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ESEMPI DI INTEGRALI DOPPI CALCOLABILI CON LE FORMULE DI RIDUZIONE

ESEMPIO 1)

Sia

$$T = \{(x,y) \in \mathbb{R}^2 \mid x \in [0,1] \quad ; \quad e^{-x} \le y \le e^{x} \ \} \ .$$

Calcolare gli integrali

$$\iint_T f_i(x,y)\ dx\ dy \ , \qquad i = 1,2,3$$

essendo

$$\textbf{I)}\ f_1(x,y) = x^2 + y^2 \quad ; \quad \textbf{II)}\ f_2(x,y) = \frac{e^x}{(1+y)^2} \quad ; \quad \textbf{III)}\ f_3(x,y) = y^2 \cos x \ .$$

I)

$$\iint_T f_1(x,y)\ dx\ dy \ = \int_0^1 \ dx \ \int_{e^{-x}}^{e^x} \ (x^2+y^2)\ dy \ =$$
$$= \int_0^1 \ \left[\int_{e^{-x}}^{e^x} \ x^2 \ \ dy \ + \int_{e^{-x}}^{e^x} \ y^2 \ dy \right] \ dx \ =$$
$$= \int_0^1 \ x^2 \left(e^x - e^{-x} \right) \ dx \ + \int_0^1 \frac{1}{3} \left(e^{3x} - e^{-3x} \right) \ dx \ = ...$$

II)

$$\iint_T f_2(x,y)\ dx\ dy \ = \int_0^1 \ dx \ \int_{e^{-x}}^{e^x} \frac{e^x}{(1+y)^2} \ dx \ =$$
$$= \int_0^1 \ e^x \ dx \ \int_{e^{-x}}^{e^x} \frac{1}{(1+y)^2} \ \ dy \ = \ \int_0^1 \ e^x \left[\frac{1}{1+e^{-x}} - \frac{1}{1+e^x} \right] \ dx \ =$$
$$= \int_0^1 \ \frac{e^{2x}}{1+e^x} \ \ dx \ - \int_0^1 \ \frac{e^x}{1+e^x} \ \ dx \ = ...$$

$$[\ \text{R.} : 2\log 2 - 2\log(1+e) + e - 1 \ ]$$

III)

$$\iint_T f_3(x,y)\ dx\ dy\ = \int_0^1\ dx\ \int_{e^{-x}}^{e^x}\ y^2 \cos x\ dy\ =$$
$$= \int_0^1\ \cos x\ dx\ \int_{e^{-x}}^{e^x}\ y^2\ dy\ =\ \frac{1}{3}\ \int_0^1\ \cos x \left(e^{3x} - e^{-3x}\right)\ dx\ =$$
$$= \frac{1}{3}\ \int_0^1\ e^{3x} \cos x\ dx\ - \frac{1}{3}\ \int_0^1\ e^{-3x} \cos x\ dx\ = ...$$

ESEMPIO 2)

Sia

$$T = \{(x,y) \in I\!R^2 \mid y \in [0, \pi] \quad ; \quad \sin^2 y \le x \le \sin y\ \}\ .$$

Calcolare

$$\iint_T f_i(x,y)\ dx\ dy\ , \qquad i = 1, 2, 3 \quad ,$$

essendo

$$\textbf{I)}\ f_1(x,y) = x^2 + y^2 \quad ; \quad \textbf{II)}\ f_2(x,y) = x \cos^2 y \quad ; \quad \textbf{III)}\ f_3(x,y) = \frac{|\cos y|}{(1+x)^2}\ .$$

I)

$$\iint_T f_1(x,y)\ dx\ dy\ = \int_0^\pi\ dy\ \int_{\sin^2 y}^{\sin y}\ (x^2 + y^2)\ dx\ =$$
$$= \int_0^\pi\ \frac{1}{3} \left(\sin^3 y - \sin^6 y\right)\ dy\ + \int_0^\pi\ y^2 \left(\sin y - \sin^2 y\right)\ dy\ =$$
$$= \frac{1}{3} \int_0^\pi\ \sin^3 y\ dy\ - \frac{1}{3}\ \int_0^\pi\ \sin^6 y\ dy\ + \int_0^\pi\ y^2 \sin y\ dy\ - \int_0^\pi\ y^2 \sin^2 y\ dy\ =\ ...$$

II)

$$\iint_T f_2(x,y)\ dx\ dy\ =\ \iint_T x \cos^2 y\ dx\ dy\ = \int_0^\pi \cos^2 y\ dy\ \int_{\sin^2 y}^{\sin y}\ x\ dx\ =$$
$$= \frac{1}{2}\ \int_0^\pi\ \cos^2 y \left[\sin^2 y - \sin^4 y\right]\ dy\ =\ ...$$

III)

$$\iint_T f_3(x,y)\ dx\ dy\ = \int_0^\pi\ dy\ \int_{\sin^2 y}^{\sin y} \frac{|\cos y|}{(1+x)^2}\ dx\ =$$
$$= \int_0^\pi\ |\cos y|\ dy\ \int_{\sin^2 y}^{\sin y} \frac{1}{(1+x)^2}\ dx\ = \int_0^\pi |\cos y| \left[\frac{1}{1+\sin^2 y} - \frac{1}{1+\sin y}\right]\ dy\ =$$
$$= \int_0^{\frac{\pi}{2}}\ \cos y \left[\frac{1}{1+\sin^2 y} - \frac{1}{1+\sin y}\right]\ dy\ - \int_{\frac{\pi}{2}}^{\pi}\ \cos y \left[\frac{1}{1+\sin^2 y} - \frac{1}{1+\sin y}\right]\ dy\ = \ ...$$

ESEMPI DI INTEGRALI DOPPI CALCOLABILI CON LE FORMULE DI TRASFORMAZIONE DELLE COORDINATE POLARI

ESEMPIO 1)

Sia

$$T = \{(x,y) \in I\!R^2 \mid 1 \leq x^2 + y^2 \leq 4\ \}\ .$$

Comunque si scelga la funzione $f(x,y) \in C^0(T)$, si ha

$$\iint_T f(x,y)\ dx\ dy\ =\ \iint_D f(\rho\cos\theta, \rho\sin\theta)\ \rho\ d\rho\ d\theta\ = \int_0^{2\pi}\ d\theta \int_1^2 f(\rho\cos\theta, \rho\sin\theta)\ \rho\ d\rho\ ,$$

dove il dominio D del piano $0\rho\theta$, che viene trasformato in T dalla trasformazione

$$(\rho,\theta) \longrightarrow (\rho\cos\theta, \rho\sin\theta)$$

delle **coordinate polari**

$$\begin{cases} x & = \rho\cos\theta \\ y & = \rho\sin\theta \end{cases}$$

con determinante funzionale

$$\frac{\partial(x,y)}{\partial(\rho,\theta)}\ =\ \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix}\ =\ \rho\ ,$$

è

$$D = \{(\rho,\theta) \in I\!R^2 \mid \rho \in [1,2] \quad ; \quad \theta \in [0, 2\pi]\}\ .$$

I)

$$\iint_T \frac{1}{x^2+y^2+1}\ dx\ dy \ = \int_0^{2\pi} \ d\theta \int_1^2 \frac{1}{1+\rho^2}\ \rho\ d\rho \ = \ \pi \log\left(\frac{5}{2}\right) \ .$$

II)

$$\begin{aligned}\iint_T (ax^2+by^2)\ dx\ dy \ &= \int_0^{2\pi} \ d\theta \ \int_1^2 [a\rho^2\cos^2\theta + b\rho^2\sin^2\theta]\rho\ d\rho \ = \\ &= \int_0^{2\pi} \left[a\cos^2\theta \ \int_1^2 \ \rho^3\ d\rho + b\sin^2\theta \ \int_1^2 \ \rho^3\ d\rho\right]\ d\theta \ = \\ &= \frac{15}{4}a \ \int_0^{2\pi} \cos^2\theta\ d\theta + \frac{15}{4}b \ \int_0^{2\pi} \sin^2\theta\ d\theta \ = \ \frac{15}{4}(a+b)\pi \ .\end{aligned}$$

III)

$$\iint_T \frac{\sqrt{x^2+y^2}}{x^2+y^2+4}\ dx\ dy \ = \int_0^{2\pi} \ d\theta \int_1^2 \frac{\rho}{4+\rho^2}\ \rho \ \ d\rho \ =$$

$$= \int_0^{2\pi} \ d\theta \int_1^2 \frac{\rho^2}{4+\rho^2} \ \ d\rho \ = 2\pi \ \int_1^2 \frac{\rho^2}{4+\rho^2} \ \ d\rho \ = \ 2\pi(1+\arctan\frac{1}{2}-\frac{\pi}{2}) \ .$$

ESEMPIO 2)

Sia

$$T = \{(x,y)\in \mathbb{R}^2 \mid 2 \le x^2+y^2 \le 9 \quad ; \quad y-x \ge 0 \ \} \ .$$

Calcolare

$$\textbf{I)} \ \iint_T \frac{3x+y}{x^2+y^2} \ \ dx\ dy \ ;$$

$$\textbf{II)} \ \iint_T \ (x^2+y^2) \ \ dx\ dy \quad ;$$

$$\textbf{III)} \ \iint_T \ e^{\sqrt{x^2+y^2}} \ \ dx\ dy \ .$$

Il dominio D, che viene trasformato in T tramite la trasformazione delle coordinate polari, è

$$D = [\sqrt{2},3]\times\left[\frac{\pi}{4},\frac{5\pi}{4}\right] \ = \ \{(\rho,\theta)\in\mathbb{R}^2 \mid \sqrt{2}\le\rho\le 3 \quad ; \quad \frac{\pi}{4}\le\theta\le\frac{5\pi}{4} \ \} \ .$$

I)

$$\iint_T \frac{3x+y}{x^2+y^2}\ dx\ dy\ =\ \iint_D \frac{3\rho\cos\theta+\rho\sin\theta}{\rho^2}\ \rho\ \ d\rho\ d\theta\ =$$

$$=\ \iint_D (3\cos\theta+\sin\theta)\ d\rho\ d\theta\ =\ \int_{\sqrt{2}}^{3} d\rho \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (3\cos\theta+\sin\theta)\ d\theta\ =$$

$$=(3-\sqrt{2})\left[3\left(\sin\frac{5\pi}{4}-\sin\frac{\pi}{4}\right)+\left(\cos\frac{\pi}{4}-\cos\frac{5\pi}{4}\right)\ \right]\ =$$

$$=(3-\sqrt{2})(-3\sqrt{2}+\sqrt{2})\ =\ 2(2-3\sqrt{2})\ .$$

II)

$$\iint_T (x^2+y^2)\ \ dx\ dy\ =\ \iint_D \rho^2\rho\ \ d\rho\ d\theta\ =\ \iint_D \rho^3\ \ d\rho\ d\theta\ =$$

$$=\ \int_{\sqrt{2}}^{3} \rho^3\ d\rho \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\ d\theta\ =\ \pi\ \int_{\sqrt{2}}^{3} \rho^3\ d\rho\ =\ \frac{\pi}{4}(81-4)\ =\ \frac{77\pi}{4}\ .$$

III)

$$\iint_T e^{\sqrt{x^2+y^2}}\ \ dx\ dy\ =\ \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\ d\theta\ \int_{\sqrt{2}}^{3} e^{\rho}\rho\ \ d\rho\ =\ \pi\ \int_{\sqrt{2}}^{3} e^{\rho}\rho\ \ d\rho\ =$$

$$=\ \pi\left[\ 3e^3-\sqrt{2}e^{\sqrt{2}}-\ \int_{\sqrt{2}}^{3} e^{\rho}\ \ d\rho\ \right]\ =\ \pi\left(2e^3-(\sqrt{2}-1)e^{\sqrt{2}}\right)\ .$$

ESEMPIO 3)

Sia

$$T=\{(x,y)\in I\!R^2\ |\ x^2+y^2-2ax-2by\leq 0\quad ;\quad x^2+y^2-ax-by\geq 0\ \}\ .$$

Assegnata $f(x,y)$, calcolare

$$\iint_T f(x,y)\ dx\ dy\ .$$

Il dominio D, che viene trasformato in T mediante la trasformazione delle coordinate polari

$$(\rho,\theta) \longrightarrow (\rho\cos\theta, \rho\sin\theta)\ ,$$

è dato da

$$D = \{(\rho,\theta) \in I\!R^2 \mid \theta \in [\alpha,\beta] \quad ; \quad a\cos\theta + b\sin\theta \ \le\ \rho\ \le\ 2a\cos\theta + 2b\sin\theta\ \}\ .$$

α e β dipendono direttamente da a e b, come si illustra con il successivo caso particolare:

$$T = \{(x,y) \in I\!R^2 \mid x^2 + y^2 \le 2x \quad ; \quad x^2 + y^2 \ge x\ \}\ .$$

Segue che

$$D = \{(\rho,\theta) \in I\!R^2 \mid\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad ; \quad \cos\theta\ \le\ \rho\ \le\ 2\cos\theta\ \}\ .$$

$$\begin{aligned}
\iint_T (x^2+y^2)\ dx\ dy\ &=\ \iint_D \rho^2\rho\ \ d\rho\ d\theta\ = \\
&=\ \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\ d\theta \int_{\cos\theta}^{2\cos\theta}\ \rho^3\ d\rho\ =\ \frac{1}{4}\ \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \left(\left[\rho^4\right]_{\rho=\cos\theta}^{\rho=2\cos\theta} \right)\ d\theta\ = \\
&=\ \frac{15}{4}\ \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\ \cos^4\theta\ d\theta\ =\ \frac{15}{2}\ \int_0^{\frac{\pi}{2}} \cos^4\theta\ d\theta\ =\ \frac{15}{2}\ \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2\ \ d\theta\ =\ ...
\end{aligned}$$

$$
\begin{aligned}
\iint_T \frac{|y|}{4+\sqrt{x^2+y^2}}\ dx\ dy \ &= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \ d\theta \int_{\cos\theta}^{2\cos\theta} \frac{\rho^2|\sin\theta|}{4+\rho}\ d\rho \ = \\
&= \ 2\int_0^{\frac{\pi}{2}} \sin\theta\ d\theta \int_{\cos\theta}^{2\cos\theta} \frac{\rho^2}{4+\rho}\ d\rho \ = \ 2\int_0^{\frac{\pi}{2}} \sin\theta\ d\theta \int_{\cos\theta}^{2\cos\theta} \left[\rho - 4 + \frac{16}{\rho+4}\right]\ d\rho \ = \\
&= \ 2\int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{\rho^2}{2} - 4\rho + 16\log(\rho+4)\right]_{\cos\theta}^{2\cos\theta} d\theta \ = \\
&= \ 2\int_0^{\frac{\pi}{2}} \left[\ \frac{3\cos^2\theta}{2} - 4\cos\theta + 16\log(2\cos\theta+4) - 16\log(\cos\theta+4)\right]\sin\theta\ d\theta \ = \\
&= -\int_0^{\frac{\pi}{2}} \left[\ 3\cos^2\theta - 8\cos\theta + 32\log(2\cos\theta+4) - 32\log(\cos\theta+4)\right]\ d(\cos\theta) \ = \\
&= -\left[\cos^3\theta - 4\cos^2\theta\right]_0^{\frac{\pi}{2}} - 32\int_1^0 [\log(2t+4) - \log(t+4)]\ dt \ = \\
&= -3 + 32\left[t\log(2t+4) - t + 2\log(2t+4) - t\log(t+4) + t - 4\log(t+4)\right]_0^1 \ = \\
&= -3 + 32(3\log 6 - 5\log 5 + 2\log 4)\ ,
\end{aligned}
$$

avendo tenuto conto che

$$
\begin{aligned}
\int \log(at+b)\ dt \ &= \ t\log(at+b) - \int \frac{at}{at+b}\ dt \ = \\
&= \ t\log(at+b) - \int \left[1 - \frac{b}{at+b}\right] dt \ = \ t\log(at+b) - t + \frac{b}{a}\log(at+b) + C\ .
\end{aligned}
$$