SVOLGIMENTI PROVA SCRITTA di ANALISI 1 del 10/10/2019

1) OMOGENEA ASSOCIATA:

$$\alpha^2-4\alpha+4=(\alpha-2)^2=0$$

$$\Rightarrow \alpha_{1,2}=2$$

$$\Rightarrow y_0(x)=C_1 e^{2x}+C_2 x e^{2x}.$$

NON OMOGENEA:

$$\underbrace{\alpha=2}_{\text{radice}} \text{ con } m_a=2 \Rightarrow y_p(x)=Ax^2e^{2x}$$

$$y_p'=A\left[(2x+2x^2)\right]e^{2x}=2A(x+x^2)e^{2x}$$

$$y_p''=2A\left[1+2x+2x+2x^2\right]e^{2x}=2A(2x^2+4x+1)e^{2x}$$

$$\Rightarrow A\left[\cancel{4x^2}+\cancel{8x}+2\cancel{-8x}\cancel{-8x^2}+\cancel{4x^2}\right]e^{2x}=e^{2x}$$

$$\Rightarrow 2A=1 \qquad \Rightarrow A=\frac{1}{2}$$

$$\Rightarrow y(x)=C_1e^{2x}+C_2xe^{2x}+\frac{1}{2}x^2e^{2x}$$

$$y(x)\underset{x\to+\infty}{\sim}\frac{1}{2}x^2e^{2x}\xrightarrow[x\to+\infty]{}+\infty \quad \forall C_1,C_2\in\mathbb{R}$$

$\nexists$ soluzioni limitate per $x\to+\infty$.

$$\lim_{x \to -\infty} x^n e^{2x} = 0 \quad \forall n \in \mathbb{N}$$ ②

$\Rightarrow$ tutte le soluzioni particolari sono limitate per $x \to -\infty$.

2) $D = \mathbb{R}$

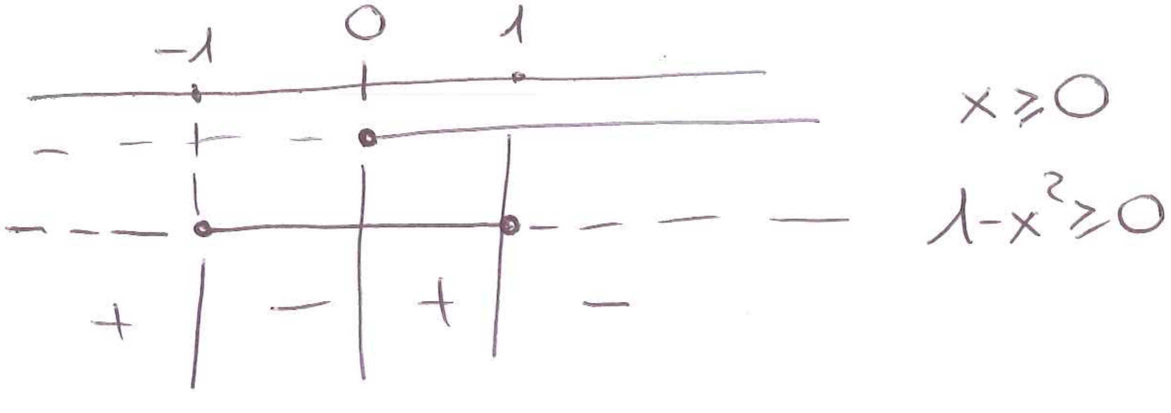
$f(x) \geq 0 \quad \forall x \in \mathbb{R} \qquad f(x) = 0 \iff x = 0$

UNICA INTERSEZIONE CON GLI ASSI: $x = 0$

$f(x)$ SIMMETRICA : $f(x) = f(-x)$.

$$\lim_{x \to \pm\infty} f(x) = 0$$ AS. ORIZZ. per $x \to \pm\infty$: $y = 0$.

$$f'(x) = (2x - 2x^3)e^{-x^2}$$
$$= 2(x - x^3)e^{-x^2} = 2x(1 - x^2)e^{-x^2}$$



f cresce in $(-\infty, -1)$; decresce in $(-1, 0)$; cresce in $(0, 1)$; decresce in $(1, +\infty)$.

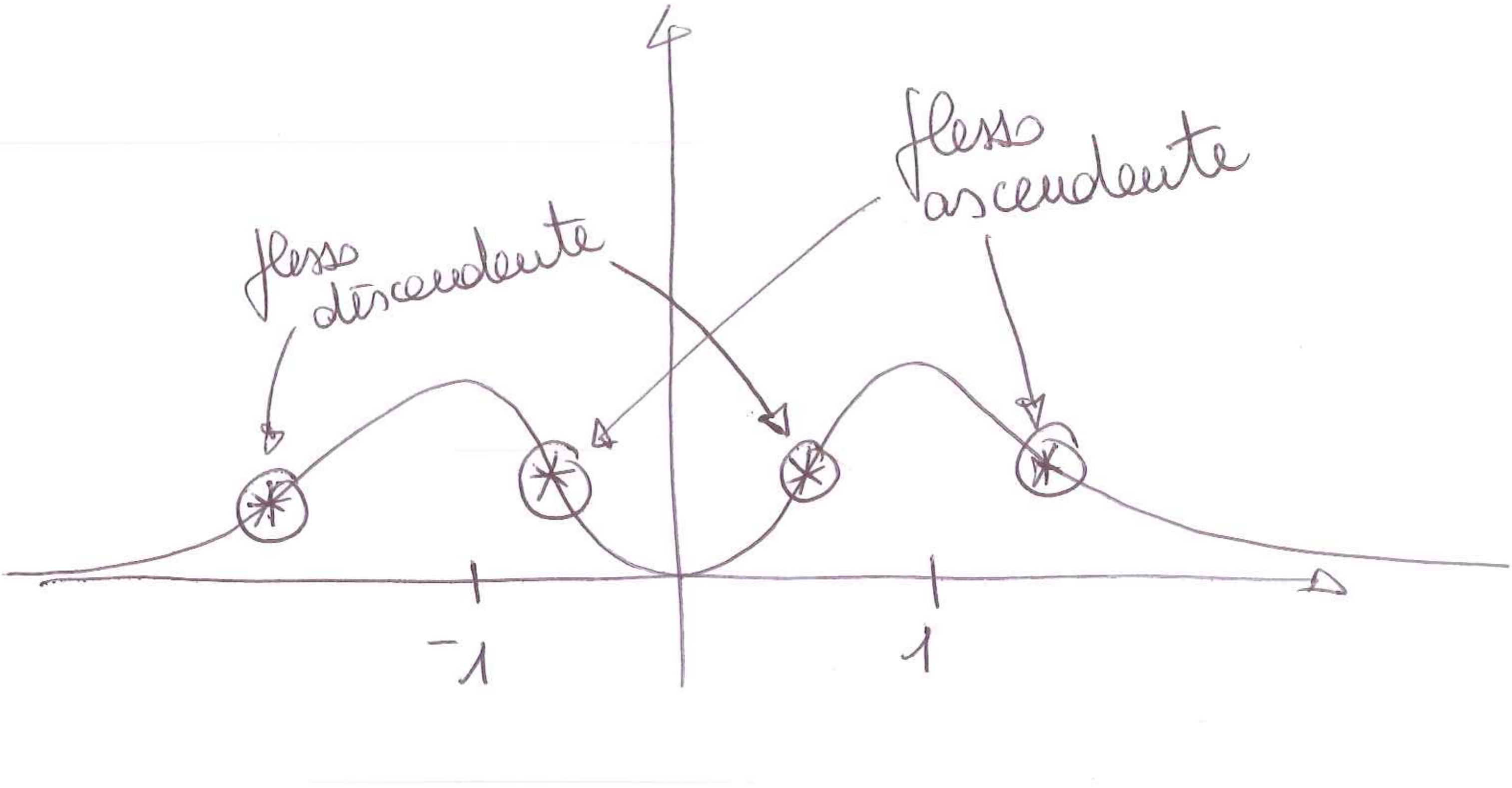
$x = -1$ punto di MAX. REL.
$x = 1$ punto di MAX. REL.
per la simmetria, entrambi sono di MAX. ASS.

$$f(\pm 1) = \frac{1}{e}$$

$x = 0$ punto di MIN. REL. e ASS.

$$f(0) = 0.$$

In ipotesi di numero minimo di flessi, si ha



Infatti:

$$f''(x) = 2\left[1-3x^2-2x^2+2x^4\right]e^{-x^2}$$

$$= 2\left(2x^4-5x^2+1\right)e^{-x^2} \geq 0$$

$$\Longleftrightarrow 2x^4-5x^2+1>0$$ BIQUADRATICA

$$2t^2-5t+1=0 \quad \Rightarrow \quad t_{1,2} = \frac{5\pm\sqrt{17}}{4} > 0$$

$$\Rightarrow \quad x_{1,2,3,4} = \pm\sqrt{\frac{5\pm\sqrt{17}}{4}}$$

4 flessi, come da grafico.

3) $$\frac{z-2i+1}{z+i} = \frac{x+iy-2i+1}{x+iy+i} = \frac{(x+1)+(y-2)i}{x+(y+1)i}$$

$\boxed{z \neq -i}$

$$= \frac{\left[(x+1)+(y-2)i\right]\left[x-(y+1)i\right]}{x^2+(y+1)^2}$$

$$= \frac{\left[(x+1)x+(y-2)\right]+i\left[(y-2)x-(x+1)(y+1)\right]}{x^2+(y+1)^2}$$

$$\Longrightarrow \operatorname{Im} = \frac{(y-2)x-(x+1)(y+1)}{x^2+(y+1)^2}$$

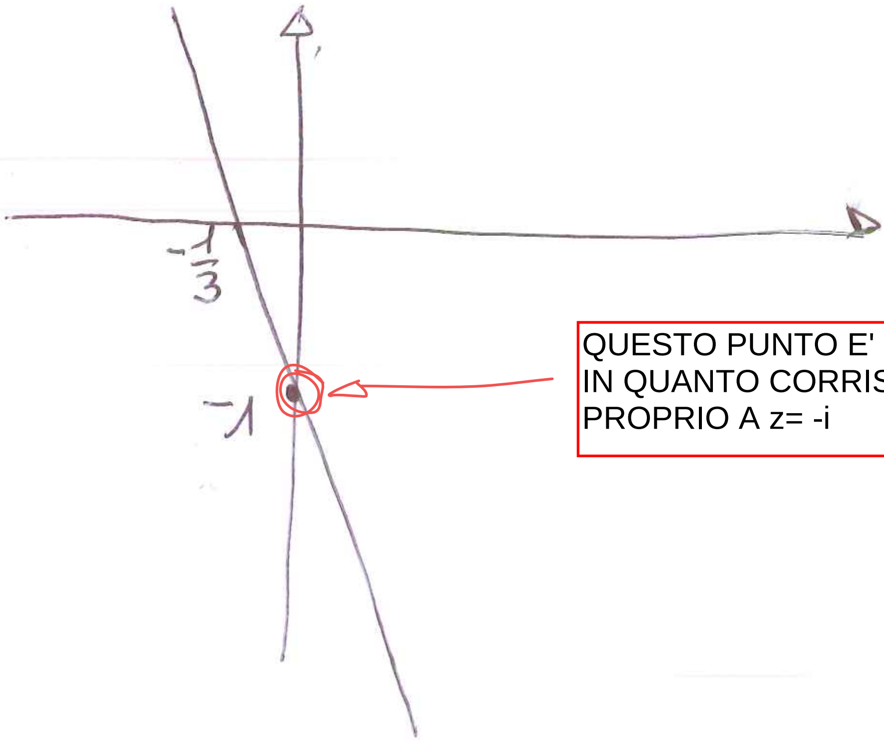
Quindi

$$\operatorname{Im}\left(\frac{z-2i+1}{z+i}\right)=0$$

$$\iff x(y-2)-(x+1)(y+1)=0$$

$$\iff \cancel{xy}-2x-\cancel{xy}-x-y-1=0$$

$$\iff y=-3x-1 \qquad \text{retta}$$



QUESTO PUNTO E' ESCLUSO, IN QUANTO CORRISPONDE PROPRIO A z= -i

4)

$$\sqrt{e^{-\frac{1}{n}}} = e^{-\frac{1}{2n}} = 1 - \frac{1}{2n} + \frac{1}{2}\left(-\frac{1}{2n}\right)^2 + o\left(\frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{2n} + \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right)$$

$$\Rightarrow a_n = \cancel{1} - \cancel{\frac{1}{2n}} + \frac{1}{8n^2} - \cancel{1} + \cancel{\frac{1}{2n}} - \frac{1}{4!\,n^2} + o\left(\frac{1}{n^2}\right)$$

$$= \left(\frac{1}{8} - \frac{1}{24}\right)\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$\Rightarrow \sum a_n \approx \sum \frac{1}{12n^2}$$ convergente.

5)

$$\frac{\operatorname{arctg} x}{x^2} \sim \frac{\pi}{2} \cdot \frac{1}{x^2}$$ che è integrabile in $[1, +\infty)$.

(criterio del confronto asintotico)

Altrimenti, in $[1,+\infty)$

$$0 < \frac{\operatorname{arctg} x}{x^2} < \frac{\pi}{2x^2}$$ che è integrabile

(criterio del confronto).

$$\int_1^{+\infty} \frac{\operatorname{arctg} x}{x^2}\,dx = \left[-\frac{\operatorname{arctg} x}{x} \Big|_1^{+\infty} + \int_1^{+\infty} \frac{1}{x(1+x^2)}\,dx \right]$$

$$= \left[\operatorname{arctg} 1 + \int_1^{+\infty} \underbrace{\left[\frac{1}{x} - \frac{x}{1+x^2} \right]}_{\text{fratti semplici}} dx \right]$$

$$= \frac{\pi}{4} + \lim_{x \to +\infty} \left[\log x - \frac{1}{2} \log(1+x^2) \right]$$

$$- \left[\log 1 - \frac{1}{2} \log 2 \right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\log 2 + \lim_{x \to +\infty}\left[\log\left(\frac{x}{\sqrt{1+x^2}}\right)\right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\log 2 + \lim_{x \to +\infty}\left[\log\left(\frac{x}{x}\right)\right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\log 2.$$