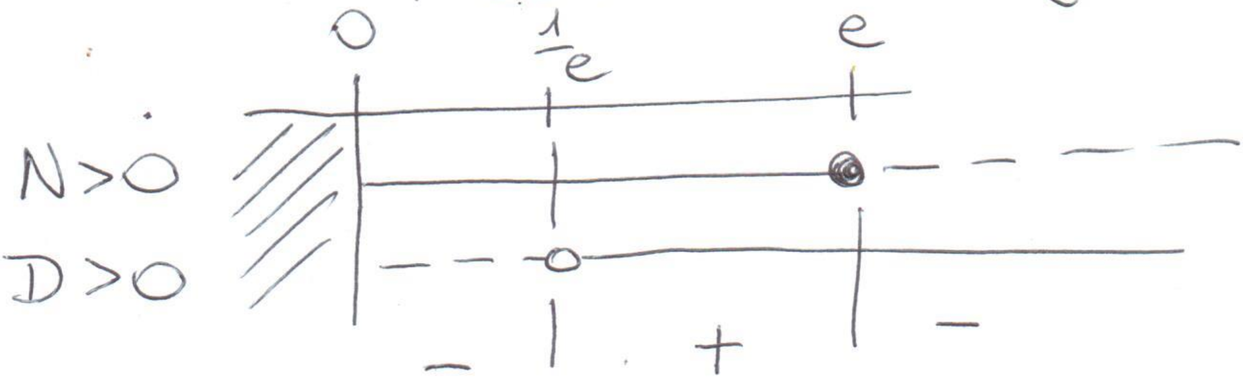
SVOLGIMENTI PROVA SCRITTA di ANALISI 1 - 1°/7/2021.

1) $\begin{cases} x>0 \\ 1+\ln x \neq 0 \end{cases} \Rightarrow \begin{cases} x>0 \\ x \neq \frac{1}{e} \end{cases}$

$D = \left(0, \frac{1}{e}\right) \cup \left(\frac{1}{e}, +\infty\right).$

Segno: $1-\ln x > 0 \iff 0<x<e$

$1+\ln x > 0 \iff x > \frac{1}{e}$



$f(x)<0$ per $x \in \left(0, \frac{1}{e}\right) \cup \left(e, +\infty\right)$

$f(x)>0$ per $x \in \left(\frac{1}{e}, e\right)$

No intersezione asse y.

Intersezione asse x: $f(x)=0 \iff x=e$.

2

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{-\ln x}{\ln x} = -1$$

NO asintoto verticale

$$\lim_{x\to \frac{1}{e}^\pm} f(x) = \frac{2}{0^\pm} = \pm\infty$$

asintoto verticale $x = \frac{1}{e}$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} \frac{-\ln x}{\ln x} = -1$$

asintoto orizzontale $y=-1$

$$f'(x) = \frac{-\frac{1}{x}(1+\ln x) - \frac{1}{x}(1-\ln x)}{(1+\ln x)^2} = \frac{-2}{x(1+\ln x)^2} < 0$$

$\forall x \in D$.

$\Rightarrow$ f decresce in $\left(0, \frac{1}{e}\right)$ e in $\left(\frac{1}{e}, +\infty\right)$

Poichè f NON è definita in $x=0$, non abbiamo MAX. REL.

Poichè $\lim_{x\to \frac{1}{e}^\pm} f(x) = \pm\infty$, non abbiamo MAX. o MIN. ASS.

Anche se non richiesto, completiamo il grafico. ③

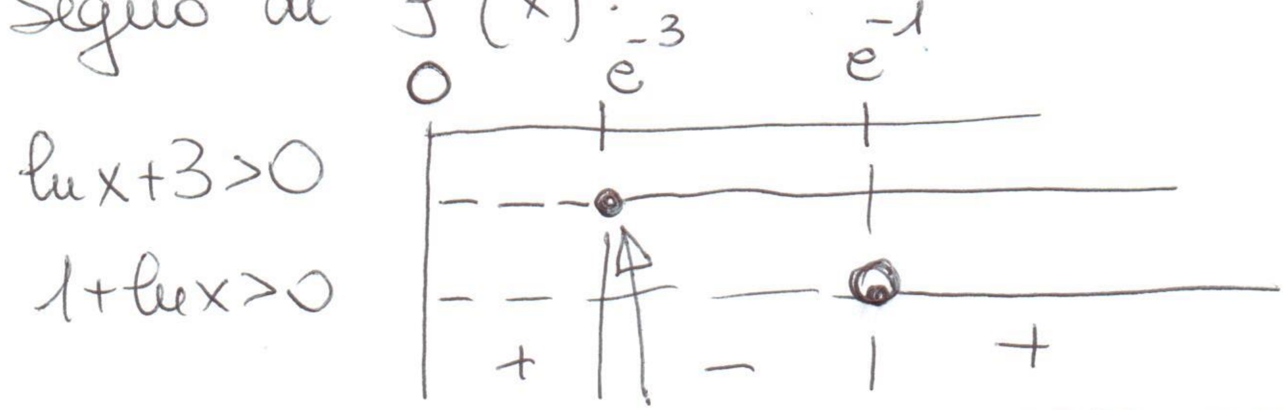
$$f''(x) = \frac{2}{\left[x(1+\ln x)^2\right]^2}\left(x(1+\ln x)^2\right)'$$

$$= \frac{2}{\left[x^2(1+\ln x)^4\right]}\left[(1+\ln x)^2 + \cancel{x\frac{1}{x}}\, x\,\frac{1}{x}\,2(1+\ln x)\right]$$

$$\cancel{= \frac{2(2+\ln x)}{x^2(1+\ln x)^4}} \qquad = \frac{2}{x^2(1+\ln x)^3}\left[1+\ln x+2\right]$$

$$= \frac{2(\ln x+3)}{x^2(1+\ln x)^3}$$

Segno di $f''(x)$:



FLESSO DISCENDENTE

f convessa in $\left(0, \frac{1}{e^3}\right)$; concava in $\left(\frac{1}{e^3}, \frac{1}{e}\right)$

convessa in $\left(\frac{1}{e}, +\infty\right)$.

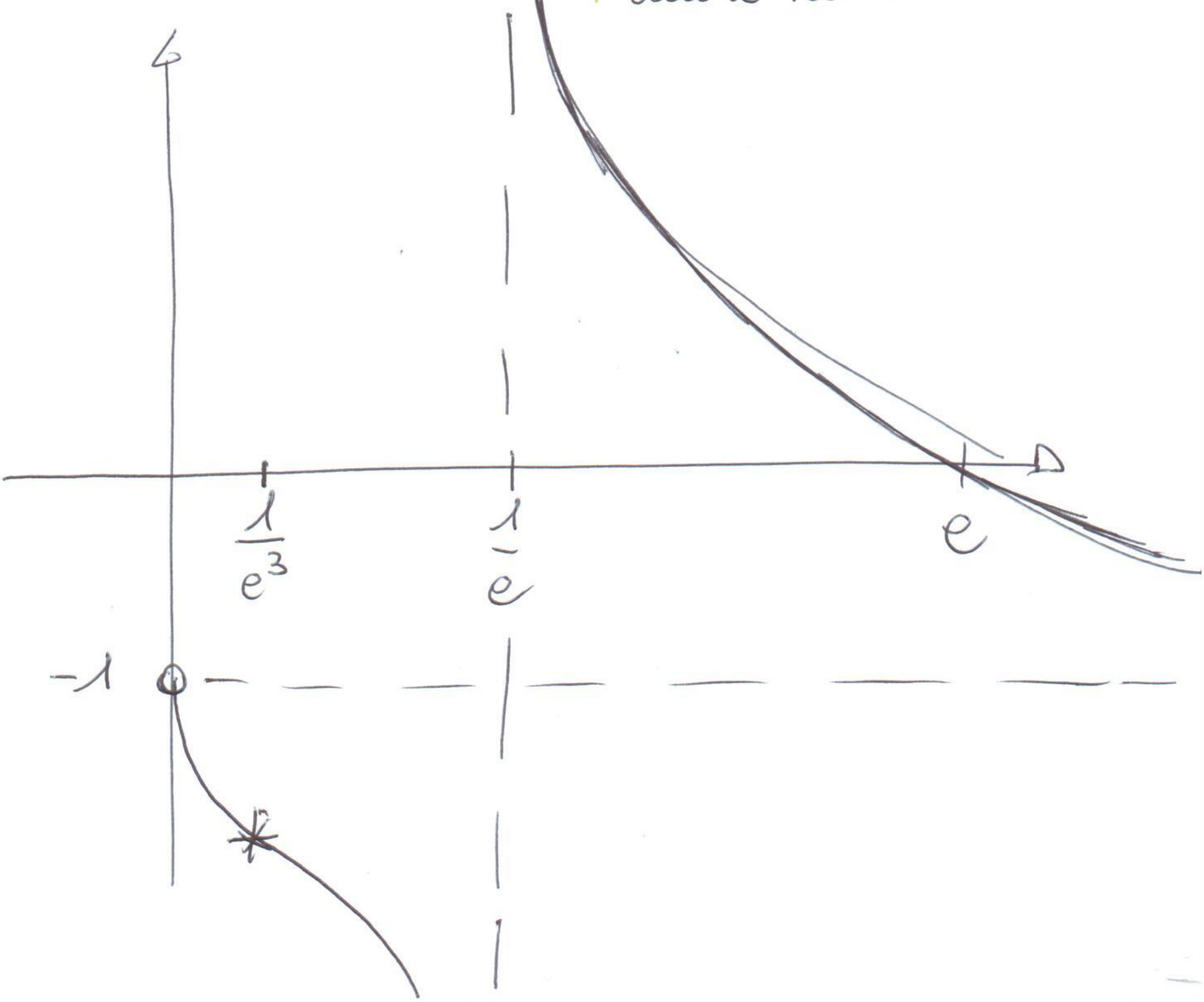
4

Flesso discendente in $x = \frac{1}{e^3}$

$$f\left(\frac{1}{e^3}\right) = \frac{4}{-2} = -2$$

Grafico: si osservi che

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \frac{-2}{\underbrace{x \cdot \ln^2 x}_{\text{limite notevole}}} = \frac{-2}{0^+} = -\infty$$



2) $\lim_{x\to 0^+} \frac{\sin x - x}{x^2} = \lim_{x\to 0^+} \frac{\cancel{x} - \frac{x^3}{6} + o(x^3) \cancel{-x}}{x^2}$

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$= \lim_{x\to 0^+} \left(-\frac{x}{6}\right) = 0$

$\lim_{x\to 0^-} f(x) = f(0) = 0$

$\Rightarrow f$ CONTINUA in $x=0$.

Derivabilità:

$f'_+(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{\sin x - x}{x^3}$

$= -\frac{1}{6}$

$f'_-(0) = \frac{1}{6}\left[\frac{\ln(1-x)}{x}\right] = -\frac{1}{6}$

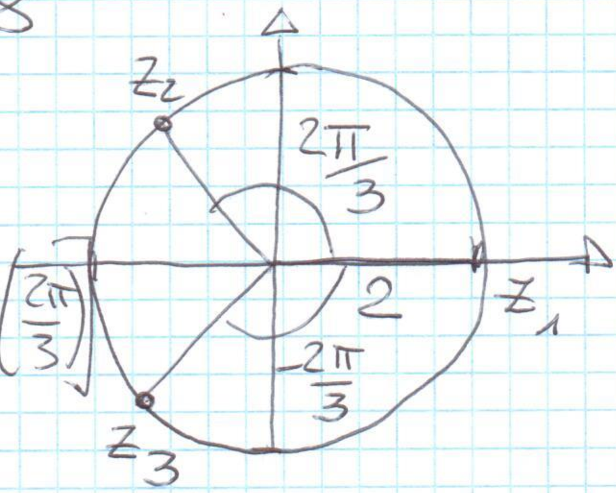
f DERIVABILE in $x=0$.

3) Poniamo $z^3 = t$

$\Rightarrow t^2 - 7t - 8 = (t-8)(t+1) = 0$

$t_1 = 8$; $t_2 = -1$

1° ~~[illegible]~~ caso: $z^3 = 8$



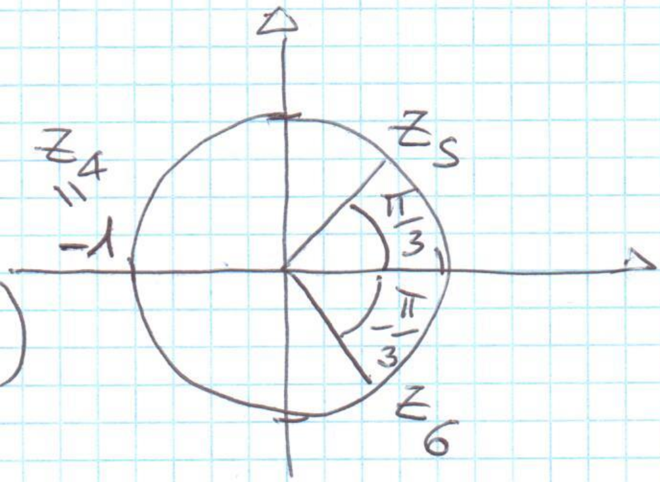
$z_1 = 2$

$z_2 = 2\left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right]$

$= -1 + \sqrt{3}\, i$

$z_3 = \overline{z}_2 = -1 - \sqrt{3}\, i$

2° caso: $z^3 = -1$



$z_4 = -1$

$z_5 = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$

$= \frac{1}{2} + i \frac{\sqrt{3}}{2}$

$z_6 = \overline{z}_5 = \frac{1}{2} - i \frac{\sqrt{3}}{2}$

4) $\alpha = 0$: $a_n = \operatorname{arctg}(\sqrt{2}-1) \xrightarrow[n\to\infty]{} \operatorname{arctg}(\sqrt{2}-1)$ (7)

$\alpha < 0$:

$$a_n \xrightarrow[n\to+\infty]{} \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$

$\alpha > 0$:

$$a_n \underset{n\to\infty}{\sim} \operatorname{arctg}\left(\frac{1}{2n^\alpha}\right) \sim \frac{1}{2n^\alpha}$$

$$a_n \xrightarrow[n\to\infty]{} 0 \quad \forall \alpha > 0.$$

Serie: per $\alpha \leq 0$ $a_n \not\to 0$

$\Rightarrow$ la serie (a termini positivi) diverge a $+\infty$.

Per $\alpha > 0$: $a_n \sim \dfrac{1}{2n^\alpha}$

$\Rightarrow$ la serie diverge per $\alpha \leq 1$

converge per $\alpha > 1$.

5) Poniamo $t = x^5 \Rightarrow dt = 5x^4 dx$

$t(0) = 0 \quad ; \quad t(+\infty) = +\infty$

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$$\Rightarrow \int_0^{+\infty} f(x)\,dx = \int_0^{+\infty} \frac{t\,dt}{5(1+t)^3}$$

$$\frac{t}{(1+t)^3} = \frac{t+1-1}{(t+1)^3} = \frac{1}{(t+1)^2} - \frac{1}{(t+1)^3}$$

(oppure con i fratti semplici)

$$\Rightarrow \int_0^{+\infty} f(x)\,dx = \frac{1}{5}\int_0^{+\infty} \left[\frac{1}{(t+1)^2} - \frac{1}{(t+1)^3}\right] dt$$

$$= \frac{1}{5}\left[\frac{-1}{t+1} + \frac{1}{2(t+1)^2}\right]_0^{+\infty}$$

$$= \frac{1}{5}\left[1 - \frac{1}{2}\right] = \frac{1}{10}.$$

5 bis) L'equazione è lineare, con coefficienti definiti e C^{∞} in $(-\infty, 0) \cup (0, +\infty)$

Poiché $x_0 = -1$ $\Rightarrow$ la soluzione esiste ed è unica in tutto $(-\infty, 0)$ (soluzione globale).

Poniamo $z = y'$

$$\begin{cases} z' - \frac{1}{x} z = x \\ z(-1) = 0 \end{cases} \qquad x<0 \Rightarrow t<0$$

$$z(x) = e^{\int_{-1}^{x} \frac{1}{t} dt} \left[\int_{-1}^{x} e^{-\int_{-1}^{t} \frac{1}{s} ds} \, t \, dt \right]$$

$$= e^{\ln(|t|)\big|_{-1}^{x}} \cdot \left[\int_{-1}^{x} e^{-\ln(|s|)\big|_{-1}^{t}} \, t \, dt \right]$$

$$= e^{\ln(-x)} \left[\int_{-1}^{x} e^{-\ln(-t)} \, t \, dt \right]$$

$$= -x \left[\int_{-1}^{x} -\frac{1}{t} \, t \, dt \right] = -x \left[-t \Big|_{-1}^{x} \right] = -x(-x-1)$$

$$= x^2 + x$$

$$\begin{cases} y' = z = x^2 + x \\ y(-1) = 0 \end{cases} \Rightarrow \begin{cases} y(x) = \frac{x^3}{3} + \frac{x^2}{2} + C \\ y(-1) = 0 \end{cases}$$

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$$\Rightarrow 0 = -\frac{1}{3} + \frac{1}{2} + C \Rightarrow C = -\frac{1}{6}$$

$$\Rightarrow y(x) = \frac{x^3}{3} + \frac{x^2}{2} - \frac{1}{6}$$