x^2}-1)}{x^4} = \frac{2\left[x^2e^{x^2} - e^{x^2} + 1\right]}{x^3} & x<0 \\ \frac{\cos(x)\cdot x - \sin x}{x^2} & x>0 \end{cases}$$

$$= \lim_{x \to 0^-} \frac{2}{x^3}\left[x^4 \cancel{-1} - \frac{x^4}{2} + o(x^4) \cancel{+1}\right]$$

$$= \lim_{x \to 0^-} \frac{2}{x^3} \cdot \frac{x^4}{2} = \lim_{x \to 0^-} x = 0 = f'_-(0)$$

$$\lim_{x \to 0^+} \frac{x\cos x - \sin x}{x^2} = \lim_{x \to 0^+} \frac{x\left[1 - \frac{x^2}{2} + o(x^2)\right] - \left(x - \frac{x^3}{6} + o(x^3)\right)}{x^2}$$

$$= \lim_{x \to 0^+} \frac{\cancel{x} - \frac{x^3}{2} \cancel{-x} + \frac{x^3}{6} + o(x^3)}{x^2} =$$

$$= \lim_{x \to 0^+} \frac{-\frac{x^3}{3}}{x^2} = \lim_{x \to 0^+} -\frac{x}{3} = 0 = f'_+(0)$$

f è derivabile in $x=0$ per $\alpha = \beta = 1$.

Poichè per $x \neq 0$ $f \in C^\infty$, allora f è continua e derivabile in $\mathbb{R}$ solo per $\alpha = \beta = 1$.

5) f è definita se $|1-e^{2x}| > 0$

$$\Rightarrow 1 - e^{2x} \neq 0 \quad \Rightarrow x \neq 0.$$

$D = \mathbb{R} - \{0\} = (-\infty, 0) \cup (0, +\infty)$.

$$f(x) = \begin{cases} \log(1-e^{2x}) & \text{se } 1-e^{2x}>0 \Rightarrow x<0 \\ \log(e^{2x}-1) & \text{se } e^{2x}-1>0 \Rightarrow x>0 \end{cases}$$

CONTINUA e DERIVABILE in D.

$$\left.\begin{aligned} &\lim_{x\to 0^+} f(x) = \log(0^+) = -\infty \\ &\lim_{x\to 0^-} f(x) = \log(0^+) = -\infty \end{aligned}\right\}$$

$x=0$ ASINTOTO VERTICALE DX e SX

6A

Per $x \to +\infty$

$$f(x) = \log\left(e^{2x}\left(1 - e^{-2x}\right)\right)$$
$$= \log\left(e^{2x}\right) + \underbrace{\log\left(1 - e^{-2x}\right)}_{\to 0}$$
$$= 2x + \log\left(1 - e^{-2x}\right)$$

$$\Rightarrow \quad f(x) \underset{x \to +\infty}{\sim} 2x$$

$y = 2x$ ASINTOTO OBLIQUO per $x \to +\infty$.
(la distanza $f(x) - 2x \xrightarrow[x \to +\infty]{} 0$)

Per $x \to -\infty$

$$\lim_{x \to -\infty} f(x) = \log 1 = 0$$

$y = 0$ ASINTOTO ORIZZONTALE per $x \to -\infty$.

ANCHE SE NON RICHIESTO, completiamo lo studio del grafico.

$$f(x) = 0 \iff \left|1 - e^{2x}\right| = 1 \iff 1 - e^{2x} = \pm 1$$
$$\Rightarrow \quad e^{2x} = 2 \Rightarrow x = \frac{\log 2}{2}.$$

$f(x) > 0 \iff |1-e^{2x}| > 1$

$\iff 1-e^{2x} < -1$ oppure $\underbrace{1-e^{2x} > 1}_{\text{MAI!}}$

$\iff e^{2x} > 2 \iff x > \frac{\log 2}{2}$

$f'(x) > 0 \iff e^{2x} > 1 \iff x > 0$

x decrescente in $(-\infty, 0)$; crescente in $(0, +\infty)$.

$f'(x) = 2 + \frac{2}{e^{2x}-1} \Rightarrow f''(x) = \frac{-2}{(e^{2x}-1)^2} 2e^{2x} < 0$ $\forall x \in D$

$\Rightarrow f$ concava in $(-\infty, 0)$ e in $(0, +\infty)$.