COMPITO B

B_1

1) $$a_n = n^\alpha \left[\cancel{\frac{1}{3n^2}} - \frac{1}{3!(3n^2)^3} + o\left(\frac{1}{n^6}\right) - \cancel{\frac{1}{3n^2}} \right]$$

$$\sim \frac{-1}{162\, n^{6-\alpha}}$$

Pertanto

$$a_n \longrightarrow \begin{cases} 0 & \text{se } \alpha < 6 \\ -\frac{1}{162} & \text{se } \alpha = 6 \\ -\infty & \text{se } \alpha > 6 \end{cases}$$

Poichè, per $t > 0$, $\sin t < t$, allora $a_n < 0$ $\forall n \in \mathbb{N}$.

$$\sum a_n \approx -\frac{1}{162} \sum \frac{1}{n^{6-\alpha}}$$

Quindi la serie converge se $\alpha < 5$
diverge a $-\infty$ se $\alpha \geq 5$

2) $D = \{x > 0\} = (0, +\infty)$.

f è sempre positiva. Nessuna intersezione con gli assi.

$$\lim_{x \to +\infty} f(x) = +\infty^{+\infty} = +\infty$$ B2

Andamento superlineare, quindi $\nexists$ asintoto obliquo.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln\left[x^{1+\ln x}\right]} = \lim_{x \to 0^+} e^{(1+\ln x)\ln x}$$

$$= e^{(1-\infty)(-\infty)} = e^{+\infty} = +\infty.$$

AS. VERTICALE DA DX: $x = 0$.

$$f'(x) = \left[e^{\ln x + \ln^2 x}\right]' = e^{\ln x + \ln^2 x} \cdot \frac{1}{x}(1 + 2\ln x)$$

$$= x^{1+\ln x} \frac{1}{x}(1+2\ln x) = 0 \iff 2\ln x + 1 = 0$$

$$\iff \ln x = -\frac{1}{2} \iff x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$$

$$f'(x) > 0 \iff 1 + 2\ln x > 0 \iff x > \frac{1}{\sqrt{e}}$$

f decresce in $\left(0, \frac{1}{\sqrt{e}}\right)$; cresce in $\left(\frac{1}{\sqrt{e}}, +\infty\right)$

$x = \frac{1}{\sqrt{e}}$ punto di MIN. ASS.

$$f\left(\frac{1}{\sqrt{e}}\right) = \left(\frac{1}{\sqrt{e}}\right)^{\frac{1}{2}} = \frac{1}{\sqrt[4]{e}}$$

N.B.: $\underline{f'(x) = x^{\ln x}(1+2\ln x).}$ però per la derivata usiamo la precedente espressione

B_3

$$f''(x) = \left(\frac{x^{1+\ln x}(1+2\ln x)}{x} \right)'$$

$$= \frac{\left[\frac{x^{1+\ln x}(1+2\ln x)^2}{\cancel{x}} + x^{1+\ln x}\frac{2}{\cancel{x}} \right] \cancel{x} - x^{1+\ln x}(1+2\ln x)}{x^2}$$

$$= \frac{x^{1+\ln x}}{x^2} \left[(1+2\ln x)^2 + 2 - (1+2\ln x) \right]$$

$$f''(x) \geq 0 \iff (1+2\ln x)^2 - (1+2\ln x) + 2 \geq 0$$

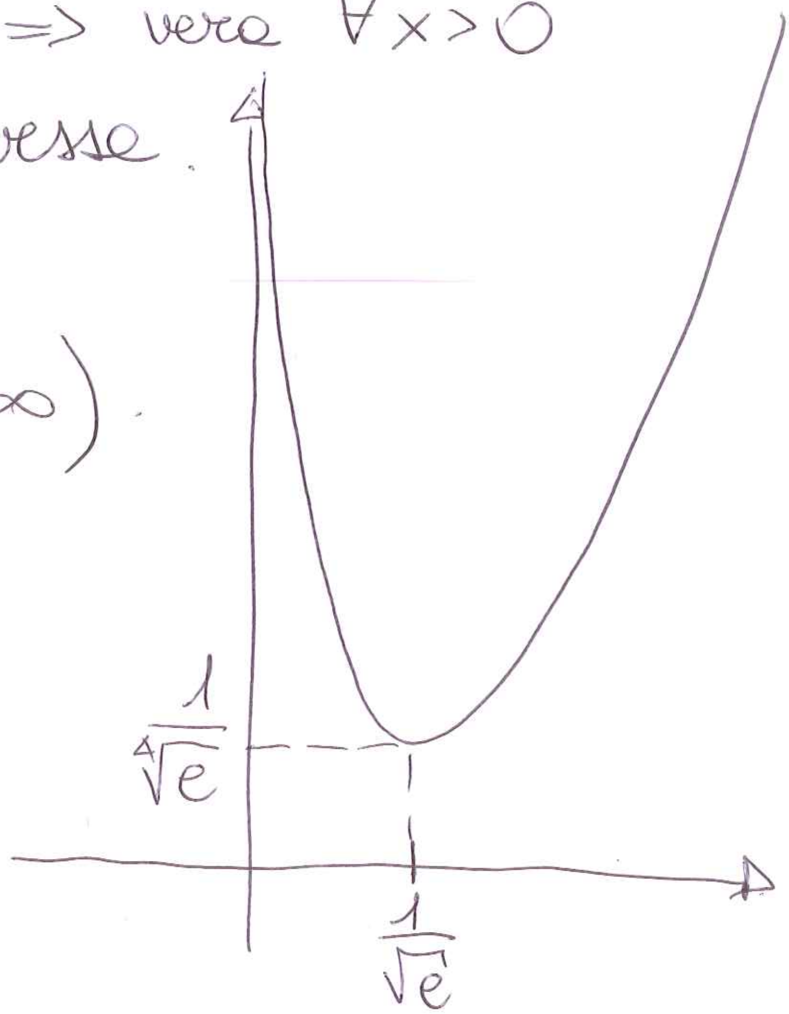
Poniamo $t = 1+2\ln x$

$$\Rightarrow t^2 - t + 2 \geq 0 \qquad \Delta = 1-8 < 0$$

$$\Rightarrow \text{vero } \forall t \quad \Rightarrow \text{vero } \forall x > 0$$

f è sempre convessa.

$$f(D) = \left[\frac{1}{\sqrt[4]{e}}, +\infty \right).$$



3) Abbassiamo l'ordine: $z = y'$

B4

$\Rightarrow z'' + 2z = 2x^2 - x$

OMO. ASSOCIATA: $\alpha^2 + 2 = 0$

$\Rightarrow \alpha_{1,2} = \pm\sqrt{2}\,i$

$\Rightarrow z_0(x) = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x)$

NON OMOGENEA: $\alpha = 0$ non e' radice del polinomio caratteristico $\Rightarrow z_p(x) = ax^2 + bx + c$

$z_p' = 2ax + b \quad ; \quad z_p'' = 2a$

$\Rightarrow 2a + 2ax^2 + 2bx + 2c = 2x^2 - x$

$$\begin{cases} 2a = 2 \\ 2b = -1 \\ 2a + 2c = 0 \end{cases} \Rightarrow \begin{cases} a = 1 \\ c = -1 \\ b = -\frac{1}{2} \end{cases}$$

$\Rightarrow z(x) = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) + x^2 - \frac{1}{2}x - 1.$

$z(0) = y'(0) = 0 = C_1 - 1 \quad \Rightarrow C_1 = 1$

$z'(x) = -\sqrt{2}C_1 \sin(\sqrt{2}x) + \sqrt{2}C_2 \cos(\sqrt{2}x) + 2x - \frac{1}{2}$

B5

$$z'(0) = y''(0) = 0 = \sqrt{2}\,c_2 - \frac{1}{2} \quad \Rightarrow \quad c_2 = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \underset{= y'(x)}{z(x)} = \cos(\sqrt{2}x) + \frac{1}{2\sqrt{2}}\sin(\sqrt{2}x) + x^2 - \frac{1}{2}x - 1$$

$$\Rightarrow y(x) = +\frac{1}{\sqrt{2}}\sin(\sqrt{2}x) - \frac{1}{4}\cos(\sqrt{2}x) + \frac{x^3}{3} - \frac{x^2}{4} - x + C_3$$

$$y(0) = 0 = -\frac{1}{4} + C_3 \quad \Rightarrow \quad C_3 = \frac{1}{4}$$

$$\Rightarrow y(x) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}x) - \frac{1}{4}\cos(\sqrt{2}x) + \frac{x^3}{3} - \frac{x^2}{4} - x + \frac{1}{4}$$

In alternativa, si sarebbe potuto risolvere l'equazione del 3° ordine, anche se non in programma.

OMO: $y''' + 2y' = 0 \Rightarrow \alpha^3 + 2\alpha = 0$

$\Rightarrow \alpha_{1,2} = \pm\sqrt{2}\,i$; $\alpha_3 = 0 \Rightarrow y_0(x) = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + C_3$.

NON OMO: poiché $\alpha = 0$ è radice del polinomio caratteristico, con $m_a = 0$, allora $y_P(x) = x(ax^2 + bx + c)$.

$y_P' = 3ax^2 + 2bx + c$; $y_P'' = 6ax + 2b$; $y_P''' = 6a$

$$\Rightarrow 6a + 2(3ax^2 + 2bx + c) = 2x^2 - x$$

B 5 bis

$$\begin{cases} 6a = 2 \\ 4b = -1 \\ 6a + 2c = 0 \end{cases} \Rightarrow \begin{cases} a = 1/3 \\ b = -1/4 \\ c = -3a = -1 \end{cases}$$

$$\Rightarrow y(x) = C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x) + C_3 + \frac{1}{3}x^3 - \frac{1}{4}x^2 - x.$$

Il resto rimane uguale.

4) Confrontare esercizio 2 del compito A.

5) Poniamo $t = \dfrac{z+3i}{2+i}$

$$\Rightarrow t^2 = 9i \Rightarrow t_{1,2} = 3\sqrt{i}$$

$$\Rightarrow t_1 = 3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$t_2 = -3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow \frac{z_1+3i}{2+i} = 3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \quad ; \quad \frac{z_2+3i}{2+i} = -3\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

B_6

$$\Rightarrow z_1 = -3i + 3(2+i)\frac{\sqrt{2}}{2}(1+i)$$

$$= 3\left[-i + \frac{\sqrt{2}}{2}(2-1+3i)\right]$$

$$= 3\left[-i + \frac{\sqrt{2}}{2} + \frac{3}{2}\sqrt{2}\,i\right]$$

$$= \frac{3}{2}\sqrt{2} + 3\left(\frac{3\sqrt{2}}{2} - 1\right)i$$

$$z_2 = -3i - 3(2+i)\frac{\sqrt{2}}{2}(1+i)$$

$$= -3\left[i + \frac{\sqrt{2}}{2}(1+3i)\right]$$

$$= -3i - \frac{3\sqrt{2}}{2} - \frac{9\sqrt{2}}{2}i = \frac{-3\sqrt{2}}{2} - 3\left(1 + \frac{3\sqrt{2}}{2}\right)i$$

Alternativamente, si sarebbe potuto risolvere direttamente l'equazione:

$$(z+3i)^2 = 9i(2+i)^2$$

$$z^2 + 6iz - 9 = 9i(3+4i)$$

$$z^2 + 6iz + 27(-i+1) = 0$$

$$\Rightarrow z_{1,2} = -3i + \sqrt{-9 + 27(i-1)}$$

$$\Rightarrow \quad z_{1,2} = -3i + \sqrt{-36+27i}$$

$$= -3i + \frac{3}{\sqrt{2}}\sqrt{-8+6i}$$

$$= -3i \pm \frac{3}{\sqrt{2}}\sqrt{(1+3i)^2}$$

$$= -3i \pm \frac{3}{\sqrt{2}}(1+3i).$$

B7