COMPITO B

B_1

1) $$(z-1)^2\left[3(z-1)^2+1\right]=0$$

$$\{z=1 \quad m_a=2\} \cup \left\{(z-1)^2=-\frac{1}{3}\right\}$$

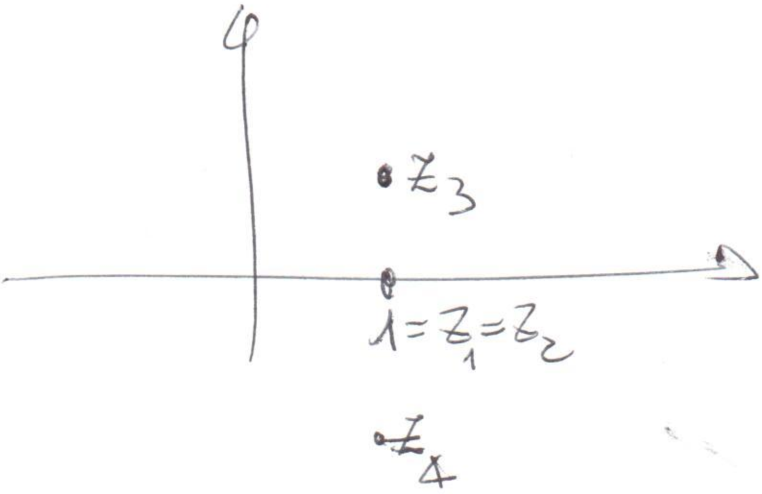
$$\downarrow$$

$$z-1=\pm\sqrt{-\frac{1}{3}}$$

$$\{z_{1,2}=1\} \quad \cup \quad \left\{z_3=1+i\sqrt{\frac{1}{3}}\right\}$$

$$\cup \left\{z_4=1-i\sqrt{\frac{1}{3}}\right\}$$

L'equazione algebrica di 4ª grado ammette 4 soluzioni in accordo col Teorema Fondamentale dell'Algebra.



B2

2) a) n pari: $a_n = \frac{1}{n^2+2}$; $a_{n+1} = \frac{1}{(n+1)^2-2}$

$a_n \geq a_{n+1} \iff \frac{1}{n^2+2} \geq \frac{1}{(n+1)^2-2}$

$\iff (n+1)^2 - 2 \geq n^2 + 2$

$\iff 2n - 1 \geq 2 \iff n \geq \frac{3}{2}$

$\forall n \geq 2$

n dispari:

$a_n = \frac{1}{n^2-2}$; $a_{n+1} = \frac{1}{(n+1)^2+2}$

$a_n \geq a_{n+1} \iff \frac{1}{n^2-2} \geq \frac{1}{(n+1)^2+2}$

$\iff (n+1)^2 + 2 \geq n^2 - 2$

$\iff 2n + 2 \geq -2 \iff n \geq -2$

$\forall n \in 2\mathbb{N}+1.$

$\Rightarrow \{a_n\}$ non crescente $\forall n \in \mathbb{N}$

$a_n \xrightarrow[n \to \infty]{} 0$

B3

b) Poiché

$\lim_{n\to\infty} a_n = 0$; $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$

$\Rightarrow$ per il Criterio di Leibniz

$$\sum (-1)^n \frac{1}{n^2+2(-1)^n} \text{ converge.}$$

3)

$$f(x) = \left|\log\left[(x-2)^2\right]\right| \geq 0$$

$$D = \{x \neq 2\}$$

$$\lim_{x\to 2^\pm} f(x) = \left|\log(0^+)\right| = +\infty$$

~~$f(x) = \begin{cases} \log(\\ \end{cases}$~~

$$\log\left[(x-2)^2\right] \geq 0$$

$$\Leftrightarrow (x-2)^2 \geq 1$$

$$\Leftrightarrow x \leq 1 \ ; \ x \geq 3.$$

$$f(x) = 0 \Leftrightarrow x = 1 \ ; \ x = 3$$

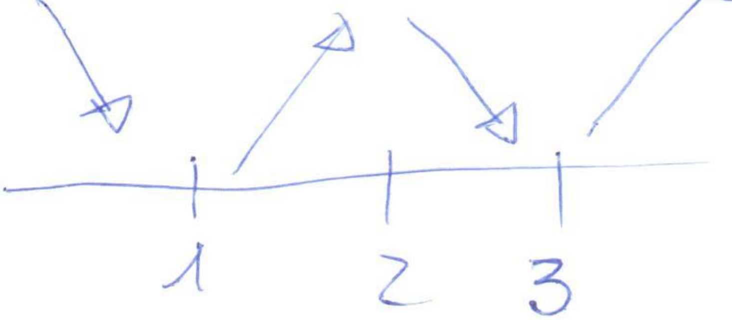
PUNTI DI MIN. ASSOLUTO

B_4

$$f(x) = \begin{cases} \log\left[(x-2)^2\right] & x \leq 1 \,;\, x \geq 3 \\ -\log\left[(x-2)^2\right] & 1 < x < 3 \end{cases} \qquad x \neq 2$$

$$= \begin{cases} 2\log(x-2) & x \geq 3 \\ -2\log(x-2) & 2 \leq x < 3 \\ -2\log(2-x) & 1 < x < 2 \\ 2\log(2-x) & x \leq 1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{2}{x-2} > 0 & x > 3 \\ -\dfrac{2}{x-2} < 0 & 2 < x < 3 \\ \dfrac{2}{2-x} > 0 & 1 < x < 2 \\ \dfrac{-2}{2-x} < 0 & x < 1 \end{cases}$$



B_5

$$\lim_{x \to 1^+} f'(x) = 2 \; ; \; \lim_{x \to 1^-} f'(x) = -2$$

PUNTO ANGOLOSO

$$\lim_{x \to 3^+} f'(x) = 2 \; ; \; \lim_{x \to 3^-} f'(x) = -2$$

PUNTO ANGOLOSO

In $(2, 4]$:

$\lim_{x \to 2^+} f(x) = +\infty$ $\quad \nexists$ MAX. ASS.

$x = 3$ $\quad$ MIN. ASS.

$f(4) = \log 4$ $\quad$ MAX. REL.

Più rapidamente, osservando che B 5bis

$$f(x) = \left| \log\left[(x-2)^2\right] \right|$$

e ponendo $t = x-2$, si ha

$$\tilde{f}(t) = \left| \log\left[t^2\right] \right| = 2\left| \log(|t|) \right|$$

FUNZIONE PARI.

Grafico di $g(t) = \log(|t|)$:

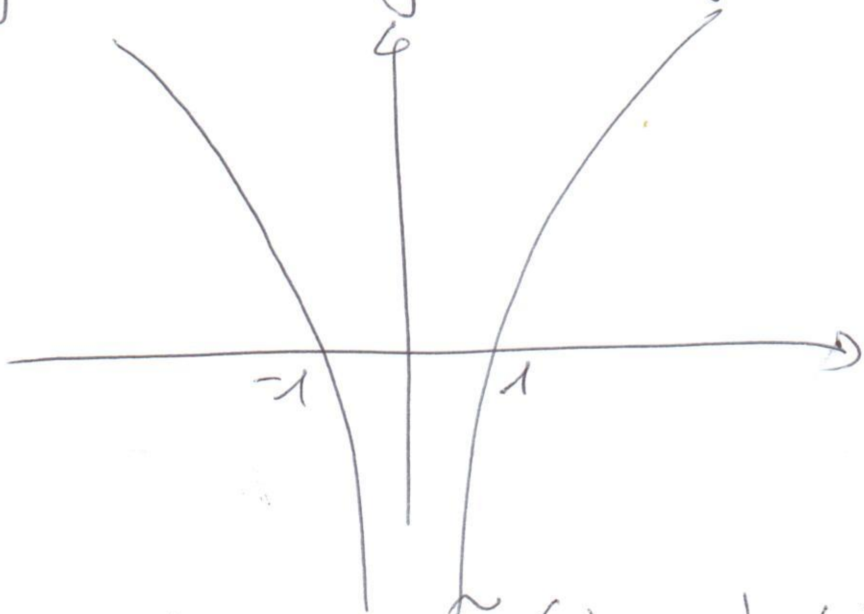
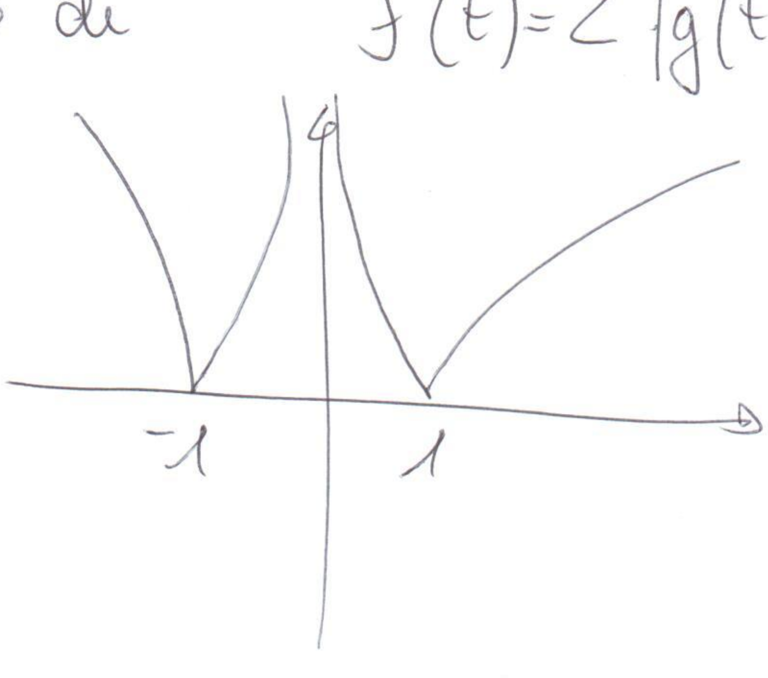
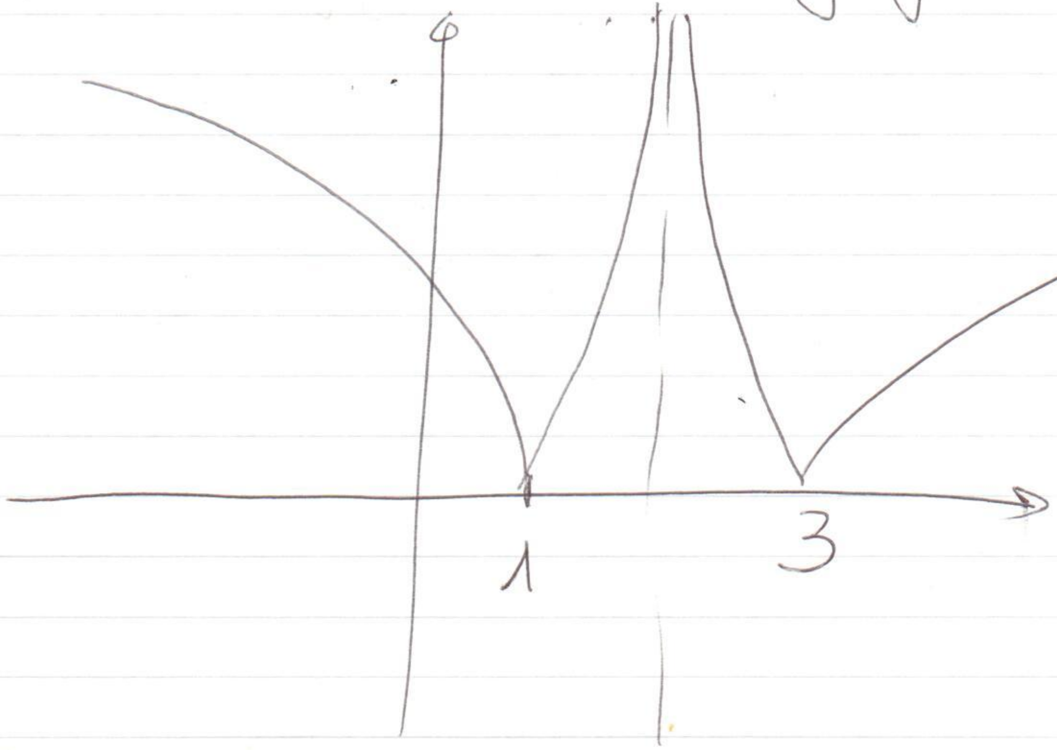


Grafico di $\tilde{f}(t) = 2|g(t)|$:



$B_{5\,ter}$

Traslando di 2, si ha il grafico di $f(x)$:



4) $a(x) = tg\,x \in C^{\infty}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \ni x_0$

$b(y) = 1 + e^{-y} \in C^{\infty}(\mathbb{R})$

$\Rightarrow \exists\, I(0) \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ t.c.

$\exists!$ sol. $y \in C^1(I)$ (SOL. LOCALE)

$b(y) = 1 + e^{-y} \neq 0 \quad \forall y \in \mathbb{R}$ B6

$\Rightarrow$ NO SOL. SING.

METODO SEPARAZIONE VARIABILI:

$$\int \frac{dy}{1+e^{-y}} = -\int tg x \, dx$$

$$\int \frac{e^y}{e^y+1} dy = \int \frac{-\sin x}{\cos x} dx$$

$$\log|e^y+1| = +\log|\cos x| + \log C$$

$C>0$

$\cos x > 0$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\Rightarrow \log(e^y+1) = +\log(\cos x) + \log C$$

$$e^y + 1 = C\cos x \quad ; C>0$$

$$e^y = C\cos x - 1$$

$y(0)=0 \Rightarrow 1 = C-1 \Rightarrow C=2$

$$\Rightarrow e^y = 2\cos x - 1$$

B7

$$\Rightarrow \quad y = \log(2\cos x - 1)$$

$$2\cos x - 1 > 0 \quad \Rightarrow \cos x > \frac{1}{2}$$

$$\Rightarrow \quad x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$$

$$y \in C^1\left(-\frac{\pi}{3}, \frac{\pi}{3}\right).$$

5) $$\lim_{x\to 0} \frac{\log(1+\cos x)}{[\cos x + \log(1+x)]} \cdot \lim_{x\to 0} \frac{\log(1+tg^5 x)}{(\sin x + \sinh x - 2xe^{x^4})}$$

$$= \frac{\log 2}{1} \cdot \lim_{x\to 0} \frac{tg^5 x}{x - \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} + x + \cancel{\frac{x^3}{6}} + \frac{x^5}{5!} - 2x(1+x^4) + o(x^5)}$$

$$= \log 2 \cdot \lim_{x\to 0} \frac{\cancel{x^5}}{\cancel{2x - 2x} + \cancel{x^5}\left(\frac{\cancel{2}\,1}{\cancel{120}\,60} - 2\right)}$$

$$= \log 2 \, \frac{1}{\left(\frac{-119}{60}\right)} = \boxed{\frac{-60 \log 2}{119}}$$