SVOLGIMENTI PROVA SCRITTA di ANALISI MAT. 1 dell'11/2/2016

A1

COMPITO A

1) Equazione a variabili separabili, definita $\forall x \in \mathbb{R}$.

$$A(x) = \frac{1}{1+x^2} \in C^0(\mathbb{R}); \quad B(y) = \frac{e^y - 1}{e^y} \in C^1(\mathbb{R})$$

$\Rightarrow$ Esistenza e unicità LOCALE per ogni Problema di Cauchy.

c) $B(y) = 0 \iff e^y - 1 = 0 \iff y = 0.$

UNICO INTEGRALE SINGOLARE

b) $$\int \frac{e^y}{e^y - 1}\, dy = \int \frac{1}{1+x^2}\, dx$$

$$\log(|e^y - 1|) = \operatorname{arctg} x + C$$

$$|e^y - 1| = e^{\operatorname{arctg} x + C} = \underset{>0}{K} e^{\operatorname{arctg} x}$$

$$\Rightarrow e^y - 1 = \underset{\neq 0}{H} e^{\operatorname{arctg} x}$$

$$\Rightarrow e^y = 1 + H e^{\operatorname{arctg} x}; \quad H \neq 0$$

$$y = \log(1 + H e^{\operatorname{arctg} x}); \quad H \neq 0$$

Aggiungendo l'integrale singolare, ~~si~~ (A₂)
$(H=0)$ si arriva all'integrale generale:

$$y(x) = \log\left(1+He^{\operatorname{arctg} x}\right) \quad ; \quad H \in \mathbb{R}$$

d) per $y(0)=0$ si ha la sola soluzione singolare

per $y(0) = \log 2$ si ha:

$$\log 2 = \log(1+H) \Rightarrow H = 1$$

$$\Rightarrow y(x) = \log\left(1+e^{\operatorname{arctg} x}\right).$$

2) $$\frac{z+i}{z-i} \underset{\boxed{(z\neq i)}}{=} \frac{x+i(y+1)}{x+i(y-1)}$$

$$= \frac{[x+i(y+1)][x-i(y-1)]}{x^2+(y-1)^2} = \frac{\left\{x^2+(y^2-1)+ix[\cancel{y}+1-\cancel{y}+1]\right\}}{x^2+(y-1)^2}$$

$$= \frac{x^2+(y^2-1)}{x^2+(y-1)^2} + i\,\frac{2x}{x^2+(y-1)^2}$$

$$\operatorname{Re}\left(\frac{z+i}{z-i}\right) = \frac{x^2+(y^2-1)}{x^2+(y-1)^2} \qquad (x,y) \neq (0,1).$$

A3

$$\Rightarrow \frac{x^2+(y^2-1)}{\underbrace{x^2+(y-1)^2}_{>0}} \le 1$$

$$\Rightarrow \cancel{x^2}+(y^2-1) \le \cancel{x^2}+(y-1)^2$$

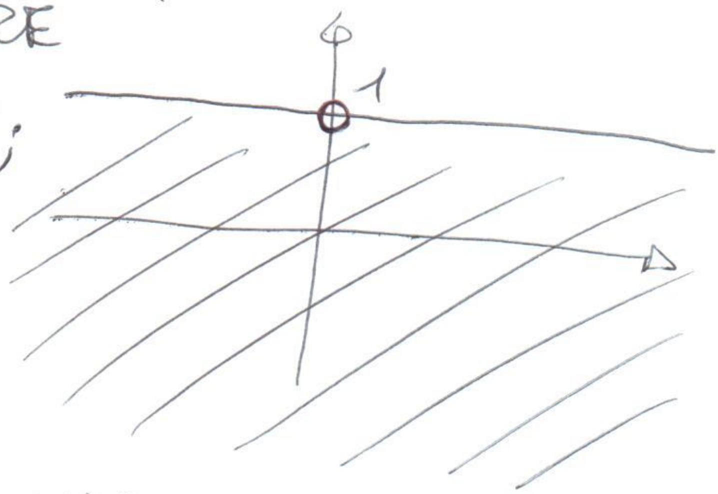
$$\Rightarrow \cancel{y^2}-1 \le \cancel{y^2}-2y+1$$

$$\Rightarrow 2y \le 2 \qquad \Rightarrow \qquad \underline{y \le 1}$$

SEMIPIANO INFERIORE

(x,y) tali che $y \le 1$;

$(x,y) \neq (0,1)$.



3) CONVERGENZA ASSOLUTA:

$$\sum \left| (-1)^n \left(1-\frac{1}{n}\right)^{n^2} \right| = \sum \left(1-\frac{1}{n}\right)^{n^2}$$

Criterio della radice:

$$\sqrt[n]{\left(1-\frac{1}{n}\right)^{n^2}} = \left(1-\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$$

$\Rightarrow$ CONV. ASSOLUTA e SEMPLICE

4) $\boxed{A_4}$

$$f(x) = \begin{cases} \dfrac{x^2}{1+x+2} = \dfrac{x^2}{x+3} = x-3+\dfrac{9}{x+3} & \text{se } x \geq -2 \\ \dfrac{x^2}{1-x-2} = \dfrac{-x^2}{x+1} = -x+1-\dfrac{1}{x+1} & \text{se } x < -2 \end{cases}$$

Definita per $\begin{cases} x \geq -2 \\ x \neq -3 \end{cases} \cup \begin{cases} x < -2 \\ x \neq -1 \end{cases}$

OK OK

$D = \mathbb{R}$. NO ASINTOTI VERTICALI:

$f \in C^0(D)$.

Per $x \to +\infty$ $f(x) \sim x-3$ (AS. OBLIQUO)

Per $x \to -\infty$ $f(x) \sim -x+1$ (AS. OBLIQUO)

$$f'(x) = \begin{cases} 1 - \dfrac{9}{(x+3)^2} & \text{se } x > -2 \\ -1 + \dfrac{1}{(x+1)^2} & \text{se } x < -2 \end{cases}$$

$$\lim_{x \to -2^+} f'(x) = -8 \neq \lim_{x \to -2^-} f'(x) = 0$$

PUNTO ANGOLOSO.

A5

Per $x > -2$: $f'(x) = 0 \iff$

$$\frac{9}{(x+3)^2} = 1 \iff (x+3)^2 = 9 \iff$$

$$x+3 = \pm 3 \iff x = 0$$

$$f'(x) > 0 \iff 9 < (x+3)^2 \iff$$

$$\begin{cases} x+3 < -3;\ x+3 > 3 \\ x > -2 \end{cases} \iff \begin{cases} x < -6;\ x > 0 \\ x > -2 \end{cases}$$

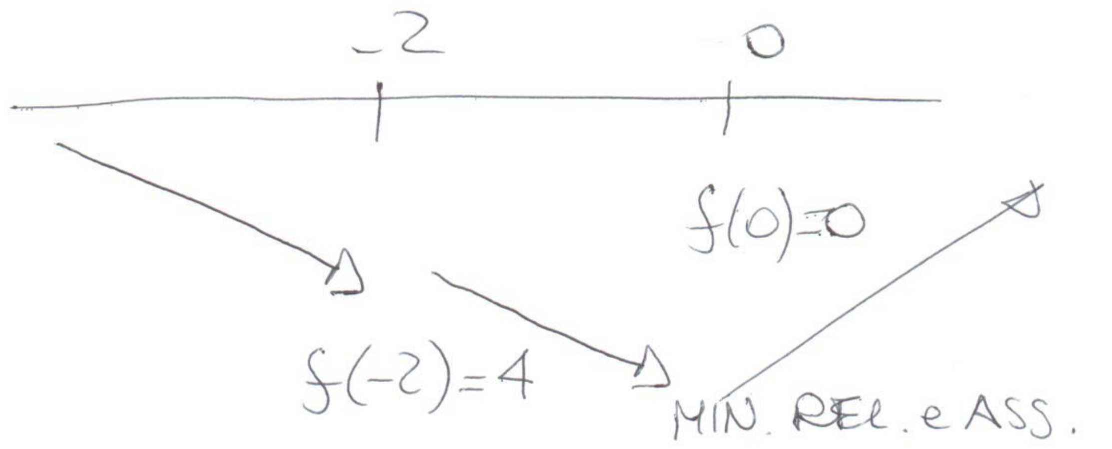
$\iff x > 0$.

Per $x < -2$: $f'(x) = 0 \iff \dfrac{1}{(x+1)^2} = 1$

$\iff (x+1)^2 = 1 \iff x+1 = \pm 1$ MAI per $x < -2$.

$$f'(x) > 0 \iff \frac{1}{(x+1)^2} > 1 \iff 1 > (x+1)^2$$

$$\iff -1 < x+1 < 1 \iff \begin{cases} -2 < x < 0 \\ x < -2 \end{cases} \text{ MAI}$$



Segno: $f(x)=\frac{x^2}{1+|x+2|}=0 \iff x=0$ (A6)

(intersezione asse x)

$f(x)>0 \iff x^2>0 \iff x\neq 0$

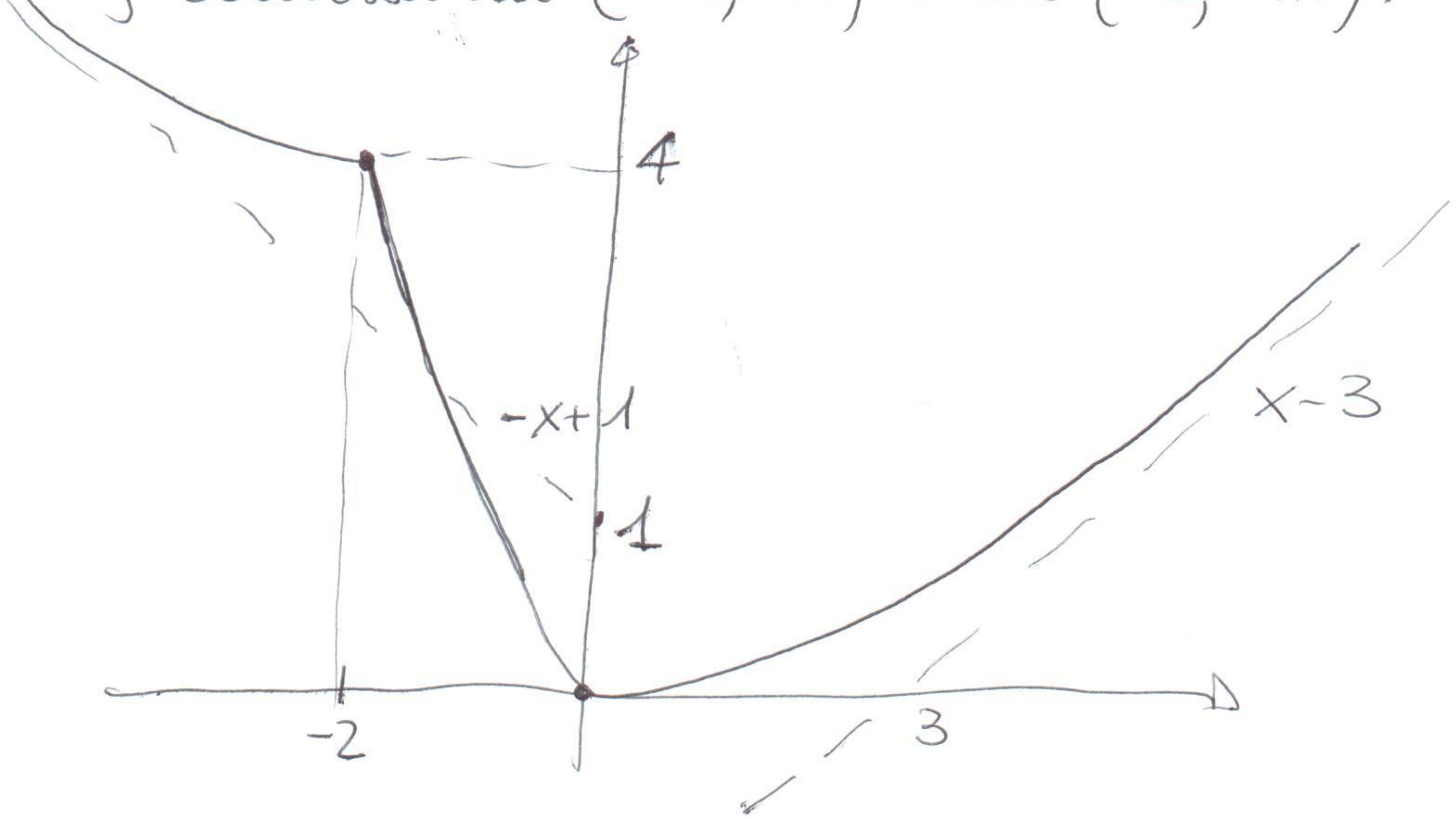
f SEMPRE NON NEGATIVA

$$f''(x)=\begin{cases}\frac{18}{(x+3)^3} & \text{se } x>-2\\ \frac{-2}{(x+1)^3} & \text{se } x<-2\end{cases}$$

$$f''(x)>0 \text{ se } \begin{cases}x>-2\\ x+3>0\end{cases} \cup \begin{cases}x<-2\\ x+1<0\end{cases}$$

cioè $\{x>-2\} \cup \{x<-2\}$

f convessa in $(-\infty,-2)$ e in $(-2,+\infty)$.



5) $\dfrac{\operatorname{senh}(x) + \log\left(1 - \frac{x^3}{6}\right) - x}{x^5} =$ A_7

$$\frac{\cancel{x} + \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} + o(x^5) + \left[-\cancel{\frac{x^3}{6}} - \frac{1}{2}\frac{x^6}{36} + o(x^6)\right] \cancel{-x}}{x^5}$$

$$= \frac{\frac{x^5}{5!} + o(x^5)}{x^5} \xrightarrow[x \to 0^-]{} \frac{1}{120}$$

$$\alpha\left[\frac{\sin\sqrt{x} - x}{\sqrt{x}}\right] = \alpha\left[\frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}} - \sqrt{x}\right] \xrightarrow[x \to 0^+]{} \alpha[1-0]$$

$$= \alpha$$

f è prolungabile per continuità se

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \in \mathbb{R}$$

quindi $\alpha = \dfrac{1}{120}$.

La prolungata per continuità è

$$\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \neq 0 \\ \dfrac{1}{120} & \text{se } x = 0. \end{cases}$$