Università degli Studi di ROMA

Mini-corso di Dottorato

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Analisi del comportamento statico e dinamico di alcuni modelli nonlineari per la trave estendibile

Claudio Giorgi

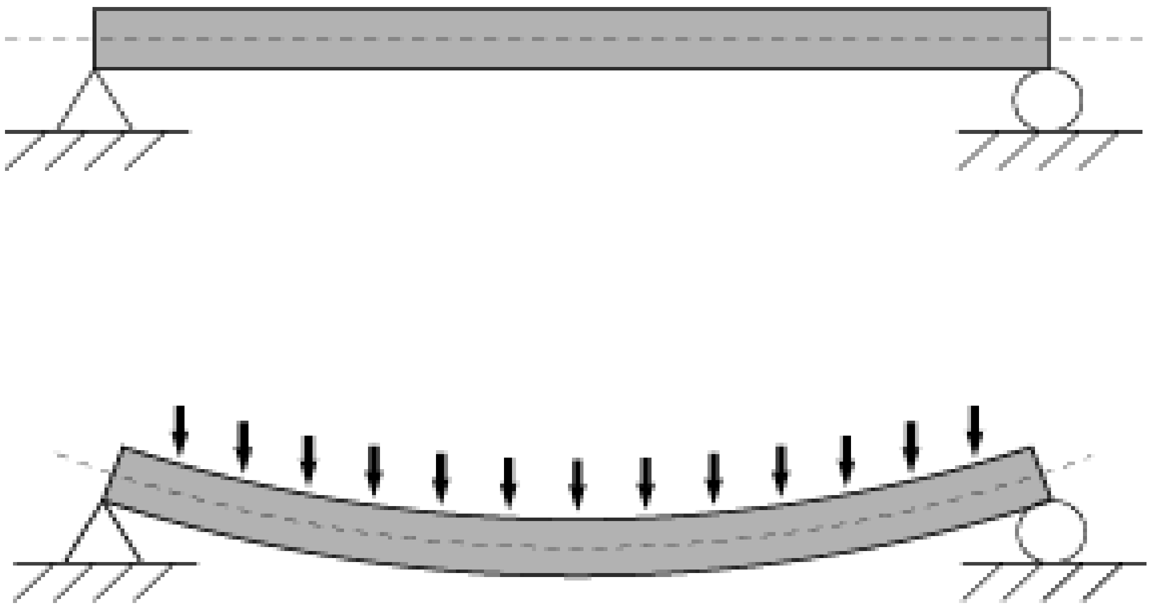
Dipartimento di Matematica, Università di Brescia

Introduzione

Il mini-corso analizza il comportamento delle soluzioni statiche e dinamiche di alcuni semplici sistemi nonlineari dipendenti dal parametro p.

Tali sistemi di equazioni alle derivate parziali si caratterizzano come modelli delle oscillazioni verticali (*bending*) di travi con estremi fissati, in cui *non vengono trascurati gli effetti dovuti all'allungamento*.
(modelli di Woinovsky-Krieger, Berger, ecc.).

In tal caso la nonlinearità ha carattere puramente geometrico: ovvero è presente anche quando il materiale che compone la trave si supponga linearmente elastico (visco-elatico, termo-elastico).



Bending

Equazione (adimensionale) dell'equilibrio per la trave elastica

$$\begin{cases} \partial_{xxxx}u + \boxed{p\,\partial_{xx}u} - \boxed{\left(\int_0^1 |\partial_\xi u(\xi)|^2 d\xi\right) \partial_{xx}u} = f, \\ u(0) = u(1) = u_{xx}(0) = u_{xx}(1) = 0, \end{cases} \tag{1}$$

dove

- $u = u(x) : [0,1] \to \mathbb{R}$: deflessione verticale della trave;
- f rappresenta il carico verticale distribuito (adim.)
- $\left(\int_0^1 |\partial_\xi u(\xi)|^2 d\xi\right) \partial_{xx}u$ rappresenta la nonlinearità geometrica

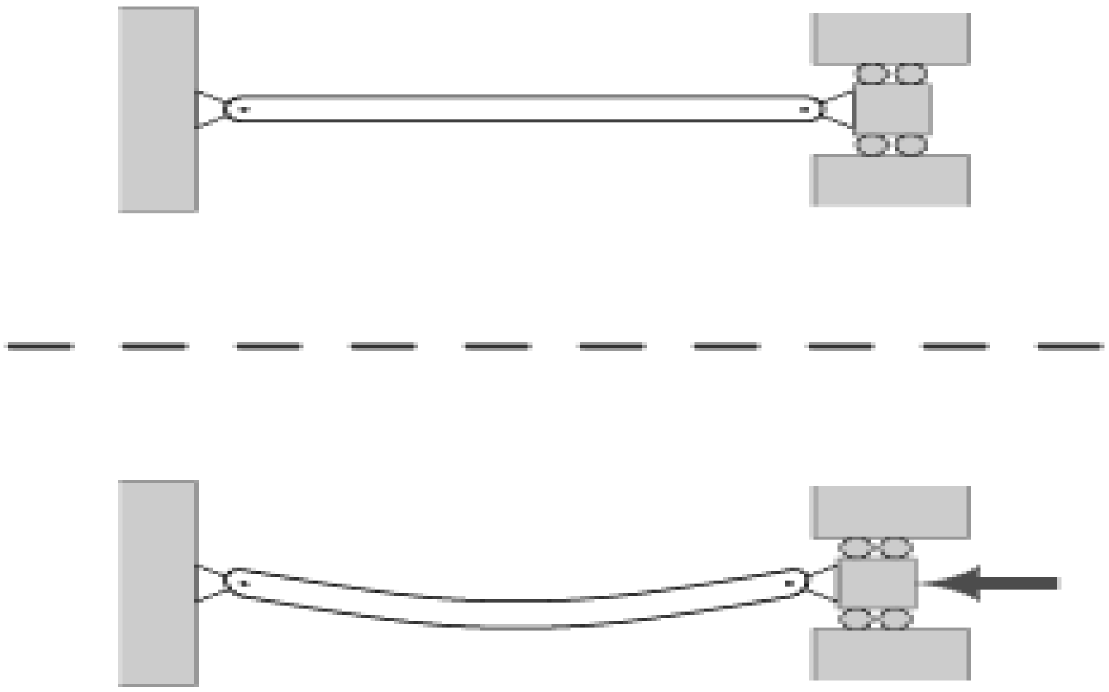
Il parametro adimensionale p è collegato allo spostamento Δ imposto all'estremo destro della trave, ovvero al *carico di punta* ad esso applicato.

Il superamento di una certa soglia critica p_0 determina la biforcazione delle soluzioni stazionarie (carico critico di Eulero).

Nel problema omogeneo ($f = 0$), quando $p > p_0$, la soluzione indeformata diventa instabile e compaiono coppie (simmetriche) di soluzioni stazionarie incurvate (*buckling*).

La determinazione del carico critico p_0 è complicata dalla presenza del termine non lineare (in rosso). Ma soprattutto, vedremo che tale termine produce una cascata di biforcazioni al crescere del parametro p.

Buckling



Equazione delle vibrazioni verticali della trave elastica

$$\partial_{tt}u + \partial_{xxxx}u + \boxed{\partial_t u} + \left(\boxed{p} - \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right)\partial_{xx}u = f, \tag{2}$$

$u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$: deflessione verticale della trave;

C.C. $\quad u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0,$

C. I. $\quad u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x).$

In presenza di termini dissipativi, il parametro $\boxed{p}$ determina anche il comportamento caotico della dinamica a lungo termine:

- se $p < p_0$, la soluzione indeformata è esponenzialmente stabile;
- se $p > p_0$, esistono più soluzioni stazionarie e la dinamica è caotica.

Programma

1. Deduzione del modello evolutivo nel caso termoelastico
La trave è costituita da un materiale termo-elastico lineare

2. Modello visco-elastico
La trave è costituita da un materiale visco-elastico lineare

3. Analisi delle soluzioni stazionarie

4. Riduzione finito-dimensionale della dinamica del modello elastico
Proiettando la dinamica sul primo autovettore, l'equazione di evoluzione diventa ODE e si visualizza la prima biforcazione

5. Analisi della dinamica in un mezzo visco-elastico
La trave vibra in un mezzo che oltre ad una resistenza viscosa produce anche una reazione elastica lineare: attrattori e risonanze

Alcuni risultati:

Analisi del modello e delle soluzioni stazionarie;

[1] C.G. - M.G.Naso, Modeling and steady states analysis of the extensible thermoelastic beam, *Math. Comp. Modeling*, to appear.

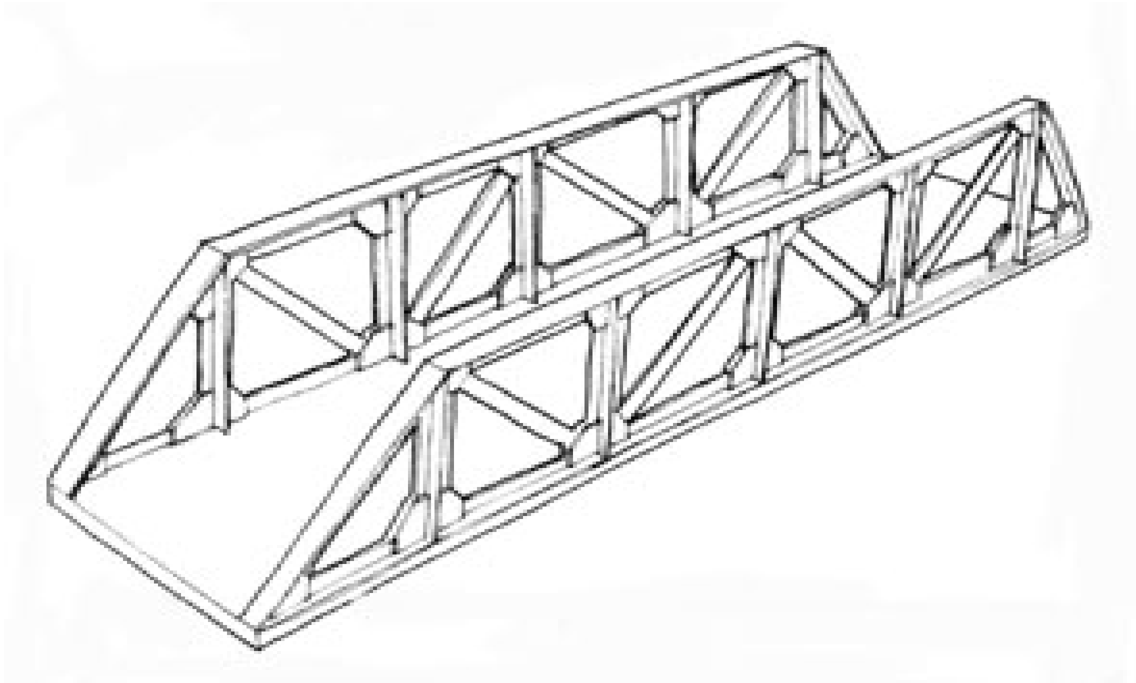
[2] M.Coti Zelati - C.G. - V.Pata, *Steady states of the hinged extensible beam with external load*, Math. Models Methods Appl. Sci., 20 (2010) 43-58

[3] I.Bochicchio - C.G. - E.Vuk, Steady states analysis and exponential stability of an extensible thermoelastic system, *Comm. SIMAI Congress*, 3 (2009) 232.1-232.12

Analisi della dinamica asintotica globale;

[4] C.G. - V.Pata - E.Vuk, *On the extensible viscoelastic beam*, Nonlinearity, 21 (2008) 713–733.

[5] C.G. - M.G.Naso - V.Pata - M.Potomkin, *Global attractors for the extensible thermoelastic beam system*, J. Differential Equations, 246 (2009) 3496-3517.



A “girder” iron bridge

Part 1. The thermoelastic model

We first present a derivation of the following thermo-elastic system

$$\begin{cases} \partial_{tt}u - \partial_{xxtt}u + \partial_{xxxx}u + \partial_{xx}\theta + \boxed{\left(p - \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right) \partial_{xx}u} = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \end{cases} \tag{3}$$

where

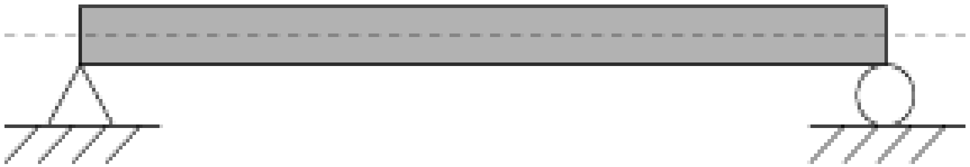
$u = u(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$: vertical deflection of the beam;

$\theta = \theta(x,t) : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$: vertical temperature **gradient**.

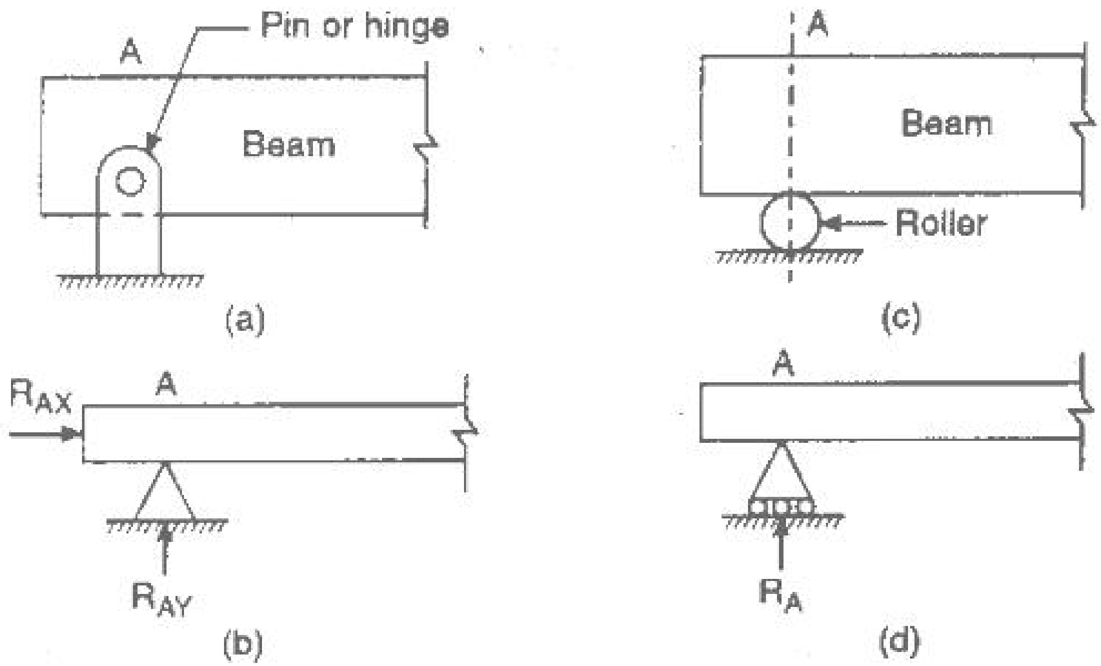
B.C. $u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0$, $\theta(0,t) = \theta(1,t) = 0$,

I.C. $u(x,0) = u_0(x)$, $\partial_t u(x,0) = u_1(x)$, $\theta(x,0) = \theta_0(x)$, Solutions to problem (3) describes the mechanical and thermal evolution (in the transversal direction) of an extensible thermoelastic beam of natural length ℓ with hinged ends. The value of the parameter p depends on Δ.

Natural length ℓ



Actual length $\ell + \Delta < \ell$



Boundary conditions

• For a general value of p, the **global dynamics** of this problem has been addressed in [5] where the existence of the global attractor is obtained jointly with its optimal regularity (see Part 5).

• The static counterpart of (3) reduces to the single equation

$$\partial_{xxxx}u + \left(p - \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right)\partial_{xx}u = f + g, \tag{4}$$

The buckled stationary states are scrutinized in [2,3] for a general value of $p \in \mathbb{R}$ and source $f + g$ with a general shape (see Part 3).

At a generic point $(x,y) \in [0,\ell] \times \left[-\frac{h}{2}, \frac{h}{2}\right]$ of the vertical section of the beam

$$\mathfrak{U}(x,y,t) = (W(x,y,t), U(x,y,t)), \quad \text{displacement vector field}$$

$$\Theta(x,y,t), \quad \text{absolute temperature field}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} = \frac{1}{2}\left[\nabla\mathfrak{U} + (\nabla\mathfrak{U})^\top\right] + \frac{1}{2}(\nabla\mathfrak{U})^\top \nabla\mathfrak{U} \quad \text{strain tensor.}$$

Let

$\Theta_0 > 0$ the reference-temperature value,

$\rho > 0$ the reference mass density.

- **The stress-strain relation** (see Carlson)

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \frac{E}{1+\nu}\left[\varepsilon + \frac{\nu}{1-2\nu}\mathrm{tr}(\varepsilon)\mathbf{1}\right] - \frac{E}{1-2\nu}\boxed{\varepsilon^{\Theta}}, \qquad \text{stress tensor}$$

where $\boxed{\varepsilon^{\Theta} = \alpha\,(\Theta - \Theta_0)\mathbf{1}}$ is the thermal strain tensor,

$E > 0$ is the Young's modulus

$\nu \in (0, \frac{1}{2})$ is the Poisson ratio

$\alpha > 0$, is the coefficient of thermal expansion

- **The entropy density (per unit mass)** (see Chadwick)

$$S = \frac{E\alpha}{\rho\,(1-2\nu)}\,\mathrm{tr}(\varepsilon) + \frac{c_v}{\Theta_0}(\Theta - \Theta_0),$$

  where $c_v > 0$ is the beam heat capacity at constant strain.

- **The Fourier law** for the heat flux vector

$$\boldsymbol{q} = -k_0 \nabla\Theta, \qquad k_0 > 0.$$

- **The entropy balance equation** (see Lagnese-Lions)

$$\boxed{\rho\,\Theta\,\partial_t S = -\nabla\cdot\boldsymbol{q} + \rho\,r}$$

  where $r(x,y,t)$ is the heat supplied (per unit mass).

It follows from the Gibbs relation and the internal energy balance,

- **The Gibbs relation** (see Carlson)

$$\rho\,(\partial_t \mathcal{E} - \Theta\,\partial_t S) - \sum_{i,j} \sigma_{ij} \partial_t \varepsilon_{ij} = 0,$$

where $\mathcal{E}$ is the internal energy density (per unit mass).

- **The internal energy balance equation**

$$\rho\,\partial_t \mathcal{E} - \sum_{i,j} \sigma_{ij} \partial_t \varepsilon_{ij} + \nabla \cdot \boldsymbol{q} = \rho\, r.$$

- **The approximation scheme** (consistent with large deformations)

  Geometrical nonlinearities, due to kinematics, are taken into account.

  Kinematic assumptions

  - the thinness of the beam: $\boxed{h \ll \ell}$,
  - the Kirchhoff hypothesis: any cross section remains perpendicular to the deformed longitudinal axis of the beam,
  - $\boxed{W(x,y,t) = w(x,t) - y\,\partial_x u(x,t)}$, $\boxed{U(x,y,t) = u(x,t)}$, where

    $w(x,t) = W(x,0,t)$ and $u(x,t) = U(x,0,t)$.
    (rigorously justified in large deflection theory by Ciarlet)

- **The approximation scheme**

  Linearization of the temperature field and source with respect to the transversal direction ($2|y| < h \ll \ell$).

  Thermal assumptions

  - $\boxed{\Theta(x,y,t) - \Theta_0 = \vartheta(x,t) + y\,\theta(x,t)}$, where
    $\vartheta(x,t) = \Theta(x,0,t) - \Theta_0$, and $\theta(x,t) = \partial_y \Theta(x,0,t)$.
  - $\boxed{r(x,y,t) = g_0(x,t) + y\,g(x,t)}$, where
    $g_0(x,t) = r(x,0,t)$, and $g(x,t) = \partial_y r(x,0,t)$.

- **The approximation scheme** (consequences)

$$\sigma_{11} = \frac{E}{1-\nu^2}\,\varepsilon_{11} - \alpha\,\frac{E}{1-\nu}[\vartheta(x,t) + y\,\theta(x,t)],$$

$$\sigma_{22} = \sigma_{12} = \sigma_{21} = 0,$$

$$S = \frac{E\alpha}{\rho\,(1-\nu)}\,\varepsilon_{11} + \varpi[\vartheta(x,t) + y\,\theta(x,t)]$$

where

$$\varepsilon_{11}(x,y,t) = \partial_x w(x,t) - y\,\partial_{xx} u(x,t) + \boxed{\frac{1}{2}\,|\partial_x u(x,t)|^2},$$

$$\varpi = \frac{E\alpha^2(1+\nu)}{\rho\,(1-2\nu)(1-\nu)} + \frac{c_v}{\Theta_0} > 0.$$

From the entropy balance equation we obtain

- **The heat equations**

$$\begin{cases} \rho\,\partial_t\vartheta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\vartheta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_t\left[\partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2\right] = \dfrac{\rho}{\Theta_0\varpi}g_0, \\ \rho\,\partial_t\theta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \dfrac{\rho}{\Theta_0\varpi}g. \end{cases} \tag{5}$$

B.C. $\quad \vartheta(0,t) = \vartheta(\ell,t) = 0$, $\theta(0,t) = \theta(\ell,t) = 0,$

I.C. $\quad \vartheta(x,0) = \vartheta_0(x), \quad \theta(x,0) = \theta_0(x).$

- **The motion equations** (variational derivation)

$$\begin{cases} \rho\,\partial_{tt}w - \dfrac{E}{1-\nu^2}\partial_x\left\{\partial_x w + \dfrac{1}{2}\,|\partial_x u|^2 - \alpha(1+\nu)\vartheta\right\} = 0, \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \dfrac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta \\ \quad -\dfrac{E}{1-\nu^2}\partial_x\left\{\left[\partial_x w + \dfrac{1}{2}\,|\partial_x u|^2 - \alpha(1+\nu)\vartheta\right]\partial_x u\right\} = \dfrac{\rho\,f}{h}. \end{cases} \tag{6}$$

B.C. $\quad u(0,t) = u(\ell,t) = \partial_{xx}u(0,t) = \partial_{xx}u(\ell,t) = 0$, and

$w(0,t) = 0,\ \boxed{w(\ell,t) = \Delta},$

I.C. $\quad u(x,0) = u_0(x),\ \ \partial_t u(x,0) = u_1(x),$

$w(x,0) = w_0(x),\ \ \partial_t w(x,0) = w_1(x).$

- **Isothermal case** $\boxed{\theta = \vartheta = 0}$:

  the model reduces to the von Kármán system.

$$\begin{cases} \rho\,\partial_{tt}w - \dfrac{E}{1-\nu^2}\partial_x\left\{\partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2\right\} = 0, \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u \\ -\dfrac{E}{1-\nu^2}\partial_x\left\{\left[\partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2\right]\partial_x u\right\} = \dfrac{\rho\,f}{h}. \end{cases}$$

B.C. $\quad u(0,t) = u(\ell,t) = \partial_{xx}u(0,t) = \partial_{xx}u(\ell,t) = 0$, and

$\quad w(0,t) = 0, \ \boxed{w(\ell,t) = \Delta},$

I.C. $\quad u(x,0) = u_0(x), \ \partial_t u(x,0) = u_1(x),$

$\quad w(x,0) = w_0(x), \ \partial_t w(x,0) = w_1(x).$

$w(\ell, t) = 0$

$w(\ell, t) = \Delta < 0$

1.1. Stationary solutions

The static counterpart of the full system (5)-(6) is the BV problem

$$\begin{cases} \partial_{xx}\vartheta = -\dfrac{\rho}{k_0} g_0, \\ \partial_{xx}\theta = -\dfrac{\rho}{k_0} g, \\ \partial_x \left\{ \partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2 - \alpha(1+\nu)\vartheta \right\} = 0, \\ \partial_{xxxx} u + \alpha(1+\nu)\partial_{xx}\theta \\ -\dfrac{12}{h^2}\partial_x \left\{ \left[\partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2 - \alpha(1+\nu)\vartheta \right] \partial_x u \right\} = \dfrac{12(1-\nu^2)\rho}{h^3 E} f \\ \vartheta(0) = \vartheta(\ell) = \theta(0) = \theta(\ell) = 0 \\ w(0) = 0, \ w(\ell) = \Delta \\ u(0) = u(\ell) = \partial_{xx} u(0) = \partial_{xx} u(\ell) = 0 \end{cases}$$

We note that the first two equations are uncoupled.

Then, performing a double integration and taking into account boundary conditions, it follows

$$\begin{cases} \vartheta(x) = \widehat{\vartheta}(x) = -\dfrac{\rho}{k_0}\left[\displaystyle\int_0^x \int_0^\xi g_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi - \frac{x}{\ell}\int_0^\ell \int_0^\xi g_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right], \\ \theta(x) = \widehat{\theta}(x) = -\dfrac{\rho}{k_0}\left[\displaystyle\int_0^x \int_0^\xi g(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi - \frac{x}{\ell}\int_0^\ell \int_0^\xi g(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right]. \end{cases}$$

Focusing on the third equation, the quantity

$$M = \partial_x w + \frac{1}{2}|\partial_x u|^2 - \alpha(1+\nu)\vartheta$$

is constant (i.e. independent of x).

Then, according to Woinovsky-Krieger (1950), we can replace it with its wean value on $(0,\ell)$. By virtue of boundary conditions and previous integrations, it follows

$$\boxed{M} = \frac{\Delta}{\ell} + \frac{1}{2\ell}\int_0^\ell |\partial_x u(x)|^2 \, \mathrm{d}x - \frac{\alpha(1+\nu)}{\ell}\int_0^\ell \widehat{\vartheta}(x) \, \mathrm{d}x.$$

Accordingly, the full system takes the form

$$\begin{cases} \vartheta(x) = \widehat{\vartheta}(x) \\ \theta(x) = \widehat{\theta}(x) \\ \partial_x w + \dfrac{1}{2}|\partial_x u|^2 = \boxed{M} + \alpha(1+\nu)\widehat{\vartheta}, \\ \partial_{xxxx} u - \dfrac{12}{h^2}\boxed{M}\,\partial_{xx} u = \dfrac{12(1-\nu^2)\rho}{h^3 E}\, f + \dfrac{\alpha(1+\nu)\rho}{k_0}\, g. \\ w(0) = 0,\ w(\ell) = \triangle \\ u(0) = u(\ell) = \partial_{xx} u(0) = \partial_{xx} u(\ell) = 0 \end{cases}$$

In particular, the mechanical equilibrium is ruled by

$$\begin{cases} \partial_{xxxx} u - \dfrac{12}{h^2} \left[-p + \dfrac{1}{2\ell} \displaystyle\int_0^\ell |\partial_x u(x)|^2 \, \mathrm{d}x \right] \partial_{xx} u = \dfrac{12(1-\nu^2)\rho}{h^3 E} f + \dfrac{\alpha(1+\nu)\rho}{k_0} g \\ \partial_x w + \dfrac{1}{2} |\partial_x u|^2 = \alpha(1+\nu)\widehat{\vartheta} - p + \dfrac{1}{2\ell} \displaystyle\int_0^\ell |\partial_x u(x)|^2 \, \mathrm{d}x \end{cases}$$

where given data are in blue and

$$p = -\frac{\Delta}{\ell} + \frac{\alpha(1+\nu)}{\ell} \int_0^\ell \widehat{\vartheta}(x) \, \mathrm{d}x$$

The first equation is uncoupled and can be solved separately in order to find stationary solution for u (cf. the single eqn (4)).

As established in [2], no buckling occurs when

$$p \le h^2\pi^2/12$$

- $\boxed{\Delta = 0}$ No buckling occurs when the mean value of $\tilde{\vartheta}$ is "small"

$$\frac{1}{\ell}\int_0^\ell \tilde{\vartheta}(x)\,\mathrm{d}x \le \frac{h^2\pi^2}{12\alpha(1+\nu)}.$$

- $\boxed{\Delta \neq 0}$ The no-buckling condition reads

$$\Delta \ge \alpha(1+\nu)\int_0^\ell \hat{\vartheta}(x)\,\mathrm{d}x - h^2\pi^2\ell/12.$$

Unlike the purely mechanical case, buckling can even occur under axial tension ($\Delta > 0$) because of the thermal axial expansion.

1.2. **A reduced dynamical model** (Woinovsky-Krieger)

We remove the dependence on ϑ and w by means of

Kinematic and Thermal assumptions

K.1 – the axial velocity component is negligible: $\boxed{\partial_t w \equiv 0}$

(physically justified by the hinged ends)

T.1 – the temperature diffusion in the axial direction is negligible:

$\boxed{\partial_{xx}\vartheta(x,t) \equiv 0}$

(physically justified by Zener in 1938)

T.2 – the external heat supply vanishes on the x-axis: $\boxed{g_0 \equiv 0}$.

The reduced system reads

$$\begin{cases} \boxed{\partial_t \left\{ \vartheta - \frac{E\alpha}{(1-\nu)\varpi\rho} \left[\partial_x w + \frac{1}{2} |\partial_x u|^2 \right] \right\} = 0}, \\ \rho\, \partial_t \theta - \frac{k_0}{\Theta_0 \varpi} \partial_{xx}\theta - \frac{E\alpha}{(1-\nu)\varpi} \partial_{xxt} u = \frac{\rho}{\Theta_0 \varpi} g, \\ \boxed{\partial_x \left\{ \partial_x w + \frac{1}{2} |\partial_x u|^2 - \alpha(1+\nu)\vartheta \right\} = 0}, \\ \rho\, \partial_{tt} u - \frac{\rho\, h^2}{12} \partial_{xxtt} u + \frac{E h^2}{12(1-\nu^2)} \partial_{xxxx} u + \frac{E\alpha h^2}{12(1-\nu)} \partial_{xx}\theta \\ \quad - \frac{E}{1-\nu^2} \partial_x \left\{ \left[\partial_x w + \frac{1}{2} |\partial_x u|^2 - \alpha(1+\nu)\vartheta \right] \partial_x u \right\} = \frac{\rho\, f}{h}. \end{cases}$$

- First equation

The constant quantity in t is replaced by its initial value

$$\boxed{\phi(x)} = \vartheta_0 - \frac{E\alpha}{(1-\nu)\varpi\rho}\left[\partial_x w_0 + \frac{1}{2}\left|\partial_x u_0\right|^2\right]$$

- Third equation

The constant quantity in x is replaced by its x-mean value

$$\boxed{\psi(t)} = \frac{\Delta}{\ell} + \frac{1}{2\ell}\int_0^\ell \left|\partial_x u(x,t)\right|^2 \mathrm{d}x - \frac{\alpha(1+\nu)}{\ell}\int_0^\ell \vartheta(x,t)\mathrm{d}x.$$

The resulting system reads

$$\begin{cases} \vartheta - \dfrac{E\alpha}{(1-\nu)\varpi\rho}\left[\partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2\right] = \boxed{\phi(x)} \\ \rho\,\partial_t\theta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \dfrac{\rho}{\Theta_0\varpi}g \\ \partial_x w + \dfrac{1}{2}\left|\partial_x u\right|^2 - \alpha(1+\nu)\vartheta = \boxed{\psi(t)} \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \dfrac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta - \dfrac{E}{1-\nu^2}\boxed{\psi(t)}\,\partial_{xx}u = \dfrac{\rho\,f}{h} \end{cases}$$

Here, the second and fourth equations (in θ and u) are coupled toghether, but are independent of the other variables (ϑ and w), except for $\psi(t)$.

In spite of its expression

$$\boxed{\psi(t)} = \frac{\Delta}{\ell} + \frac{1}{2\ell}\int_0^\ell |\partial_x u(x,t)|^2 \, dx - \frac{\alpha(1+\nu)}{\ell} \boxed{\int_0^\ell \vartheta(x,t) dx}$$

$\psi(t)$ can be shown to depend on u, only. Indeed, by taking the x-mean value of the first equation, the blue-boxed term reads

$$\boxed{\int_0^\ell \vartheta(x,t) dx} = \int_0^\ell \vartheta_0(x) dx + \frac{E\alpha}{2\rho\varpi(1-\nu)} \int_0^\ell \left[|\partial_x u(x,t)|^2 - |\partial_x u_0(x)|^2 \right] dx$$

where u_0 and ϑ_0 are given initial data.

The reduced model (after some rearrangements)

$$\begin{cases} \rho\,\partial_t\theta - \dfrac{k_0}{\Theta_0\varpi}\,\partial_{xx}\theta - \dfrac{E\alpha}{(1-\nu)\varpi}\partial_{xxt}u = \dfrac{\rho}{\Theta_0\varpi}g \\ \rho\,\partial_{tt}u - \dfrac{\rho\,h^2}{12}\partial_{xxtt}u + \dfrac{Eh^2}{12(1-\nu^2)}\partial_{xxxx}u + \dfrac{E\alpha h^2}{12(1-\nu)}\partial_{xx}\theta \\ \qquad - \dfrac{E}{\ell\,(1-\nu^2)}\left[\lambda_0 + \lambda_1 \displaystyle\int_0^\ell |\partial_\xi u(\xi,\cdot)|^2\,\mathrm{d}\xi\right]\partial_{xx}u = \dfrac{\rho\,f}{h} \end{cases}$$

$$\lambda_0 = \Delta - \alpha(1+\nu)\left[\int_0^\ell \vartheta_0(x)\mathrm{d}x - \frac{E\alpha}{2\rho\varpi(1-\nu)}\int_0^\ell |\partial_x u_0(x)|^2\,\mathrm{d}x,\right],$$

$$2\lambda_1 = 1 - \frac{\alpha^2(1+\nu)E}{\rho\,\varpi(1-\nu)} = \frac{2\nu\alpha^2(1+\nu)E\Theta_0 + \rho\,c_v(1-\nu)(1-2\nu)}{\alpha^2(1+\nu)E\Theta_0 + \rho\,c_v(1-\nu)(1-2\nu)} \boxed{> 0}$$

Part 2. The viscoelastic model

A different strategy is devised for a viscoelastic beam with length ℓ.

- ℓ natural reference length,
- $L = \ell + \Delta$ actual length
- Kirchhoff assumption

$$\boldsymbol{u}(x, z, t) = -z\partial_x u(x, t)\boldsymbol{i} + u(x, t)\boldsymbol{k},$$

- Small strains

$$\boldsymbol{\epsilon} = \{\boldsymbol{\epsilon}_{ij}\} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\top})$$

- Unidimensional strain

$$\epsilon(x, z, t) = \boldsymbol{\epsilon}_{11}(x, z, t) = -z\partial_{xx} u(x, t).$$

- The 1-D viscoelastic stress-strain constitutive relation

$$\sigma(x,z,t) = E\left[\epsilon(x,z,t) + \int_0^\infty g'(s)\epsilon(x,z,t-s)ds\right],$$

E Young's modulus

$g : \mathbb{R}^+ \to \mathbb{R}^+$ relaxation measure kernel

Substituting the expression for ϵ, we obtain

$$\sigma(x,z,t) = -Ez\left[\partial_{xx}u(x,t) + \int_0^\infty g'(s)\partial_{xx}u(x,t-s)ds\right].$$

The bending moment of the cross section Ω

$$M(x,t) = -\int_\Omega z\sigma(x,z,t)d\Omega.$$

Hence,

$$M(x,t) = EI\left[\partial_{xx}u(x,t) + \int_0^\infty g'(s)\partial_{xx}u(x,t-s)ds\right],$$

where

$$I = \int_\Omega z^2 d\Omega$$

is the moment of inertia of the cross section.

If a distributed lateral load $F(x,t)$ is applied to the beam, the balance equation at equilibrium can be written as

$$\partial_{xx} M(x,t) = \partial_x T(x,t) + F(x,t), \tag{7}$$

The shearing stress T can be expressed in terms of the axial force N by

$$T(x,t) = N(x,t)\partial_x u(x,t).$$

The lateral load F can be decomposed into the sum of the inertia force and an external load

$$F(x,t) = \rho f(x) - \rho \partial_{tt} u(x,t). \tag{8}$$

where $\rho > 0$ is the mass per unit of length

In order to consider the extensibility of the beam a specific form of the axial force N is needed. To this end, we assume (cf. Woinovsky-Krieger)

$$N(x,t) = N_0 + N_1(t),$$

- $N_0 = \dfrac{E\Delta|\Omega|}{\ell}$ applied axial load (Δ = axial displacement)
- $N_1(t) = \dfrac{E|\Omega|}{2\ell}\displaystyle\int_0^\ell |\partial_x u(y,t)|^2 dy$

N_1 is the extra-tension which takes into account the beam elongation

$$N_0 + N_1(t) = \frac{E|\Omega|}{\ell}\left(\Delta + \frac{1}{2}\int_0^\ell |\partial_x u(y,t)|^2 dy\right)$$

In conclusion, by setting

$$\alpha_0 = \frac{EI}{\rho}, \qquad \beta = \frac{E\Delta|\Omega|}{\rho\ell}, \qquad \gamma = \frac{E|\Omega|}{2\rho\ell}, \qquad \mu(s) = -\frac{EI}{\rho}g'(s),$$

the motion equation (7)-(8) transforms into

$$\partial_{tt}u + \alpha_0\,\partial_{xxxx}u - \int_0^\infty \mu(s)\partial_{xxxx}u(t-s)ds + \left[p - \gamma\int_0^\ell |\partial_x u(y,t)|^2 dy\right]\partial_{xx}u = f.$$

- $\gamma > 0$,
- p can be either negative (traction) or positive (compression).

Remark. The static counterpart of the VE model looks like eqn (4)

3. Finite-dimensional reduction

Statics. From eqn. (4), the static BV problem reads

$$\begin{cases} \partial_{xxxx} u + \left(p - \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right) \partial_{xx} u = q, \\ u(0) = u(\ell) = \partial_{xx} u(0) = \partial_{xx} u(\ell) = 0 \end{cases}$$

where $q = f + g$. It can be recast into the abstract form

$$\boxed{Au - \left(p - \|u\|_1^2\right) A^{1/2} u = q}, \qquad (9)$$

- $u \in H^2 \cap H_0^1$ (weak solution)
- $A = \partial_{xxxx}$ $\mathcal{D}(A) = \{\varphi \in H^4 : \varphi(0) = \varphi(\ell) = \partial_{xx}\varphi(0) = \partial_{xx}\varphi(\ell) = 0\}$
- $A^{1/2} = -\partial_{xx}$ $\mathcal{D}(A^{1/2}) = \{\varphi \in H^2 : \varphi(0) = \varphi(\ell) = 0\}$

$A : \mathcal{D}(A) \Subset H \to H$ is a strictly positive selfadjoint operator. Then

- $\mathcal{H}^r = \mathcal{D}(A^{r/4}), \quad \|u\|_r = \|A^{r/4}u\|, \quad \sqrt{\lambda_1}\,\|u\|_r^2 \le \|u\|_{r+1}^2.$
- $A\psi_1 = \lambda_1\psi_1, \quad A^{1/2} = \sqrt{\lambda_1}\,\psi_1, \quad ||\psi_1||_1^2 = \langle \psi_1, A^{1/2}\psi_1\rangle = \sqrt{\lambda_1}$

Eqn (9) can be reduced to an algebraic eqn by projection on the subspace spanned by the first eigenfunction ψ_1 of the operator A. Letting

$$u(x) = v\,\psi_1(x),$$

in the homogeneous case ($q = 0$) we obtain

$$\boxed{\sqrt{\lambda_1}\,v\big(\sqrt{\lambda_1} - p + \sqrt{\lambda_1}\,v^2\big) = 0},$$

- $\boxed{p < \sqrt{\lambda_1}}$: $v = 0\,;$
- $\boxed{p \ge \sqrt{\lambda_1}}$: $v = 0\,,\ v = \pm\sqrt{\dfrac{p - \sqrt{\lambda_1}}{\sqrt{\lambda_1}}}$

Dynamics. From eqn (2), the damped dynamic IBV problem reads

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \partial_t u + \left(p - \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right) \partial_{xx}u = q, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0 \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) \end{cases}$$

It can be recast into the abstract Cauchy problem

$$\boxed{\begin{cases} \partial_{tt}u + Au + \partial_t u - \left(p - \|u\|_1^2\right) A^{1/2}u = q, \quad t > 0, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \end{cases}} \tag{10}$$

on the product Hilbert space

$$\mathcal{H} = \mathcal{H}^2 \times \mathcal{H}^1$$

Eqn (10) can be reduced to an ODE by projection on the span of the first eigenfunction ψ_1 of the operator A. Letting

$$u(x) = v(t)\,\psi_1(x),$$

in the homogeneous case ($q = 0$) we obtain

$$\boxed{\ddot{v} + \dot{v} + \sqrt{\lambda_1}\,v\big(\sqrt{\lambda_1} - p + \sqrt{\lambda_1}\,v^2\big) = 0}$$

Letting $x = \sqrt{\lambda_1}\,v$, it looks like a damped Van der Pol equation

$$\boxed{\ddot{x} + \dot{x} = \varepsilon^2 x - x^3}$$

provided that $\boxed{p \geq \sqrt{\lambda_1}}$ and then $\varepsilon^2 = \sqrt{\lambda_1}(p - \sqrt{\lambda_1}) > 0$

Conserved dynamics. The Van der Pol equation

$$\ddot{x} = \varepsilon^2 x - x^3$$

Elastic (nonconvex) potential energy

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}\varepsilon^2 x^2$$

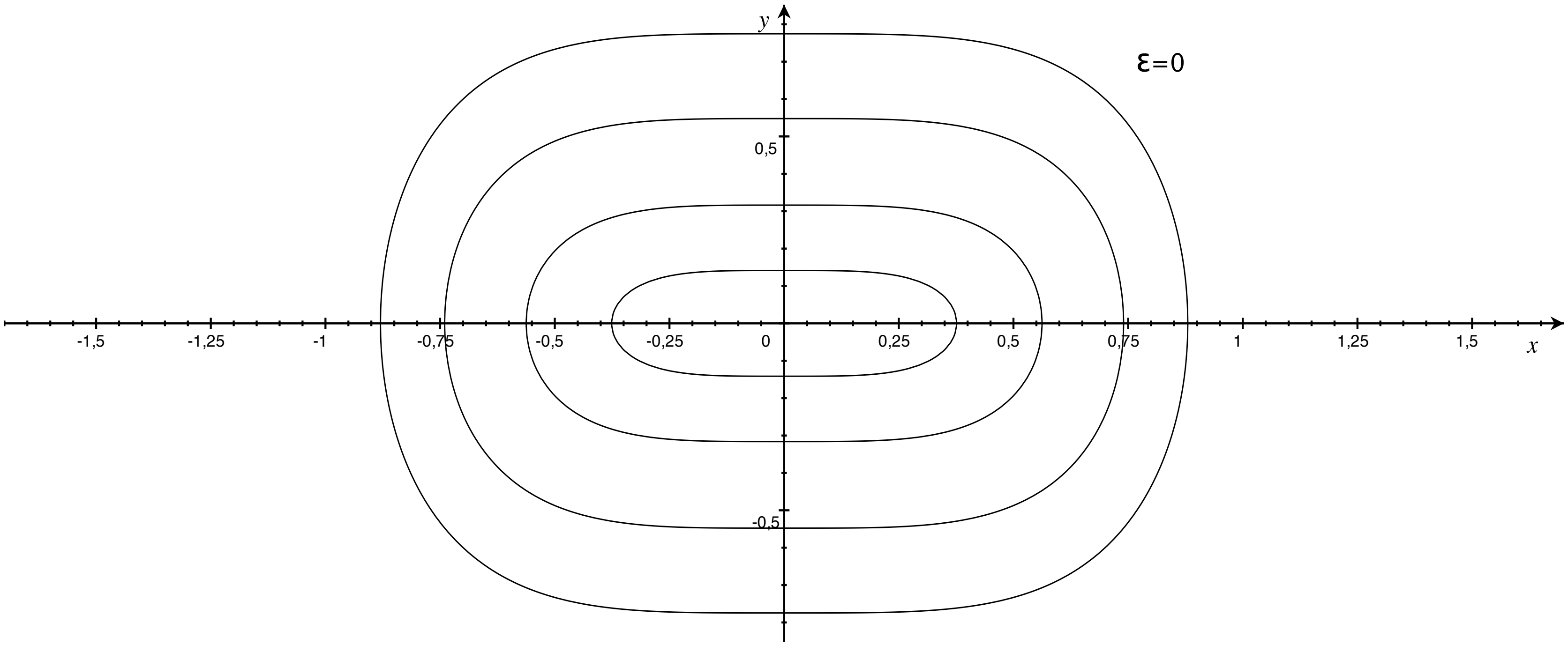
Total energy conservation

$$\frac{1}{2}\dot{x}^2 - \frac{1}{2}\varepsilon^2 x^2 + \frac{1}{4}x^4 = E$$

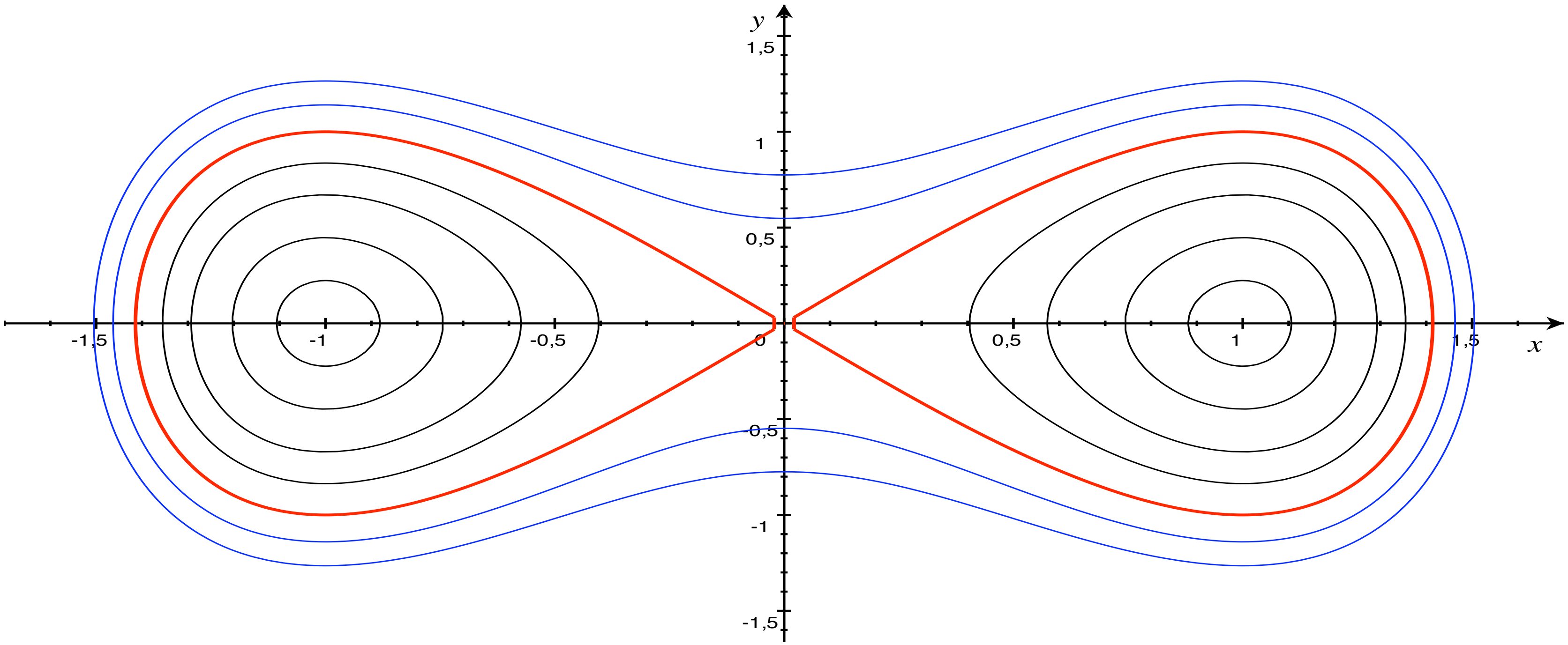
Stationary points ($\dot{x} = 0$)

$$x = 0\,, \qquad x = \pm\varepsilon$$

- $x = 0$ is **globally stable** when $\varepsilon = 0$, **unstable** when $\varepsilon > 0$.
- $x = \pm\varepsilon$ are **locally stable** when $\varepsilon > 0$



The conserved dynamics $\varepsilon = 0$



The conserved dynamics $\varepsilon = 1$. In red the separatrix curve.

Dissipative dynamics. The damped Van der Pol equation

$$\ddot{x} + \dot{x} = \varepsilon^2 x - x^3$$

Energy dissipation

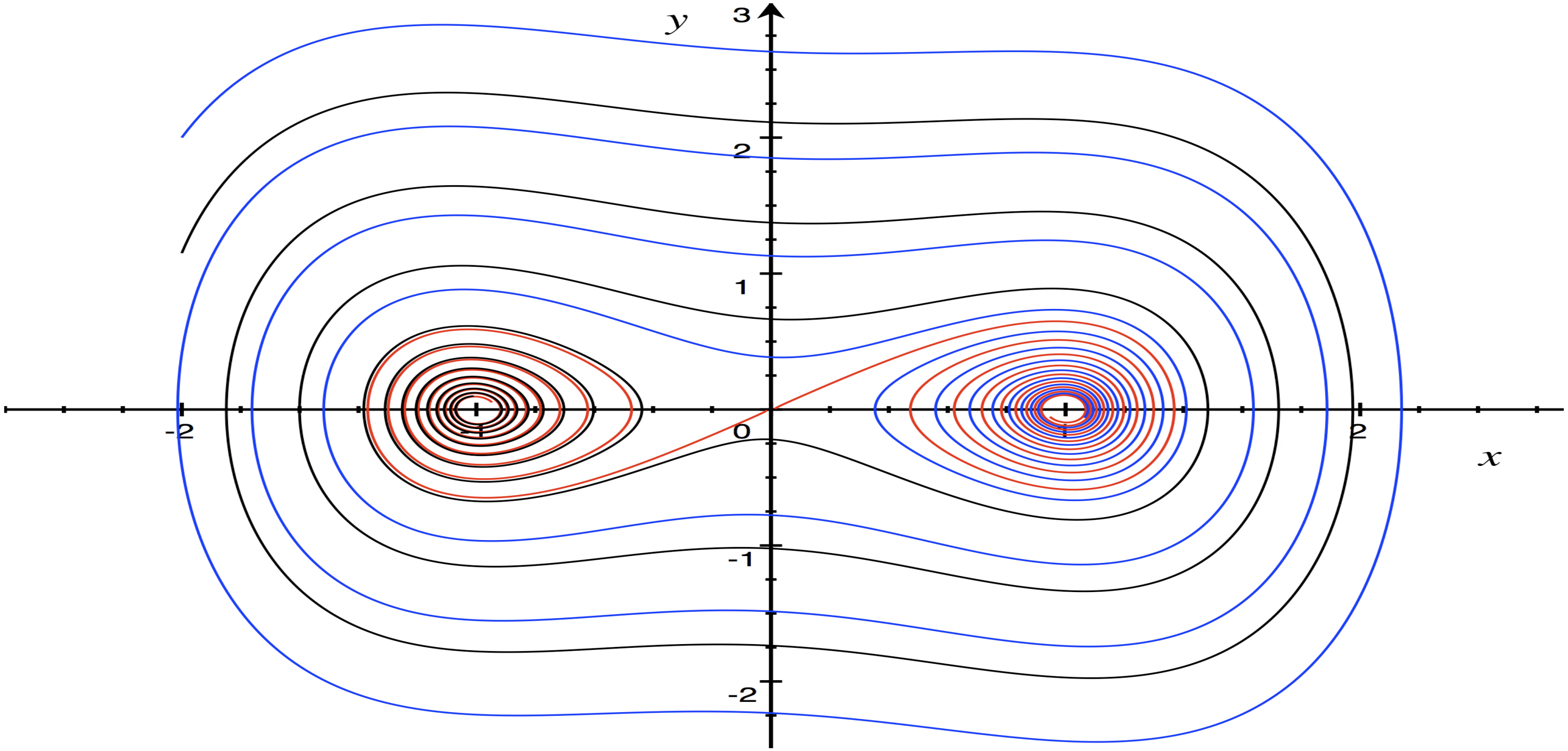
$$\frac{d}{dt}\left(\dot{x}^2 - \varepsilon^2 x^2 + \frac{1}{2}x^4\right) = -2\dot{x}^2 < 0$$

Stationary points ($\dot{x} = 0$)

$$x = 0\,, \qquad x = \pm\varepsilon$$

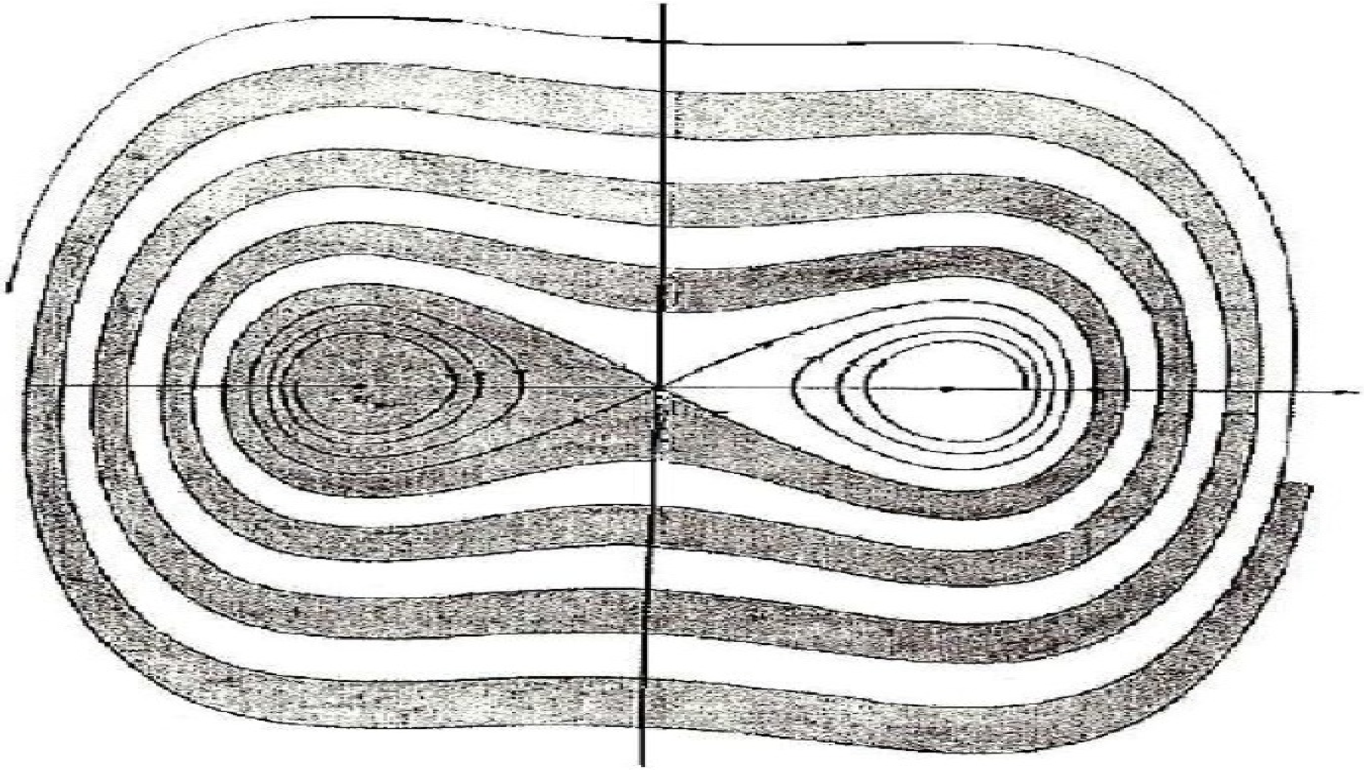
Stability:

- $x = 0$ is **globally exponentially stable** when $\varepsilon = 0$,
  unstable when $\varepsilon > 0$.
- $x = \pm\varepsilon$ are **locally exponentially stable** when $\varepsilon > 0$



The damped dynamics $\boxed{\varepsilon = 1}$.

In red the unstable manifold connecting the steady states (attractor)



Basins of attraction $\boxed{\varepsilon = 1}$

Part 4. Longtime behavior of solutions

Our goal now is to scrutinize the global longtime behavior of the IBVP

$$\begin{cases} \partial_{tt}u + \partial_{xxxx}u + \partial_{xx}\theta + \left(p - \int_0^1 |\partial_\xi u(\xi,\cdot)|^2 \mathrm{d}\xi\right) \partial_{xx}u = f, \\ \partial_t\theta - \partial_{xx}\theta - \partial_{xxt}u = g, \\ \theta(0,t) = \theta(1,t) = 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \\ \theta(x,0) = \theta_0(x), \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x). \end{cases} \tag{11}$$

for all $p \in \mathbb{R}$. For the sake of simplicity we neglect $\boxed{\partial_{xxtt}u}$.

- **The abstract setting**

We consider the abstract Cauchy problem

$$\begin{cases} \partial_{tt}u + Au - A^{1/2}\theta - \big(p - \|u\|_1^2\big)A^{1/2}u = f, \quad t > 0, \\ \partial_t\theta + A^{1/2}\theta + A^{1/2}\partial_t u = g, \quad t > 0, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \quad \theta(0) = \theta_0, \end{cases} \tag{12}$$

on the product Hilbert space

$$\mathcal{H} = H^2 \times H \times H$$

– $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ is a real Hilbert space

– $A : \mathcal{D}(A) \Subset H \to H$ a strictly positive selfadjoint operator:

$$H^r = \mathcal{D}(A^{r/4}), \quad \|u\|_r = \|A^{r/4}u\|, \quad \sqrt{\lambda_1}\,\|u\|_r^2 \leq \|u\|_{r+1}^2.$$

Remark. Problem (11) is just a particular case of (12).

(11) can be obtained from (12) by setting

– $H = L^2(0,1)$, $H^1 = H_0^1(0,1)$, $H^2 = H^2(0,1) \cap H_0^1(0,1)$

– $A = \partial_{xxxx}$, joint with

– $\mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0,1) : w(0) = w(1) = w''(0) = w''(1) = 0\}$.

– $A^{1/2} = -\partial_{xx}$, joint with

– $\mathcal{D}(-\partial_{xx}) = H^2(0,1) \cap H_0^1(0,1)$.

Proposition 1. (Non autonomous case)
Assume that

$$f \in L^1_{\text{loc}}(\mathbb{R}^+, H), \qquad g \in L^1_{\text{loc}}(\mathbb{R}^+, H) + L^2_{\text{loc}}(\mathbb{R}^+, H^{-1}).$$

For all $z = (u_0, u_1, \theta_0) \in \mathcal{H}$, the problem (12) admits a unique solution

$$(u(t), \partial_t u(t), \theta(t)) \in \mathcal{C}(\mathbb{R}^+, \mathcal{H})$$

which continuously depends on the initial data.

We define the *solution operator* $S(t) \in \mathcal{C}(\mathcal{H}, \mathcal{H})$, $\forall t \geq 0$, as

$$z = (u_0, u_1, \theta_0) \mapsto S(t)z = (u(t), \partial_t u(t), \theta(t)).$$

Proposition 2. (Autonomous case)
When both f and g are time-independent, then S is a *strongly continuous semigroup*.

For any given $z = (u_0, u_1, \theta_0) \in \mathcal{H}$, we define the *nonlinear energy* as

$$\mathcal{E}(t) = \frac{1}{2}\|S(t)z\|^2_{\mathcal{H}} + \frac{1}{4}\big(\|u(t)\|^2_1 - p\big)^2.$$

Multiplying the first equation of (12) by $\partial_t u$ and the second one by θ, we obtain the *energy identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \|\theta\|^2_1 = \langle \partial_t u, f\rangle + \langle \theta, g\rangle. \tag{13}$$

Proposition 3. The nonlinear energy $\mathcal{E}$ is bounded by an increasing function of the norms of initial-data.

For every $T > 0$, there exist a positive increasing function $\mathcal{Q}_T$ such that

$$\boxed{\mathcal{E}(t) \leq \mathcal{Q}_T(\mathcal{E}(0))} \qquad \forall t \in [0, T].$$

Theorem 4. (Absorbing set) Let $f \in H$, and $g \in H^{-1}$. Then, there exists $R_0 > 0$ such that in correspondence of every $R \geq 0$, there is

$$t_0 = t_0(R) \geq 0 : \qquad \mathcal{E}(t) \leq R_0, \qquad \forall t \geq t_0,$$

whenever $\mathcal{E}(0) \leq R$. Both R_0 and t_0 can be explicitly computed.

– All solutions that originate from some initial data in a ball of energy-radius R, after a finite time $t_0(R)$ enter into a ball of energy-radius R_0 which is called absorbing set.

– The radius R_0 of the absorbing set is independent of R!

– The entering time $t_0(R)$ is an increasing function of R.

The proof of Th. 4 requires a very special Gronwall-type lemma.

Lemma. (Gatti - Pata - Zelik 2009) Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy, for some $K \geq 0$, $Q \geq 0$, $\varepsilon_0 > 0$ and every $\varepsilon \in (0, \varepsilon_0]$, the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda(t) + \varepsilon\Lambda(t) \leq K\varepsilon^2[\Lambda(t)]^{3/2} + \varepsilon^{-2/3}\varphi(t),$$

where $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{R}^+)$ is such that $\sup_{t\geq 0} \int_t^{t+1} \varphi(\tau)\mathrm{d}\tau \leq Q$.

Then, there exist $R_1 > 0$, $\kappa > 0$ such that, for every $R \geq 0$, it follows that

$$\Lambda(t) \leq R_1, \qquad \forall t \geq R^{1/\kappa}(1 + \kappa Q)^{-1},$$

whenever $\Lambda(0) \leq R$.

Both R_1 and κ can be explicitly computed in terms of K, Q and ε_0.

4.1. The global attractor.

The strategy to prove the existence of a **global attractor** with optimal regularity (in the norm of $H^4 = \mathcal{D}(A)$) requires the existence of a Lyapunov functional for the system.

Unfortunately, the occurrence of g in eqn (13) prevents this fact. A change of variables is needed in order that g disappears from the heat equation.

An equivalent Problem. Denoting

$$\theta_g = A^{-1/2} g$$

we introduce the function

$$\omega(t) = \theta(t) - \theta_g.$$

It then is apparent that $(u(t), \partial_t u(t), \omega(t))$ solves the problem

$$\begin{cases} \partial_{tt}u + Au - A^{1/2}\omega - \big(p - \|u\|_1^2\big)A^{1/2}u = h, \\ \partial_t\omega + A^{1/2}\omega + A^{1/2}\partial_t u = 0, \end{cases}$$

where $h = f + g \in H$, with the initial conditions

$$\zeta = (u(0), \partial_t u(0), \omega(0)) = z - z_g,$$

and $z_g = (0, 0, \theta_g)$. It generates a strongly continuous semigroup $S_0(t)$ on $\mathcal{H}$, such that

$$S(t)(\zeta + z_g) = z_g + S_0(t)\zeta, \qquad \forall \zeta \in \mathcal{H}.$$

Thus, if $\mathfrak{B}$ is the absorbing set of S, $S_0(t)$ possesses the absorbing set

$$\mathfrak{B}_0 = -z_g + \mathfrak{B}.$$

Let

$$\mathcal{E}_0(t) = \frac{1}{2}\|S_0(t)z\|^2_{\mathcal{H}} + \frac{1}{4}\big(\|u(t)\|^2_1 - p\big)^2.$$

Then, the functional

$$\boxed{\mathcal{L}_0(t) = \mathcal{E}_0(t) - \langle h, u(t)\rangle}$$

is a Lyapunov functional for $S_0(t)$. It satisfies the differential equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_0 + \|\omega\|^2_1 = 0.$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_0 \leq 0.$$

Theorem 3. Let $f, g \in H$ and $p \in \mathbb{R}$. Then, the semigroup $S_0(t)$ acting on $\mathcal{H}$ possesses the (connected) global attractor $\mathfrak{A}_0$ bounded in

$$\mathcal{V} = H^4 \times H^2 \times H^2 \Subset \mathcal{H}.$$

Accodingly, the semigroup $S(t)$ acting on $\mathcal{H}$ possesses the (connected) global attractor $\mathfrak{A}$, where

$$\mathfrak{A} = z_g + \mathfrak{A}_0.$$

The regularity of $\mathfrak{A}_0$ and $\mathfrak{A}$ is optimal.

The proof follows the same arguments as devised in [GPV, *Nonlinearity*, 2008].

Remark. $\mathfrak{A}$ is as regular as f and g permit. For instance, if $f, g \in H^n$ for every $n \in \mathbb{N}$, then each component of $\mathfrak{A}$ belongs to H^{2n} for every $n \in \mathbb{N}$.

4.2. Exponential stability.

Let λ_1 the first eigenvalue of A.

Theorem 4. If $\boxed{f+g=0}$ and $\boxed{p<\sqrt{\lambda_1}}$, then $\mathfrak{A}=\{z_g\}=\{(0,0,\theta_g)\}$ and

$$\boldsymbol{\delta}_{\mathcal{H}}(S(t)B,\mathfrak{A})=\sup_{z\in B}\|S(t)z-z_g\|_{\mathcal{H}}\leq\mathcal{Q}(\|B\|_{\mathcal{H}})\mathrm{e}^{-\varkappa t},$$

for some $\varkappa>0$ and some positive increasing function $\mathcal{Q}$.

Both $\varkappa$ and $\mathcal{Q}$ can be explicitly computed.

4.3. The structure of the global attractor.

Let $p \in \mathbb{R}$ and

$$\mathcal{S} = \big\{\widehat{z} \in \mathcal{H} : S(t)\widehat{z} = \widehat{z},\ \forall t \geq 0\big\}$$

the set of stationary points of $S(t)$: $\widehat{z} = (\widehat{u}, 0, \theta_g)$, where $\widehat{u} \in H^4$ is a solution to the elliptic problem

$$\boxed{A\widehat{u} - \big(p - \|\widehat{u}\|_1^2\big)A^{1/2}\widehat{u} = f + g.}$$

$\mathcal{S}_0 = \mathcal{S} - z_g$ is the (nonempty) set of stationary points of $S_0(t)$:

$$\widehat{\zeta} = \widehat{z} = (\widehat{u}, 0, 0) - z_g.$$

- **Characterization of $\mathfrak{A}$.**

The global attractor $\mathfrak{A}$ coincides with the unstable set of $\mathcal{S}$.

$$\mathfrak{A} = \Big\{ z(0) : z(t) \text{ is a complete trajectory and } \lim_{t\to\infty} \|z(-t) - \mathcal{S}\|_{\mathcal{H}} = 0 \Big\}.$$

If $\mathcal{S}$ is finite, then

$$\mathfrak{A} = \Big\{ z(0) : \lim_{t\to\infty} \|z(-t) - z_1\|_{\mathcal{H}} = \lim_{t\to\infty} \|z(t) - z_2\|_{\mathcal{H}} = 0 \Big\},$$

for some $z_1, z_2 \in \mathcal{S}$.

If $\mathcal{S}$ consists of a single element $z_{\rm g} \in \mathcal{H}^2$, then $\mathfrak{A} = \{z_{\rm g}\}$.

Part 5. Stationary points

The set of stationary points of $S(t)$, namely

$$\mathcal{S} = \{z \in \mathcal{H} : S(t)z = z, \forall t \geq 0\}$$

consists of all vectors of the form $(u, 0, \theta_g)$, where

$$\theta_g = A^{-1/2} g \in H^2$$

and $u \in H^4$ is a solution to the elliptic problem (9), namely

$$\boxed{Au - \big(p - \|u\|_1^2\big) A^{1/2} u = q},$$

where $q = f + g \in H$. Let λ_n, $n = 1, n$, the eigenvalues of A. On $(0, 1)$

$$\lambda_n = n^4 \pi^4.$$

• Reduction of the problem

Let $h = A^{-1/2}q$. Then, problem (9) can be rewritten as

$$\boxed{A^{1/2}u - \big(p - \|u\|_1^2\big)u = h}, \tag{14}$$

which is an elliptic problem of the second order.

Weak solutions.

Let $h \in H^{-1}$. A function $\tilde{u} \in H^1$ is a weack solution to (14) if

$$\langle A^{1/2}\tilde{u}, A^{1/2}w\rangle - \big(p - \|\tilde{u}\|_1^2\big)\langle \tilde{u}, w\rangle = \langle A^{-1/2}h, A^{1/2}w\rangle,$$

for every $w \in H^1$.

- **The homogeneous case**

Theorem. Let $h = 0$ and

$$\mathcal{S}_\star = \{n : p - \sqrt{\lambda_n} > 0\}, \qquad n_\star = |\mathcal{S}_\star|$$

Then, (14) has exactly $2n_\star + 1$ solutions: the trivial one and

$$u_n^\pm = C_n^\pm \sqrt{2} \sin n\pi x,$$

for every $n \in \mathcal{S}_\star$, where

$$C_n^\pm = \pm\sqrt{\frac{p - \sqrt{\lambda_n}}{\sqrt{\lambda_n}}}.$$

Proof. For any $p \in \mathbb{R}$, $u = 0$ is a solution.

A nontrivial solution u solves the equation

$$A^{1/2}u + \mu u = 0, \qquad \mu = \|u\|_1^2 - p.$$

Hence,

$$\mu = -\sqrt{\lambda_n}, \qquad u = C\, e_n, \; C \neq 0,$$

where e_n is the eigenfunction corresponding to $\sqrt{\lambda_n}$. In particular,

$$\|u\|_1^2 = C^2\sqrt{\lambda_n}.$$

The value C is determined by the relation

$$C^2\sqrt{\lambda_n} = p - \sqrt{\lambda_n}.$$

Therefore, we have exactly $2n_\star$ nontrivial solutions if and only if $n \in \mathcal{S}_\star$.

Nontrivial solutions to the homogeneous version of (14) are given by

$$u_n^{\pm}(x) = \pm\sqrt{\frac{2\,p}{n^2\pi^2} - 2}\,\sin n\pi x.$$

From the physical viewpoint, this means that when p exceeds the first eigenvalue of the operator $A^{1/2}$, namely

$$\sqrt{\lambda_n} = n^2\pi^2$$

then nontrivial symmetric solutions pop up (the buckling states).

- **The nonhomogeneous case**

Theorem. Let $h \neq 0$, $h \in H^{-1}$ and $h_n = \langle A^{-1/2}h, A^{1/2}e_n \rangle$, where $h_n \neq 0$ for some n. We define

$$Q_j = \sum_{n \neq j} \frac{\sqrt{\lambda_n}\, h_n^2}{(\sqrt{\lambda_n} - \sqrt{\lambda_j})^2}, \qquad j \in \mathbb{N}.$$

Along with $n_\star = |\mathcal{S}_\star|$, we define

$$j_\star = \Big| \Big\{ j \in \mathbb{N} : p - \sqrt{\lambda_j} > 0,\ \ Q_j < p - \sqrt{\lambda_j},\ \ h_j = 0 \Big\} \Big|,$$
$$j_\star^0 = \Big| \Big\{ j \in \mathbb{N} : p - \sqrt{\lambda_j} > 0,\ \ Q_j = p - \sqrt{\lambda_j},\ \ h_j = 0 \Big\} \Big|,$$

Then, (14) has exactly $m_\star$ solutions, with

$$1 \leq m_\star \leq 2n_\star + 2j_\star + j_\star^0 + 1.$$

Proof. Now, $\tilde{u} = 0$ is not a solution anymore. Then, by setting

$$\nu = -p + \|\tilde{u}\|_1^2, \tag{15}$$

we have the constraint

$$p + \nu > 0. \tag{16}$$

Writing $\tilde{u} = \sum_n u_n e_n$, with $u_n = \langle \tilde{u}, e_n \rangle$, we have

$$\|\tilde{u}\|_1^2 = \sum_n \sqrt{\lambda_n} u_n^2.$$

Thus, (15) turns into

$$\nu = -p + \sum_n \sqrt{\lambda_n} u_n^2. \tag{17}$$

Projecting (14) on the orthonormal basis, we obtain,

$$(\sqrt{\lambda_n} + \nu)u_n = h_n, \qquad n \in \mathbb{N}. \tag{18}$$

Then, t solution $\tilde{u}$ is known once we determine all the coefficients u_n.

- $\nu \neq -\sqrt{\lambda_n}$, for all n.

From (18) and (17) it follows

$$u_n = \frac{h_n}{\sqrt{\lambda_n} + \nu}. \tag{19}$$

$$p + \nu = \Phi(\nu) \qquad \Phi(\nu) = \sum_n \frac{\sqrt{\lambda_n}\, h_n^2}{(\sqrt{\lambda_n} + \nu)^2}. \tag{20}$$

Setting Recalling (16), The admissible values of ν are the solutions to the equation

$$\Lambda(\nu) = \Phi(\nu) - p - \nu = 0, \qquad (21)$$

in $D = (-p, +\infty) \setminus \{-\sqrt{\lambda_n}\}$, which is the union (empty if $n_\star = 0$) of $n_\star$ bounded open interval I_n and of the open interval $I_0 = (\alpha, +\infty)$, where

$$\alpha = \begin{cases} -\sup\limits_{n \in \mathcal{S}_\star} \sqrt{\lambda_n} & \text{if } n_\star > 0, \\ -p & \text{if } n_\star = 0. \end{cases}$$

For every $\nu \in D$, we have

$$\Lambda''(\nu) = \Phi''(\nu) = 6 \sum_n \frac{\sqrt{\lambda_n} h_n^2}{(\sqrt{\lambda_n} + \nu)^4} > 0.$$

Thus, Λ is strictly convex on each $I_n \subset D$, $n \in \{1, ..., n_\star\}$ and the equation $\Lambda(\mu) = 0$ can have at most two solutions on each I_n.

In the unbounded interval I_0, the function Λ is strictly decreasing. Moreover, since $\Phi(\infty) = 0$, then $\lim_{\nu \to +\infty} \Lambda(\nu) = -\infty$, and

$$\lim_{\nu \to \alpha^+} \Lambda(\nu) = \begin{cases} +\infty & \text{if } n_\star > 0, \\ \Phi(-p) > 0 & \text{if } n_\star = 0. \end{cases}$$

So, we conclude that there is exactly one solution in I_0.

Summarizing, the equation $\Lambda(\nu) = 0$, and then (14), has at least one solution and at most $2n_\star + 1$ solutions with the property that $\nu \neq -\sqrt{\lambda_n}$.

In addition, for every $\nu \in D$ such that $\Lambda(\nu) = 0$, the vector $\tilde{u}$ with Fourier coefficients given by (19) belongs to H^1.

- $\nu = -\sqrt{\lambda_j}$, for some given j.

We preliminarily observe that, due to constrain (16),

$$p + \nu > 0$$

if $p \leq \sqrt{\lambda_j}$, no such solutions exist. In the other case, $p > \sqrt{\lambda_j}$, for $n \neq j$ the values u_n are fixed by

$$\boxed{u_n = \frac{h_n}{\sqrt{\lambda_n} - \sqrt{\lambda_j}}}$$

We are left to determine the value u_j. But (17) now reads

$$\sqrt{\lambda_j} u_j^2 + Q_j = p - \sqrt{\lambda_j}.$$

Therefore, we have no solutions whenever $Q_j > p - \sqrt{\lambda_j}$.

Assume then that $Q_j \le p - \sqrt{\lambda_j}$. From (18),

$$(\sqrt{\lambda_j} + \nu)u_j = h_j$$

has no solutions unless $h_j = 0$, since $\nu = -\sqrt{\lambda_j}$. If $Q_j = p - \sqrt{\lambda_j}$ we have only the trivial solution

$$\boxed{u_j = 0}$$

On the other hand, if $Q_j < p - \sqrt{\lambda_j}$, we have two solutions, corresponding to

$$\boxed{u_j^{\pm} = \pm\sqrt{(p - \sqrt{\lambda_j} - Q_j)/\sqrt{\lambda_j}}}$$