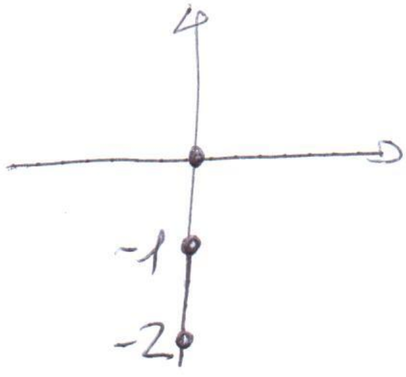
SVOLGIMENTO PROVA SCRITTA di ANALISI MAT. 1 del 28/10/2014 ①

1)
$$(z+i)\left[(z+i)^2+1\right]=0$$
$$(z+i)\left(z^2+2zi\right)=0$$
$$z(z+i)(z+2i)=0$$




$z_1=0$ ; $z_2=-i$ ; $z_3=-2i$.

$$z_2=\cos\left(-\frac{\pi}{2}\right)+i\,\text{sen}\left(-\frac{\pi}{2}\right)=e^{-\frac{\pi}{2}i}$$
$$z_3=2\left[\cos\left(-\frac{\pi}{2}\right)+i\,\text{sen}\left(-\frac{\pi}{2}\right)\right]=2e^{-\frac{\pi}{2}i}$$

2) Studiamo la convergenza assoluta, col criterio del rapporto:

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{|a|^{n+1}\,(2(n+1))!}{(3(n+1))!}\cdot\frac{(3n)!}{|a|^n(2n)!}$$

$$=|a|\cdot\frac{(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)}\xrightarrow[n\to\infty]{}0$$

$\forall a\in\mathbb{R}$.

Convergente assoluta e semplice $\forall a\in\mathbb{R}$.

3) $$\frac{1}{x\left[\log x+1\right]^2} \underset{x \to +\infty}{\sim} \frac{1}{x \log^2 x}$$

$\displaystyle\int_1^{+\infty} \frac{1}{x\log^2 x}\,dx$ converge, ad esempio per confronto con la serie di Abel

$$\sum \frac{1}{n \log^2 n}.$$

$$\int_1^{+\infty} \frac{1}{x\left[\log x+1\right]^2}\,dx = \left.\frac{-1}{\left[\log x+1\right]}\right|_1^{+\infty} = 1.$$

3 bis) $f(x,y) = x^2(y^2+1)\,\mathrm{arctg}(y)$ è

tale che $a(x) = x^2 \in C^\infty(\mathbb{R})$

$b(y) = (y^2+1)\,\mathrm{arctg}(y) \in C^\infty(\mathbb{R})$

dunque le condizioni per l'esistenza e unicità (locale) sono garantite.

Poiché $y \equiv 0$ è integrale singolare e soddisfa la condizione iniziale, allora $y \equiv 0$ è LA soluzione cercata.

4)

$$\left(e^{-x^2}-1\right)\log\left(1+2x^2\right)=$$

$$\left(-x^2+\frac{x^4}{2}-\frac{x^6}{6}+o(x^6)\right)\left(2x^2-\frac{(2x^2)^2}{2}+\frac{(2x^2)^3}{3}+o(x^6)\right)$$

$$=\left(-x^2+\frac{x^4}{2}-\frac{x^6}{6}+o(x^6)\right)\left(2x^2-2x^4+\frac{8}{3}x^6+o(x^6)\right)$$

$$=-2x^4+2x^6+x^6+o(x^6)$$

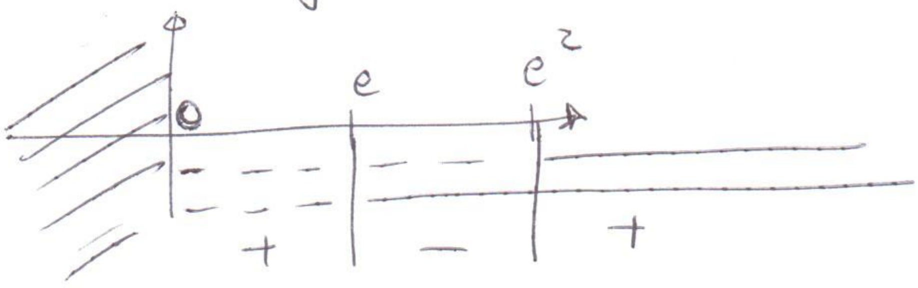
$$=-2x^4+3x^6+o(x^6)$$

$$\lim_{x\to 0}\frac{\left(e^{-x^2}-1\right)\log\left(1+2x^2\right)+2x^4}{x^6}=\lim_{x\to 0}\frac{3x^6}{x^6}=3.$$

5) $I_{def}=\left\{x\in\mathbb{R}\ \middle|\ \begin{array}{l}\log x-1\neq 0\\ x>0\end{array}\right\}=(0,e)\cup(e,+\infty)$

Segno: $\log x-2>0 \iff x>e^2$

$\log x-1>0 \iff x>e$




$f(e^2)=0$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\log x}{\log x} = 1$$

$y = 1$ ASINTOTO ORIZZONTALE per $x \to +\infty$

$$f(x) = \frac{\log x - 1 - 1}{\log x - 1} = 1 - \frac{1}{\log x - 1}$$

$$\lim_{x \to 0^+} f(x) = 1$$

NO ASINTOTO VERTICALE in $x = 0$

$f$ è prolungabile per continuità in $x = 0$.

$$f'(x) = \frac{1}{(\log x - 1)^2} \frac{1}{x} > 0 \quad \forall x \in I_{def}.$$

$f$ cresce in $(0, e)$ e in $(e, +\infty)$.

$$\lim_{x \to e^+} f(x) = 1 - \frac{1}{0^+} = 1 - \infty = -\infty$$

$$\lim_{x \to e^-} f(x) = 1 - \frac{1}{0^-} = 1 + \infty = +\infty.$$

Per $x \to 0^+$

$$f'(x) \sim \frac{1}{x \log^2 x} \longrightarrow +\infty$$

perché $\lim_{x \to 0^+} x^\alpha \log^\beta x = 0 \qquad \forall \alpha, \beta > 0.$

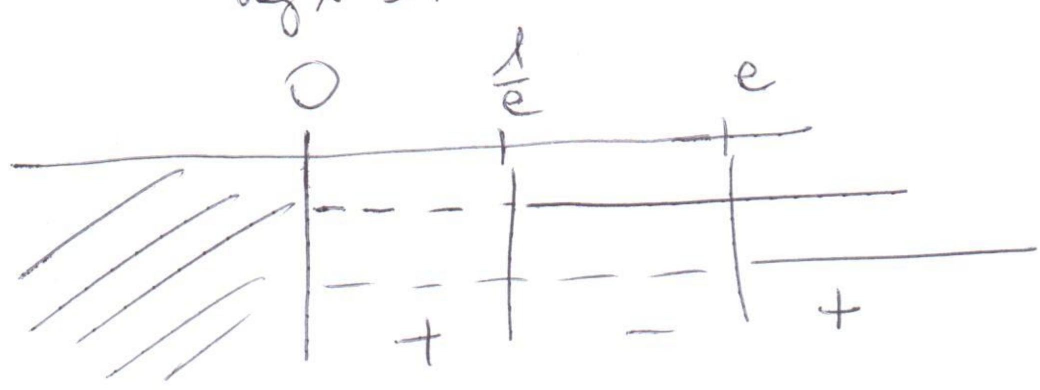
Inoltre

$$f''(x) = \frac{-1}{(\log x - 1)^4 x^2} \left[ (\log x - 1)^2 x \right]' =$$

$$= \frac{-1}{(\log x - 1)^4 x^2} \left[ 2(\log x - 1) \frac{1}{\cancel{x}} \cdot \cancel{x} + (\log x - 1)^2 \right]$$

$$= \frac{-1}{(\log x - 1)^3 x^2} (\log x + 1) > 0$$

$$\iff \frac{\log x + 1}{\log x - 1} < 0$$




concava convessa concava

$x = \frac{1}{e}$ punto di flesso discendente

$$f\left(\frac{1}{e}\right) = \frac{\log\left(\frac{1}{e}\right)-2}{\log\left(\frac{1}{e}\right)-1} = \frac{-3}{-2} = \frac{3}{2}$$

Grafico: