SVOLGIMENTI PROVA SCRITTA di ANALISI 1 del 10/1/2019

A_1

COMPITO A

1) Abbassiamo innanzitutto l'ordine:

$$y'(x) = z(x)$$

$$\Rightarrow \quad z''(x) + 4z(x) = x^2 - 1$$

Omogenea associata: $\alpha^2 + 4 = 0 \Rightarrow \alpha_{1,2} = \pm 2i$

$$\Rightarrow \quad z_0(x) = C_1 \cos(2x) + C_2 \sin(2x).$$

Non omogenea: metodo somiglianza.

Poichè $\alpha = 0$ NON è radice del polinomio caratteristico, allora $z_p(x) = ax^2 + bx + c$

$$\Rightarrow z_p' = 2ax + b \quad ; \quad z_p'' = 2a$$

$$\Rightarrow \quad 2a + 4ax^2 + 4bx + 4c = x^2 - 1$$

$$\Rightarrow \begin{cases} 4a = 1 \\ b = 0 \\ 2a + 4c = -1 \end{cases} \Rightarrow \begin{cases} a = 1/4 \\ b = 0 \\ 4c = -1 - \frac{1}{2} \end{cases} \Rightarrow \begin{cases} a = 1/4 \\ b = 0 \\ c = -\frac{3}{8} \end{cases}$$

$$\Rightarrow \underbrace{z(x)}_{=\,y'(x)} = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{4}x^2 - \frac{3}{8}.$$

$$z'(x) = y''(x) = -2C_1 \sin(2x) + 2C_2 \cos(2x) + \frac{1}{2}x$$

A_2

$$z(0) = y'(0) = 0 = C_1 - \frac{3}{8}$$

$$\Rightarrow C_1 = \frac{3}{8}$$

$$z'(0) = y''(0) = 0 = 2C_2 \quad \Rightarrow \quad C_2 = 0$$

$$\Rightarrow z(x) = \frac{3}{8}\cos(2x) + \frac{1}{4}x^2 - \frac{3}{8}$$

$$\Rightarrow y(x) = \int \left[\frac{3}{8}\cos(2x) + \frac{1}{4}x^2 - \frac{3}{8}\right] dx$$

$$= \frac{3}{16}\sin(2x) + \frac{1}{12}x^3 - \frac{3}{8}x + C_3$$

$$y(0) = 0 = C_3$$

$$\Rightarrow y(x) = \frac{3}{16}\sin(2x) + \frac{1}{12}x^3 - \frac{3}{8}x$$

In alternativa, si può studiare direttamente l'equazione di 3° ordine, anche se non in programma:

OMO.: $\quad y''' + 4y' = 0 \quad \Rightarrow \alpha^3 + 4\alpha = 0$

$$\Rightarrow \alpha_{1,2} = \pm 2i \quad ; \quad \alpha_3 = 0.$$

$$\Rightarrow y_0(x) = C_1\cos(2x) + C_2\sin(2x) + C_3$$

NON OMO.: $\quad \alpha = 0$ radice con $m_a = 1$

A2bis

$$\Rightarrow y_p(x) = x\,(ax^2+bx+c)$$

$$y_p'(x) = 3ax^2 + 2bx + c$$

$$y_p''(x) = 6ax + 2b \qquad y_p''' = 6a$$

$$\Rightarrow \quad 6a + 4(3ax^2 + 2bx + c) = x^2 - 1$$

$$\begin{cases} 12a = 1 \\ 8b = 0 \\ 6a + 4c = -1 \end{cases} \Rightarrow \begin{cases} a = 1/12 \\ b = 0 \\ c = 1/4\left[-1-6a\right] = -\frac{3}{8} \end{cases}$$

$$\Rightarrow y(x) = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) + C_3 + \frac{1}{12}x^3 - \frac{3}{8}x$$

Il resto rimane uguale.

2) Integriamo per parti:

$$\int x \cdot \left(\frac{\cos x}{\sin^3 x}\right) dx = \int x \left(-\frac{1}{2\sin^2 x}\right)' dx$$

$$= -\frac{1}{2\sin^2 x}x + \int \frac{1}{2\sin^2 x} dx$$

$$= -\frac{\ast}{2\sin^2 x} - \frac{1}{2}\operatorname{cotg} x + C$$

A3

Si osservi che, per $x \to 0$,

$$\frac{x\cos x}{\sin^3 x} \underset{x\to 0}{\sim} \frac{x}{x^3} = \frac{1}{x^2}$$ che NON è integrabile in $\left(0, \frac{\pi}{2}\right]$.

Altrimenti, calcoliamo direttamente l'integrale improprio:

$$\int_0^{\frac{\pi}{2}} f(x)dx = -\frac{1}{2}\left[\frac{x}{\sin^2 x} + \operatorname{cotg} x\right]_0^{\frac{\pi}{2}} =$$

$$= -\frac{1}{2}\left[\frac{\pi}{2} - \lim_{x\to 0}\left[\frac{x+\cos x \sin x}{\sin^2 x}\right]\right] = -\frac{1}{2}\left(\frac{\pi}{2} - \infty\right) = +\infty$$

in quanto

$$\frac{x+\cos x\sin x}{(\sin x)^2} \underset{x\to 0}{\sim} \frac{x+x+o(x)}{x^2} \sim \frac{2}{x} \longrightarrow +\infty$$

L'integrale, dunque, DIVERGE a $+\infty$.

3) Chiamiamo $t = \frac{2z-i}{3-i}$

$$\Rightarrow t^2 = 4i \quad \Rightarrow \quad t_{1,2} = \pm 2\sqrt{i}$$

$$= \pm 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \pm\sqrt{2}(1+i)$$

$$\Rightarrow \frac{2z-i}{3-i} = \pm\sqrt{2}(1+i)$$

A4

$$2z_1 - i = \sqrt{2}\,(1+i)(3-i)$$

$$\Rightarrow z_1 = \frac{1}{2}\left[i + \sqrt{2}\,(3+2i+1)\right] = \frac{1}{2}\left[4\sqrt{2} + (2\sqrt{2}+1)i\right];$$

$$2z_2 - i = -\sqrt{2}\,(1+i)(3-i)$$

$$\Rightarrow z_2 = \frac{1}{2}\left[i - \sqrt{2}\,(3+2i+1)\right] = \frac{1}{2}\left[-4\sqrt{2} + (1-2\sqrt{2})i\right]$$

Qualora si fosse scelta la via della risoluzione dell'equazione del 2° grado

$$\Rightarrow (2z-i)^2 = 4i(3-i)^2$$

$$\Rightarrow 4z^2 - 4iz - 1 = 4i(8-6i)$$

$$\Rightarrow 4z^2 - 4iz - 25 - 32i = 0$$

$$\Rightarrow z_{1,2} = \frac{2i + \sqrt{-4+100+128i}}{4} = \frac{1}{2}i + \sqrt{2}\sqrt{(3+4i)}$$

Ma $3+4i = (2+i)^2$

$$\Rightarrow z_{1,2} = \frac{1}{2}i \pm \sqrt{2}\,(2+i)$$

4) A4b

$$a_n = n^\alpha \left[\frac{1}{2n^2} - \frac{1}{3(2n^2)^3} - \frac{1}{2n^2} + o\left(\frac{1}{n^6}\right) \right] =$$

$$\sim -\frac{1}{24\, n^{6-\alpha}}$$

$$a_n \longrightarrow \begin{cases} 0 & \text{se } \alpha < 6 \\ -\frac{1}{24} & \text{se } \alpha = 6 \\ -\infty & \text{se } \alpha > 6 \end{cases}$$

Poichè, per $t > 0$, $\operatorname{arctg} t < t$, allora $a_n < 0$ $\forall n \in \mathbb{N}$.

$$\sum a_n \approx -\frac{1}{24} \sum \frac{1}{n^{6-\alpha}}$$

la serie converge se $\alpha < 5$ $\boxed{A_5}$

diverge negativamente se $\alpha \geq 5$

5) DOMINIO: $D = \{x > 0\} = (0, +\infty)$

f è sempre positiva.

$$\lim_{x \to +\infty} f(x) = (+\infty)^{+\infty} = +\infty$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln(x^{\ln x})} = \lim_{x \to 0^+} e^{(\ln x)^2}$$

$$= e^{(-\infty)^2} = +\infty.$$

AS. VERT. DX

$x = 0$.

$$f(x) = e^{(\ln x)^2} \Rightarrow f'(x) = e^{(\ln x)^2} \cdot \left[(\ln x)^2\right]'$$

$$= x^{\ln x} \left[2 \frac{\ln x}{x}\right]$$

$$f'(x) = 0 \iff \ln x = 0 \iff x = 1$$

$$f'(x) > 0 \iff \ln x > 0 \iff x > 1$$

f decresce in $(0,1)$; cresce in $(1,+\infty)$

$x=1$ punto di MINIMO ASSOLUTO (A6)

$$f(1)=1^{\ln 1}=1.$$

$$f''(x)=2\left[\left(x^{\ln x}\right)'\left(\frac{\ln x}{x}\right)+x^{\ln x}\left(\frac{\ln x}{x}\right)'\right]=$$

$$=2\left[2x^{\ln x}\left(\frac{\ln x}{x}\right)^2+x^{\ln x}\left[\frac{\frac{1}{\cancel{x}}\cdot\cancel{x}-\ln x}{x^2}\right]\right]$$

$$=\frac{2}{x^2}x^{\ln x}\left[2\ln^2 x-\ln x+1\right]$$

$$f''(x)>0 \iff 2\ln^2 x-\ln x+1>0$$

Poniamo $t=\ln x$

$$2t^2-t+1>0 \qquad \Delta=1-8<0$$

$\Rightarrow$ vero $\forall t$ $\Rightarrow$ vero $\forall x>0$.

f sempre convessa.

Grafico:

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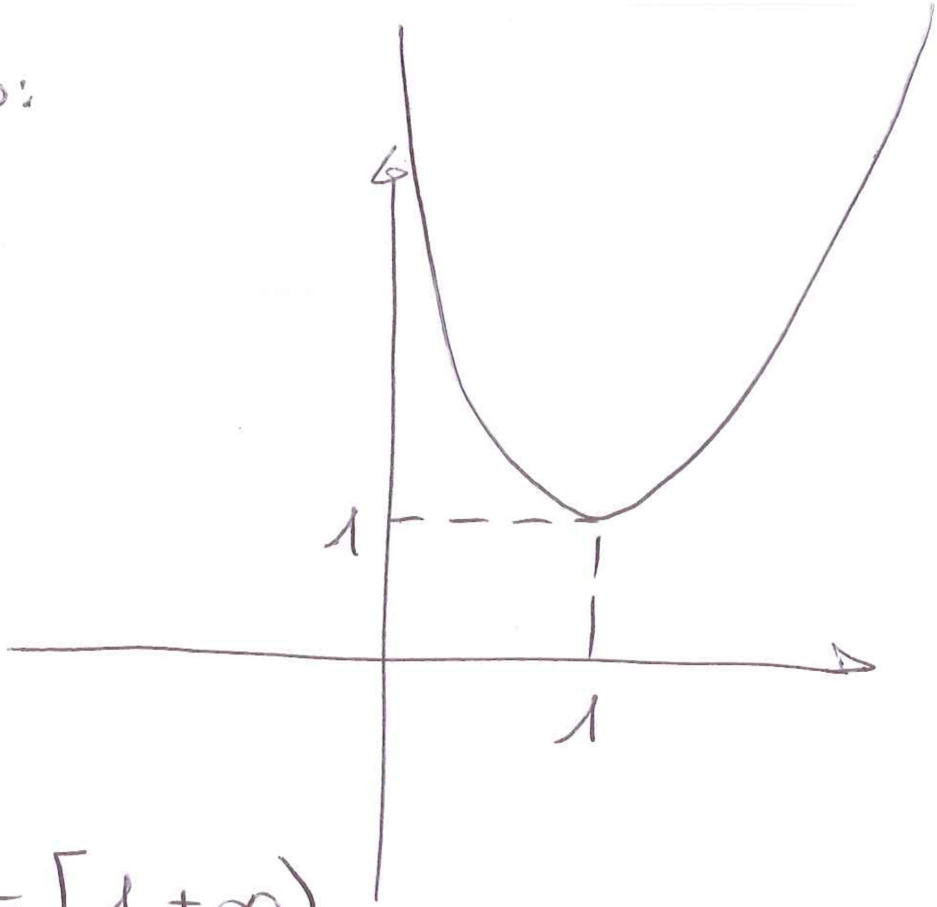

$f(D) = [1, +\infty)$.