Carleman estimates and nonlocal severely ill-posed linear parabolic problems

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Abstract

Via Carleman estimates we can prove uniqueness and continuous dependence results for solutions to overdetermined ill-posed linear integro-differential parabolic problems. Similar results can be proved for ill-posed linear differential parabolic problems with deviating arguments. The overdetermination is prescribed either in an open subset of the (geometric) domain or on an open subset of its boundary.

1 Introduction

Severely ill-posed problems for PDE's are well-known and studied as for uniqueness and continuous dependence on the data. Each mathematician working with PDE's perfectly knows that the Cauchy problem for elliptic equations, the *spatial* boundary (*not* the initial-boundary) problem for hyperbolic equations and the backward initial-boundary problem for parabolic equations are ill-posed, i.e. the contradict Hadamard's celebrated definition of a well-posed problem that greatly affected the Mathematics of the first half of the twentieth century.
On the contrary, in the second half of the last century a lot of interest, due to the rushing on of Technology, was devoted to Inverse Problems, a branch of which consists just of severely ill-posed problems, where *severely* means that no transformation can be found in order to change such problems to well-posed ones, at least, say, when working in classical or Sobolev function spaces of *finite order*.
Of course, in this situation lesser interest was devoted to severely ill-posed integrodifferential problems or to differential problem with deviating arguments.
This paper is just devoted to shed some light on such problems, mainly on the questions of uniqueness and continuous dependence on the data, two fundamental topics for people working in Applied Mathematics.
More exactly we will deal here with four parabolic problems, three of them being integrodifferential, the remaining being differential, but with deviating arguments. In both problems *no* initial condition will be supplied. It will be replaced by the requirement that the “temperature” u should either assume: (i) *prescribed*

values $u(t,x) = u_0(t,x)$ for all $(t,x) \in (0,T) \times \omega$ ω being a subdomain of the spatial domain Ω where the parabolic equation is assigned. or (ii) should satisfy *prescribed Cauchy conditions* on the lateral boundary of Ω, Problems of this type seem, to the author's knowledge, not to have been yet studied, but in [20] and [21]. More exactly, in [20] the case (i) is studied for both an integrodifferential equation and a differential equation, but with deviating arguments. Instead, in [21] the case (ii) is analyzed for several integrodifferential problems and two differential equations with deviating arguments. The motivation of these papers is to collect some information about the general problem stated in Subsection 1.3 of this paper. The main task of this paper consists in finding out estimates in L^2 for the traces $u(t_0,\cdot)$, $t_0 \in (0,T]$, of our solution in terms of suitable norms of the data as well as in showing that the *unique continuation property* holds for our ill-posed problem (cf. Remark 2.2 in Subsection 2.2). As far as unique continuation for PDE's is concerned, we quote the papers [1], [5], [10], [11], [12], [13], [17], [23], [24], [25]. The fundamental tool to give some positive answer to our problem will be deduced by adapting to our case Carleman's celebrated estimates for PDE's - of use both in Control and Inverse Problem Theory -.

1.1 Plan of the paper

Section 1 is devoted to exhibiting general parabolic integrodifferential ill-posed problems with an additional condition on a open subset of Ω and showing the related (admissible) linear integral operators. Most of them are, at present, open problems. Section 2 is concerned with the unique extension property and the solvability - i.e. with uniqueness and continuous dependence on the data - of one of such problems via Carleman estimates related to the associated differential operator. Section 3 deals with similar questions for a differential equation with deviating arguments. Sections 4 and 5 are devoted to integrodifferential ill-posed problems when the Cauchy condition is given on a part of the lateral boundary. They are concerned with the same questions as above.

1.2 A second-order linear operator

Let ω and Ω be two bounded open sets in $\mathbb{R}^n$ such that $\omega \subset\subset \Omega$, when needed $\partial\omega$ and $\partial\Omega$ being of C^l-class, with suitable l. Let $A(x,D)$ be the (formal) uniformly elliptic linear operator, with principal part in divergence form, defined by

$$A(x,D) = \sum_{i,j=1}^{n} D_{x_i}[a_{i,j}(x)D_{x_j}] + \sum_{j=1}^{n} a_j(x)D_{x_j} + a_0(x),$$

where

$$a_{i,j} \in C^1(\overline{\Omega}),\ i,j = 1,\dots,n, \quad a_j \in L^\infty(\Omega),\ j = 0,\dots,n,$$

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \ge \mu_1|\xi|^n, \quad x \in \overline{\Omega},\ \xi \in \mathbb{R}^n,$$

for some positive constant μ_1.

1.3 The general linear ill-posed problem

The question we are concerned with consists in solving a parabolic integrodifferential (or differential) problem when the initial condition is missing. To overcome this big trouble - making our problem *ill-posed* - we need to have at our disposal at least a suitable additional information. In this case our fundamental aim consists in recovering, at least, the uniqueness of the solution and its continuous dependence on the data in suitable metrics to be determined. To fix ideas, we will be concerned with the problem of *estimating the trace* $u(t_0,\cdot)$, $t_0 \in (0,T)$, *of the solution*[1] $u : [0,T]\times\Omega \to \mathbb{R}$ *to the problem*

$$(IP)\quad \begin{cases} D_t u(t,x) - A(x,D)u(t,x) \\ = B_0 u(t,x) + Ku(t,x) + f(t,x), & (t,x) \in (0,T)\times\Omega =: Q_T, \\ u(t,x) = u_0(t,x) + B_1 u(t,x), & (t,x) \in (0,T)\times\omega, \\ u(t,x) = g(t,x) + B_2 u(t,x), & (t,x) \in (0,T)\times\partial\Omega, \end{cases}$$

where $f \in L^2((0,T)\times\Omega)$, $u_0 \in L^2((0,T)\times\omega)$, $g \in L^2((0,T)\times\partial\Omega)$ and K, B_0, B_1, B_2 are linear operators with domain in $L^2(\Omega)$ defined by

$$Ku(t,x) = k(t,x)u(t,\rho x), \quad \rho \in (0,1),$$

$$B_0 u(t,x) = \int_{\overline{Q}_T} k_{0,1}(t,x,s,y)u(s,y)\, d(\nu_1(s)\times\nu_2(y)) + \int_{\overline{\Omega}} k_{0,2}(t,x,y)u(t,y)\, d\nu_3(y),$$

$$B_j u(t,x) = \int_{(0,T)\times\Gamma} k_{j,1}(t,x,s,y)u(s,y)\, ds d\sigma(y) + \int_{\Gamma} k_{j,2}(t,x,y)u(s,y)\, d\sigma(y), \quad j = 1,2,$$

Γ being an open subset in $\partial\Omega$ and ν_j, $j = 1,2,3$, standing for three positive measures such that

$$\nu_1 \in \{\delta_t, m_1\},\ t \in (0,T), \quad \nu_2,\nu_3 \in \{\delta_x, m_n\},\ x \in \Omega, \quad (\nu_1,\nu_2) \neq (\delta_t,\delta_x).$$

$$\nu_3 \in \{\delta_x, \sigma, m_n\},$$

Here δ, m_k and σ denote, respectively, the Dirac measure, the k-dimensional Lebesgue measure and the surface Lebesgue measure.

[1] in a sense to be made precise for each specific problem. Similarly the open set Ω will be assumed to be convex with respect to 0, if needed.

Remark Of course, Γ could be a (smooth) submanifold in $\partial\Omega$, e.g., a curve in $\partial\Omega$, when $n = 3$. $\square$

2 The first (interior) ill-posed problem

In this section we consider a particular case of problem (IP) [2]. We assume that $B_1 = B_2 = O$ - O denoting the null-operator - so that our task consists in *estimating the trace* $u(t_0, \cdot)$, $t_0 \in (0, T)$, *of the* weak *solution* $u : [0, T] \times \Omega \to \mathbb{R}$ *to the problem*

$$(IP1) \begin{cases} D_t u(t, x) - A(x, D)u(t, x), & \\ = B(u)(t, x) + f(t, x), & (t, x) \in (0, T) \times \Omega =: Q_T, \\ u(t, x) = u_0(t, x), & (t, x) \in (0, T) \times \omega, \\ u(t, x) = g(t, x), & (t, x) \in (0, T) \times \partial\Omega, \end{cases} \quad (2.1)$$

where $f \in L^2((0, T)\times\Omega)$, $u_0 \in H^1((0, T); L^2(\omega))\cap L^2((0, T); H^1(\omega))$, $g \in H^{1/4,1/2}((0, T)\times \partial\Omega)$ and $B = B_0$ is the linear operator with domain in $L^2(\Omega)$ defined by

$$Bu(t, x) = \int_\Omega k_0(t, x, y)u(t, y)\, dy, \quad (t, x) \in (0, T) \times \partial\Omega. \quad (2.2)$$

First of all we need to recall the Carleman estimates related to problem (2.1) when B is dealt with as a perturbation of the differential operator $D_t - A(x, D)$. For this purpose, taking [16] into account, we know that it is possible to construct a function

$$\psi \in C^2(\overline{\Omega}), \qquad \psi(x) > 0,\ \forall x \in \Omega, \qquad |\nabla\psi(x)| > 0,\ \forall x \in \overline{\Omega \setminus \omega}. \quad (2.3)$$

Introduce then the functions $\varphi_{\beta,\lambda} : [0, T] \times \Omega \to \mathbb{R}_+$ and $\alpha_{\beta,\lambda} : [0, T] \times \Omega \to \mathbb{R}_-$, depending on the parameters $\beta \in [2, +\infty)$ and $\lambda \in [1, +\infty)$, defined by

$$\varphi_{\beta,\lambda}(t, x) = \frac{e^{\lambda\psi(x)}}{l(t)^\beta}, \quad \alpha_{\beta,\lambda}(t, x) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_\infty}}{l(t)^\beta}, \quad (t, x) \in (0, T) \times \overline{\Omega}, \quad (2.4)$$

where $\| \cdot \|_\infty$ denotes the norm in $L^\infty(\Omega)$ and

$$l(t) = t(T - t).$$

Making use of the Carleman estimate in [16], related to the differential case $B = O$, any *weak* solution $u \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of (2.1) satisfies the

[2]For the missing computations and proofs the reader is referred to [20].

following estimate: *there exist $\lambda_0 > 0$ and a positive constant $C \geq 1$ such that for any $\lambda > \lambda_0$ there exists $\widehat{s}_0 = \widehat{s}_0(\lambda) \geq 1$ such that*

$$\begin{aligned}
& s\int_{Q_T} \varphi_{\beta,\lambda}(t,x)|u(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
& +s^{-1}\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-1}|\nabla u(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
\leq\; & Cs^{-2}\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-2}|B(u)(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
& +Cs\int_{(0,T)\times\omega} \varphi_{\beta,\lambda}(t,x)|u_0(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
& +Cs^{-2}\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-2}|f(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
& +Cs^{-1/2}\|\varphi_{\beta,\lambda}^{-1/4} g \exp(\alpha_{\beta,\lambda})\|_{H^{1/4,1/2}((0,T)\times\partial\Omega)} \\
& +Cs^{-1/2}\|\varphi_{\beta,\lambda}^{-1/4+1/\beta} g \exp(\alpha_{\beta,\lambda})\|_{L^2((0,T)\times\partial\Omega)}, \quad \forall s \geq \widehat{s}_0. \qquad (2.5)
\end{aligned}$$

Our first task consists in determining sufficient conditions on the operator B ensuring the estimate

$$\begin{aligned}
& \int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-2}|B(u)(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
\leq\; & C_0\int_{Q_T} \varphi_{\beta,\lambda}(t,x)|u(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx \\
& + C_1\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-1}|\nabla u(t,x)|^2 \exp\left[2s\alpha_{\beta,\lambda}(t,x)\right] dtdx, \qquad (2.6)
\end{aligned}$$

so that the term containing $B(u)$ can be absorbed by the first two integrals in the left-hand side in (2.5).
Since in our case u is known on the sub-cylinder $(0,T)\times\omega$, we easily deduce the following equality, where $\widehat{\Omega} = \Omega \setminus \omega$:

$$\begin{aligned}
Bu(t,x) &= \int_{\widehat{\Omega}} k_0(t,x,y)u(t,y)\,dy + \int_{\omega} k_0(t,x,y)u_0(t,y)\,dy \\
&= B_{\widehat{\Omega}}u(t,x) + B_{\omega}u(t,x), \quad (t,x) \in (0,T)\times\Omega. \qquad (2.7)
\end{aligned}$$

Remark We can now explain more clearly why our problem (IP) is severely ill-posed, if we assume, for the sake of simplicity, that $u_0 \in H^1((0,T);L^2(\omega)) \cap$

$L^2((0,T);H^2(\omega))$. Indeed, problem (IP1) can be rewritten in the following form:

$$(IP1)\quad \begin{cases} D_t u(t,x) - A(x,D)u(t,x) & \\ = B_{\widehat{\Omega}} u(t,x) + \overline{f}(t,x), & (t,x) \in (0,T)\times\widehat{\Omega}, \\ u(t,x) = u_0(t,x),\ D_n u(t,x) = D_n u_0(t,x), & (t,x) \in (0,T)\times\partial\omega, \\ u(t,x) = g(t,x), & (t,x) \in (0,T)\times\partial\Omega, \end{cases}$$

where $\overline{f}(t,x) = f(t,x) + B_\omega u_0(t,x)$. In this problem the initial condition is missing and is replaced by the Cauchy condition on the inner surface $(0,T)\times\partial\omega$ of the open cylinder $(0,T)\times\widehat{\Omega}$. It is well-known that prescribing the Cauchy condition on the surface $(0,T)\times\partial\omega$ makes problem (IP1) *severely ill-posed*. Concerning this question see, e.g., the book [4] and the papers [3, 7, 8, 9, 15, 18]. □

Owing to formula (2.7), we can rewrite inequality (2.5) in the form

$$\begin{aligned} & s\int_{Q_T} \varphi_{\beta,\lambda}(t,x)|u(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\ & + s^{-1}\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-1}|\nabla u(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\ \le\ & Cs^{-2}\int_{\widehat{Q}_T} \varphi_{\beta,\lambda}(t,x)^{-2}|B_{\widehat{\Omega}} u(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\ & + Cs^{-2}\int_{(0,T)\times\omega} \varphi_{\beta,\lambda}(t,x)^{-2}|B_\omega u_0(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\ & + \text{terms involving the data}, \quad \forall s \ge \widehat{s}_0. \end{aligned}$$

Finally, the kernel k_0 has to be determined so as to satisfy condition (2.6).
First we assume that, for some $\gamma \in \mathbb{R}_+$, the kernel k_0 satisfies

$$K_0 := \sup_{(t,x)\in\widehat{Q}_T} l(t)^\gamma \int_{\widehat{\Omega}} |k_0(t,x,y)|\,dy < +\infty. \tag{2.8}$$

From (2.8) and the simple inequality

$$\int_{\widehat{\Omega}} |k_0(t,x,y)v(t,y)|\,dy \le K_0^{1/2} l(t)^{-\gamma}\Big[\int_{\widehat{\Omega}} |k_0(t,x,y)||v(t,y)|^2\,dy\Big]^{1/2},$$

we easily deduce the estimates

$$\begin{aligned} & \int_{\widehat{Q}_T} \varphi_{\beta,\lambda}(t,x)^{-2} \exp[2s\alpha_{\beta,\lambda}(t,x)]\Big|\int_{\widehat{\Omega}} |k_0(t,x,y)v(t,y)|\,dy\Big|^2 dtdx \\ \le\ & K_0 \int_{\widehat{Q}_T} \varphi_{\beta,\lambda}(t,y)^{-1}|v(t,y)|^2\,dtdy \int_{\widehat{\Omega}} h_{0,\beta,s,\lambda}(t,x,\sigma,y)|k_0(t,x,\sigma,y)|\,dx, \end{aligned} \tag{2.9}$$

where the kernel $h_{0,\beta,s,\lambda}$ is defined by

$$h_{0,\beta,s,\lambda}(t,x,y) = l(t)^{-\gamma}\varphi_{\beta,\lambda}(t,x)^{-2}\varphi_{\beta,\lambda}(t,y)^{-1}\exp\{2s[\alpha_{\beta,\lambda}(t,x) - \alpha_{\beta,\lambda}(t,y)]\}.$$

Assume now that kernel k_0 satisfies the additional condition

$$\sup_{(s,t,y)\in[1,+\infty)\times\widehat{Q}_T}\int_{\widehat{\Omega}} h_{0,\beta,s,\lambda}(t,x,y)|k_0(t,x,y)|\,dy =: K_1 < +\infty. \tag{2.10}$$

Under conditions (2.10), from (2.8), (2.9) we easily deduce the desired estimate

$$\int_{\widehat{Q}_T}\varphi_{\beta,\lambda}(t,x)^{-2}|Bu(t,x)|^2\exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx$$

$$\leq K_0K_1\int_{\widehat{Q}_T}\varphi_{\beta,\lambda}(t,x)|u(t,x)|^2\exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx. \tag{2.11}$$

We try now to simplify conditions (2.10). For this task first we note that

$$\alpha_{\beta,\lambda}(t,x) - \alpha_{\beta,\lambda}(t,y) = l(t)^{-\beta}\{\exp[\lambda\psi(x)] - \exp[\lambda\psi(y)]\} \leq 0 \Longleftrightarrow \psi(x) \leq \psi(y).$$

Consequently,

$$0 < t < T,\ x \in \Omega,\ y \in \widehat{\Omega},\ \psi(x) \leq \psi(y) \quad \Longrightarrow$$
$$h_{0,\beta,s,\lambda}(t,x,y) \leq l(t)^{3\beta-\gamma}\exp\{-\lambda[2\psi(x)+\psi(y)]\}.$$

Whence we deduce

$$\int_{\{x\in\Omega:\,\psi(x)\leq\psi(y)\}} h_{0,\beta,s,\lambda}(t,x,y)|k_0(t,x,y)|\,dx \leq l(t)^{3\beta-\gamma}\int_{\widehat{\Omega}}|k_0(t,x,y)|\,dx.$$

Let now assume

$$k_0 = 0 \text{ on } E_{0,T} = \{(t,x,y) \in (0,T)\times\Omega\times\widehat{\Omega} : \psi(y) > \psi(x)\}, \tag{2.12}$$

This implies the following representation for B:

$$Bu(t,x) = \int_{\widehat{\Omega}_{\psi}(x)} k_0(t,x,y)u(t,y)\,dy, \tag{2.13}$$

where

$$\widehat{\Omega}_{\psi}(x) = \{y \in \widehat{\Omega} : \psi(y) \leq \psi(x)\}, \quad x \in \Omega.$$

Therefore, we can conclude that, under condition (2.8), inequality (2.10) is fulfilled, if kernel k_0 satisfies the following inequality for some pair $(\beta,\lambda) \in [2,+\infty)\times[1,+\infty)$ and a positive constant K_1:

$$\int_{\Omega}|k_0(t,x,y)|\,dx \leq K_1 l(t)^{\gamma-3\beta}, \quad (t,y) \in \widehat{Q}_T. \tag{2.14}$$

Choose now s to be a solution to the inequalities

$$Cs^{-2}K_0K_1 \le \frac{1}{2}s \Longleftrightarrow s \ge \max\{2CK_0K_1, \widehat{s}_0\} =: s_0.$$

Therefore, owing to (2.7), the term containing Bu can be absorbed from the first integral in the left-hand side in (2.5). Then from (2.5) and (2.11) we deduce

$$\begin{aligned}
&\frac{1}{2}s\int_{Q_T}\varphi_{\beta,\lambda}(t,x)|u(t,x)|^2\exp\left[2s\alpha_{\beta,\lambda}(t,x)\right]dtdx\\
&+\frac{1}{2}s^{-1}\int_{Q_T}\varphi_{\beta,\lambda}(t,x)^{-1}|\nabla u(t,x)|^2\exp\left[2s\alpha_{\beta,\lambda}(t,x)\right]dtdx\\
\le\ & Cs^{-2}\int_{(0,T)\times\omega}\varphi_{\beta,\lambda}(t,x)^{-2}|B_\omega u_0(t,x)|^2\exp\left[2s\alpha_{\beta,\lambda}(t,x)\right]dtdx\\
&+s\int_{(0,T)\times\omega}\varphi_{\beta,\lambda}(t,x)|u_0(t,x)|^2\exp\left[2s\alpha_{\beta,\lambda}(t,x)\right]dtdx\\
&+Cs^{-2}\int_{Q_T}\varphi_{\beta,\lambda}(t,x)^{-2}|f(t,x)|^2\exp\left[2s\alpha_{\beta,\lambda}(t,x)\right]dtdx\\
&+Cs^{-1/2}\|\varphi_{\beta,\lambda}^{-1/4}g\exp(\alpha_{\beta,\lambda})\|_{H^{1/4,1/2}((0,T)\times\partial\Omega)}\\
&+Cs^{-1/2}\|\varphi_{\beta,\lambda}^{-1/4+1/\beta}g\exp(\alpha_{\beta,\lambda})\|_{L^2((0,T)\times\partial\Omega)} =: J_1(s,u_0,f,g), \quad s\ge s_0. \quad (2.15)
\end{aligned}$$

We collect the result of this subsection in the following theorem

Theorem 2.2 *Let the kernels k_0 satisfy conditions (2.8), (2.12), (2.14). Then the weak solution u to problem (IP) satisfy the Carleman estimate (2.15).*

2.1 A continuous dependence result

Using the techniques developed in [6] we can estimate the trace $u(t_0,\cdot)$ in $L^2(\Omega)$ for all $t_0 \in (0,T]$. More precisely, we can estimate u in $C((0,T];L^2(\Omega))$. For this purpose, we have to introduce in some way the missing initial condition at $t=0$. This can can be done by the aid of the auxiliary function

$$v_\varepsilon = \sigma_\varepsilon(u-g),$$

where g denotes now a fixed extension of the previous g to $H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H^1(\Omega))$. Moreover, $\{\sigma_\varepsilon\}_{\varepsilon\in(0,1/4)}$ is a family of functions in $W^{1,\infty}((0,T);[0,1])$ defined by

$$\sigma_\varepsilon(t) = 0,\ t\in[0,\varepsilon T], \quad \sigma_\varepsilon(t)=1,\ t\in[2\varepsilon T,T].$$

First we need the following lower and upper bounds for functions $\varphi_{\beta,\lambda}$ and $\alpha_{\beta,\lambda}$:

$$l(t)^{-\beta} \le \varphi_{\beta,\lambda}(t,x) \le e^{\lambda\|\psi\|_\infty} l(t)^{-\beta},$$

$$-[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m}]l(t)^{-\beta} \le \alpha_{\beta,\lambda}(t,x) \le -[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\|\psi\|_\infty}]l(t)^{-\beta},$$

where $(t,x) \in Q_T$ and $\psi_m = \min_{x\in\overline{\Omega}} \psi(x)$.
Observe now that, for $s \in [s_0, 2s_0]$ and $(t,x) \in Q_T$, we have

$$\begin{aligned}\rho_{1,\lambda}(t) &:= \exp\big\{-4s_0[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi_m}]l(t)^{-\beta}\big\} \le \exp\big[2s\alpha_{\beta,\lambda}(t,x)\big] \\ &\le \exp\big\{-2s_0\big[e^{2\lambda\|\psi\|_\infty} - e^{\lambda\|\psi\|_\infty}\big]l(t)^{-\beta}\big\} =: \rho_{2,\lambda}(t). \qquad (2.16)\end{aligned}$$

Consequently, from (2.15)-(2.16), with $s \in [s_0, 2s_0]$, we easily deduce the estimate

$$\begin{aligned}&\max\Big\{\frac{1}{2}s_0, \frac{e^{-\lambda\|\psi\|_\infty}}{4s_0}\Big\}\int_0^T \big[l(t)^{-\beta}\|u(t,\cdot)\|^2_{L^2(\Omega)} + l(t)^{\beta}\|\nabla u(t,\cdot)\|^2_{L^2(\Omega)}\big]\rho_{1,\lambda}(t)\,dt \\ &\le C\int_0^T l(t)^{-\beta}\rho_{2,\lambda}(t)\|u_0(t,\cdot)\|^2_{L^2(\omega)}\,dt \\ &\quad + C\int_0^T \rho_{2,\lambda}(t)\Big[s_0^{-2}l(t)^{-2\beta}\|f_0(t,\cdot)\|^2_{L^2(\Omega)} + \sum_{j=1}^n \|f(t,\cdot)\|^2_{L^2(\Omega)}\Big]\,dt \\ &\quad + C\big[\|l^{-1/4}\rho_{2,\lambda}g\|_{H^{1/4,1/2}((0,T)\times\partial\Omega)} + \|l^{-1/4+1/\beta}\rho_{2,\lambda}g\|_{L^2((0,T)\times\partial\Omega)}\big]. \qquad (2.17)\end{aligned}$$

Since σ_ε commutes with B, i.e. $\sigma_\varepsilon(t)Bu(t,x) = B(\sigma_\varepsilon u)(t,x)$, it is a simple task to show that v_ε solves the following initial and boundary-value problem:

$$(DP)\quad\begin{cases} D_t v_\varepsilon(t,x) - A(x,D)v_\varepsilon(t,x) &= Bv_\varepsilon(t,x) + \sigma'_\varepsilon(t)u(t,x) \\ +\widetilde{g}_\varepsilon(t,x) + f_\varepsilon(t,x), & (t,x)\in(0,T)\times\Omega, \\ v_\varepsilon(0,x) = 0, & x\in\Omega, \\ v_\varepsilon(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega,\end{cases}$$

where

$$f_{0,\varepsilon} = \sigma_\varepsilon f_0, \quad g_\varepsilon = \sigma_\varepsilon g, \quad \widetilde{g}_\varepsilon = -D_t g_\varepsilon + A(\cdot,D)g_\varepsilon + Bg_\varepsilon.$$

Recall now that $-A(\cdot,D)$ satisfies the following estimate for all $v \in H^1_0(\Omega)$:

$$-\langle A(\cdot,D)v, v\rangle \ge \mu_3\|\nabla v\|^2_{L^2(\Omega)} - \mu_4\|v\|^2_{L^2(\Omega)},$$

for some positive constants μ_3 and μ_4.

Using standard energy estimates and denoting by $(\cdot,\cdot)$ the usual inner product in $L^2(\Omega)$, we get [3]

$$D_t\|v_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)} + \mu_3\|\nabla v_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)} - \mu_4\|v_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)}$$

$$\leq (Bv_\varepsilon(t,\cdot), v_\varepsilon(t,\cdot)) + \|v_\varepsilon(t,\cdot)\|_{L^2(\Omega)}\Big\{\|\sigma'_\varepsilon(t)u(t,\cdot)\|_{L^2(\Omega)}$$

$$+ \|\widetilde{g}_\varepsilon(t,\cdot)\|_{L^2(\Omega)} + \|f_\varepsilon(t,\cdot)\|_{L^2(\Omega)}\Big\}, \tag{2.18}$$

Assume now that the kernels k_0 satisfies also the inequality

$$H_0 := \sup_{(t,y)\in\widehat{Q}_T} l(t)^\delta \int_{\widehat{\Omega}} |k_0(t,x,y)|\, dx < +\infty, \tag{2.19}$$

where $\gamma + \delta < 2$, Then it is well-known that the norm of $B_{\widehat{\Omega}}v_\varepsilon(t\cdot)$ in $L^2(\Omega)$ can be estimated by $(H_0K_0)^{1/2}l(t)^{-\kappa}$ (cf., e.g., [19, Chapter 16]), $\kappa = (\gamma+\delta)/2$. Therefore, integrating both sides of estimate (2.18) over the interval $(0,\tau)$, $\tau \in (0,T)$, we easily deduce the integral inequality:

$$\|v_\varepsilon(\tau,\cdot)\|^2_{L^2(\Omega)} + \mu_3\int_0^\tau \|\nabla v_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)}\, dt$$

$$\leq J_2(\sigma'_\varepsilon)\int_0^\tau l(t)^{-\kappa}\|v(t,\cdot)\|^2_{L^2(\Omega)}\, dt + \int_0^\tau \kappa_\varepsilon(t)\|v_\varepsilon(t,\cdot)\|_{L^2(\Omega)}\, dt$$

$$+ \frac{1}{2}\|\sigma'_\varepsilon\|_{L^\infty(0,T)}\int_{\varepsilon T}^{2\varepsilon T} \|u(t,\cdot)\|^2_{L^2(\Omega)}\, dt. \tag{2.20}$$

Here we have set

$$J_2(\sigma'_\varepsilon) = \Big[\mu_4 + \frac{1}{2}\|\sigma'_\varepsilon\|_{L^\infty(0,T)}\Big](T^2/4)^\kappa + (H_0K_0)^{1/2},$$

$$\kappa_\varepsilon(t) = \|\widetilde{g}_\varepsilon(t,\cdot)\|_{L^2(\Omega)} + \|f_\varepsilon(t,\cdot)\|_{L^2(\Omega)},$$

and have used the inclusion $\operatorname{supp}\sigma'_\varepsilon \subset [\varepsilon T, 2\varepsilon T]$.
Then, taking advantage of (2.15) and of the inequality

$$l(t)^{-\beta}\rho_{1,\lambda}(t) \geq C(\varepsilon,T) > 0, \quad t \in [\varepsilon T, 2\varepsilon T],$$

we can estimate u in terms of the data

$$\int_{\varepsilon T}^{2\varepsilon T} \|u(t,\cdot)\|^2_{L^2(\Omega)}\, dt \leq C(\varepsilon,T)^{-1}\int_{\varepsilon T}^{2\varepsilon T} l(t)^{-\beta}\rho_{1,\lambda}(t)\|u(t,\cdot)\|^2_{L^2(\Omega)}\, dt$$

$$\leq C(\varepsilon,T)^{-1}J_1(u_0,f,g). \tag{2.21}$$

[3] For the missing computations and proofs the reader is referred to [20].

Finally, from (2.20) and (2.21) we deduce the fundamental integrodifferential inequality

$$\|v_\varepsilon(\tau,\cdot)\|^2_{L^2(\Omega)} \le J_2(\sigma'_\varepsilon)\int_0^\tau l(t)^{-\kappa}\|v_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)}\,dt + \int_0^\tau \kappa_\varepsilon(t)\|v_\varepsilon(t,\cdot)\|_{L^2(\Omega)}\,dt + J_3(\sigma'_\varepsilon,u_0,f,g), \quad \tau\in(0,T), \tag{2.22}$$

where

$$J_3(\sigma'_\varepsilon,u_0,f,g) = 2C(\varepsilon,T)^{-1}\|\sigma'_\varepsilon\|_{L^\infty(0,T)}J_1(u_0,f,g). \tag{2.23}$$

Then we need a simple variant of Theorem 4.9 in [2], with $p = 1/2$, which we report here as a lemma.

Lemma 2.2 *Let z be a nonnegative $C([0,T])$–function and let b, k be nonnegative $L^1((0,T))$–functions satisfying*

$$z(t) \le a + \int_0^t b(s)z(s)\,ds + \int_0^t k(s)z(s)^p\,ds, \quad t\in[0,T],$$

where $p\in(0,1)$ and $a\ge 0$ are given constants. Then for all $t\in[0,T]$

$$z(t) \le \exp\left(\int_0^t b(s)\,ds\right) \times\left[a^{1-p} + (1-p)\int_0^t k(s)\exp\left((p-1)\int_0^s b(\sigma)\,d\sigma\right)\,ds\right]^{1/(1-p)}.$$

From this lemma and the integral inequality (2.22) we immediately deduce the estimate

$$\|v_\varepsilon(\tau,\cdot)\|_{L^2(\Omega)} \le \exp\Big[\frac{1}{2}J_2(\sigma'_\varepsilon)\int_0^\tau l(r)^{-\kappa}\Big]\Big\{J_3(\sigma'_\varepsilon,u_0,f,g)^{1/2} + \frac{1}{2}\int_0^\tau \exp\Big[-\frac{1}{2}J_2(\sigma'_\varepsilon)\int_0^s l(r)^{-\kappa}\,dr\Big]\kappa_\varepsilon(s)\,ds\Big\}^{1/2}, \quad \tau\in[0,T]. \tag{2.24}$$

In particular, for all $\tau\in[2\varepsilon T,T]$ from (2.17) we find the desired estimate for u:

$$\|u(\tau,\cdot)\|_{L^2(\Omega)} \le \|g(\tau,\cdot)\|_{L^2(\Omega)} + \exp\Big[\frac{1}{2}J_2(\sigma'_\varepsilon)\int_0^\tau l(r)^{-\kappa}\,dr\Big]\Big\{J_3(\sigma'_\varepsilon,u_0,f,g)^{1/2} + \frac{1}{2}\int_0^\tau \exp\Big[-\frac{1}{2}J_2(\sigma'_\varepsilon)\int_0^s l(r)^{-\kappa}\,dr\Big]\kappa_\varepsilon(s)\,ds\Big\}^{1/2}. \tag{2.25}$$

Remark If $u_0 = 0$, $g = 0$ and $f = 0$, then $J_3(\sigma'_\varepsilon,u_0,f,,g) = 0$ and $\kappa_\varepsilon = 0$ so that $u = 0$ in $[2\varepsilon T,T]\times\Omega$ for all $\varepsilon\in(0,1/2)$. This implies $u = 0$ in $(0,T]\times\Omega$. In particular, since $u \in H^1((0,T);L^2(\Omega))$ and $H^1((0,T);L^2(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$, we can conclude that $u = 0$ in Q_T, i.e. that a *unique continuation property* holds true for the solution to problem (2.1. □

Remark By virtue of (2.20), (2.21), (2.24) we can also estimate the spatial gradient ∇u in $L^2((2\varepsilon T,T)\times\Omega)$, $\varepsilon \in (0,1/4)$, in terms of the data. □

Remark Assume that Ω is the ball $B(0,r_2)$ containing the smaller ball $B(0,r_1) = \omega$, $0 < r_1 < r_2$. Define $\psi(x) = r_2^2 - |x|^2$. Then function ψ satisfies all properties in (2.5). Observe that condition

$$\psi(x) > \psi(x), \quad x,y \neq B(0,r_1) \quad \Longleftrightarrow \quad |x| < |y|, \quad x,y \in B(0,r_1).$$

Therefore the condition to be imposed on the kernel k_0 is

$$k_0 = 0 \text{ on } E_{0,T} = \{(t,x,y) \in (0,T)\times\Omega\times\widehat{\Omega} : |x| < |y|\}.$$

□

We conclude this subsection by stating the results so far proved.

Theorem 2.6 *Let the kernel k_0 satisfy conditions (2.8), (2.12), (2.14) and (2.19), with $\gamma,\delta \in [0,2)$ and $\gamma+\delta < 2$. Then the weak solution u to problem (IP1) satisfy the continuous dependence estimate (2.25).*

3 The second ill-posed problem

We consider here the ill-posed problem (IP) [4], where Ω is convex with respect to $x = 0$ and the linear operator K is defined by the formula

$$Ku(t,x) = k(t,x)u(t,\rho x).$$

for some fixed $\rho \in (0,1)$ and a given function $k \in L^\infty(Q_T)$. Moreover, we assume $(1/\rho)\omega \subset\subset \Omega$.
Since $u = u_0$ in $(0,T)\times\omega$, we immediately deduce that $u(t,\rho x) = u_0(t,\rho x)$ if, and only if, $x \in (1/\rho)\omega$. Whence we derive

$$\begin{aligned} Ku(t,x) &= k(t,x)u_0(t,\rho x)\chi_{(1/\rho)\omega}(x) + k(t,x)u(t,\rho x)\chi_{\Omega\setminus(1/\rho)\omega}(x) \\ &= K_0u(t,x) + K_1u(t,x). \end{aligned}$$

[4] For the missing computations and proofs the reader is referred to [20].

Assume that k satisfies, for all $(t,x) \in (0,T)\times(\rho\Omega)$ and $s \in [1,+\infty)$, the inequality

$$\frac{\varphi_{\beta,\lambda}(t,\rho^{-1}x)^{-1}}{\varphi_{\beta,\lambda}(t,x)^{1/2}} \exp\{s[\alpha_\lambda(t,\rho^{-1}x) - \alpha_\lambda(t,x)]\}|k(t,\rho^{-1}x)| \le C_0, \tag{3.1}$$

for a suitable positive constant C_0 to be determined later on.
By a simple change of variables we easily deduce the following estimates, where we have set $\widehat{Q}_{T,\rho} = (0,T)\times[\Omega\setminus(1/\rho)\omega]$:

$$\begin{aligned}
&\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-2}|Ku(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\
&\le \int_{(0,T)\times(1/\rho)\omega} \varphi_{\beta,\lambda}(t,x)^{-2}|K_0u_0(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\
&\quad + \int_{\widehat{Q}_{T,\rho}} \varphi_{\beta,\lambda}(t,x)^{-2}|K_1u(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx.
\end{aligned} \tag{3.2}$$

By standard computations we get

$$\begin{aligned}
&\int_{\widehat{Q}_{T,\rho}} \varphi_{\beta,\lambda}(t,x)^{-2}|K_1u(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\,dtdx \\
&\le C_0\rho^{-n}\int_{Q_T} \varphi_{\beta,\lambda}(t,x)\exp[2s\alpha_\lambda(t,x)]|u(t,x)|^2\,dtdx \\
&\quad + C_1\rho^{-n}\int_{Q_T} \varphi_{\beta,\lambda}(t,x)^{-1}\exp[2s\alpha_\lambda(t,x)]|\nabla u(t,x)|^2\,dtdx.
\end{aligned}$$

Therefore the term containing K_1u in (3.2) can be absorbed by the the first two integrals in the left-hand side in (2.5), with $B = K_0 + K_1$, if we choose

$$s \ge \max\{(CC_0\rho^{-n})^{1/3}, CC_1\rho^{-n}, \widehat{s}_0\} =: s_0.$$

To simplify condition (3.1) we note that

$$\alpha_{\beta,\lambda}(t,\rho^{-1}x) - \alpha_{\beta,\lambda}(t,x) = \{\exp[\lambda\psi(\rho^{-1}x)] - \exp[\lambda\psi(x)]\}l(t)^{-\beta}$$

$$\begin{cases} \le 0, & \text{if } \psi(x) \ge \psi(\rho^{-1}x), \\ > 0, & \text{if } \psi(\rho^{-1}x) > \psi(x), \end{cases} \qquad (t,x) \in (0,T)\times(\rho\Omega\setminus\omega).$$

Since $\exp\{2s[\alpha_{\beta,\lambda}(t,\rho^{-1}x) - \alpha_{\beta,\lambda}(t,x)]\} \to +\infty$ as $s \to +\infty$ if $t \in (0,T)$ and $\psi(\rho^{-1}x) > \psi(x)$, we are forced to assume that k vanishes on $(0,T)\times\Omega(\psi,\rho)$, where

$$\Omega(\psi,\rho) = \{y \in \Omega\setminus(1/\rho)\omega : \psi(y) > \psi(\rho y)\}. \tag{3.3}$$

Further, observe that

$$\frac{\varphi_{\beta,\lambda}(t,\rho^{-1}x)^{-1}}{\varphi_{\beta,\lambda}(t,x)^{1/2}} = l(t)^{3\beta/2}\exp\{-\lambda[\psi(\rho^{-1}x) + \psi(x)/2]\} \le l(t)^{3\beta/2}.$$

So, from (3.2) we get the following inequalities for all $(t,x) \in (0,T) \times (\rho\Omega \setminus \omega)$ and $s \in [1,+\infty)$:

$$\frac{\varphi_{\beta,\lambda}(t,\rho^{-1}x)^{-1}}{\varphi_{\beta,\lambda}(t,x)^{1/2}} \exp\{s[\alpha_\lambda(t,\rho^{-1}x)] - \alpha_\lambda(t,x)\}|k(t,\rho^{-1}x)|$$
$$\leq l(t)^{3\beta/2} \exp\{\lambda[-\psi(\rho^{-1}x) + \psi(x)/2]\}|k_l(t,\rho^{-1}x)|\chi_{\rho\Omega(\psi,\rho)^c}(\rho^{-1}x).$$

Consequently, from (3.3) we conclude that the function k satisfies (3.1), if it does for all $(t,x) \in (0,T) \times (\rho\Omega \setminus \omega)$ and $s \in [1,+\infty)$:

$$|k(t,x)| \leq C_0 l(t)^{-3\beta/2}\chi(\Omega(\psi,\rho)^c), \tag{3.4}$$

for some positive constant C_0.
Summing up, we have proved the following result.

Theorem 3.1 *Let $k \in L^\infty(Q_T)$ satisfy relation (3.4). Then Carleman estimate (2.5) holds true for $s \geq \widehat{s}_0$, when the third integral is replaced by*

$$\int_{\widehat{Q}_{T,\rho}} \varphi_{\beta,\lambda}(t,x)^{-2}|K_0u_0(t,x)|^2 \exp[2s\alpha_{\beta,\lambda}(t,x)]\, dtdx.$$

Then, proceeding as in Subsection 2.1, we can prove the following continuous dependence result.

Theorem 3.2 *Let $k \in L^\infty(Q_T)$ satisfy assumption (3.4). Then problem (IP) admits at most one solution u continuously depending on the data. More explicitly, for all $\tau \in [2\varepsilon T, T]$, $\varepsilon \in (0,1)$, the following estimate holds:*

$$\|u(\tau,\cdot)\|_{L^2(\Omega)} \leq \|g(\tau,\cdot)\|_{L^2(\Omega)} + J_3(\sigma'_\varepsilon, u_0, f, g)^{1/2} \exp[J_2(\sigma'_\varepsilon)\tau/2]$$
$$+ \int_0^\tau \exp[J_2(\sigma'_\varepsilon)(\tau - t)/2]\big\{\|\widetilde{g}(t,\cdot)\|_{L^2(\Omega)} + \|f(t,\cdot)\|_{L^2(\Omega)}\big\}\, dt,$$

where

$$\widetilde{g} = -D_t g + \sigma'_\varepsilon g + A(\cdot,D)g + Bg, \quad J_2(\sigma'_\varepsilon) = \mu_1 + C(k) + \frac{1}{2}\|\sigma'_\varepsilon\|_{L^\infty(0,T)},$$
$$J_3(\sigma'_\varepsilon, u_0, f, g) = 2C(\varepsilon,T)^{-1}\|\sigma'_\varepsilon\|_{L^\infty(0,T)} J_1(u_0, f, g),$$

4 The first ill-posed problem with Cauchy conditions on the lateral boundary

As in the previous sections for the problem we are dealing here *no* initial condition will be supplied. It will be replaced by the requirement that the "temperature" u should assume prescribed values on an open subsurface of the lateral boundary $(0,T)\times\Gamma$ of $(0,T)\times\partial\Omega$, while the flux of u should either assume prescribed values on $(0,T)\times\partial\Omega$ or should satisfy there a linear integrodifferential equation. In other terms, u is required to solve a Cauchy problem on $(0,T)\times\Gamma$ as well as to satisfy additional conditions on the remaining part of the lateral boundary.
We consider the ill-posed problem consisting in *estimating the trace* $u(t_0,\cdot)$, $t_0\in(0,T)$, *of the solution* $u:[0,T]\times\Omega\to\mathbb{R}$ *to the linear integrodifferential parabolic problem*

$$(IP2)\begin{cases} u\in H^1((0,T);L^2(\Omega))\cap L^2((0,T);H^2(\Omega)), \\ D_tu(t,x)-A(x,D)u(t,x) \\ = Bu(t,x)+f(t,x), & (t,x)\in(0,T)\times\Omega, \\ u(t,x)=g(t,x), & (t,x)\in(0,T)\times\partial\Omega, \\ D_{\nu_A}u(t,x)=D_{\nu_A}g(t,x), & (t,x)\in(0,T)\times\Gamma, \end{cases} \tag{4.1}$$

where

$$Bu(t,x)=\int_S k(t,x,y)u(t,y)\,d\sigma(y). \tag{4.2}$$

Here D_{ν_A} denotes the conormal (outer) derivative related to the differential operator $A(\cdot,D)$ and Γ is an open subset in $\partial\Omega$, while S is (open or closed) surface in $\mathbb{R}^n$, $S\subset\partial\omega\setminus\Gamma$, where $\omega\subset\Omega$ with $\partial\omega\in C^1$. Moreover, σ denotes the Lebesgue surface measure and $g\in H^1((0,T);L^2(\Omega))\cap L^2((0,T);H^2(\Omega))$, while the kernel $k:(0,T)\times\Omega\times S\to\mathbb{R}$ is measurable and, for the time being, separably integrable with respect to $x\in\Omega$ and $y\in S$.
Introduce the function

$$v=u-g, \tag{4.3}$$

where u is the solution to problem (4.1). It is a simply task to show that v solves the following boundary-value problem:

$$(IP3)\begin{cases} v\in H^1((0,T);L^2(\Omega))\cap L^2((0,T);H^2(\Omega)), \\ D_tv(t,x)-A(x,D)v(t,x)=Bv(t,x)+\widetilde{f}(t,x), & (t,x)\in(0,T)\times\Omega, \\ v(t,x)=0, & (t,x)\in(0,T)\times\partial\Omega, \\ D_{\nu_A}v(t,x)=0, & (t,x)\in(0,T)\times\Gamma, \end{cases} \tag{4.4}$$

where

$$\widetilde{f}=f-D_tg+A(\cdot,D)g+Bg. \tag{4.5}$$

We need here the functions φ_λ and α_λ defined by (2.4) with $\beta = 1$, where now function ψ belongs to $C^4(\overline{\Omega})$ and satisfies (cf. [14]) the properties

$$\psi(x) > 0,\ x \in \Omega, \quad |\nabla\psi(x)| \geq \mu_2 > 0,\ x \in \overline{\Omega}, \quad D_{\nu_A}\psi(x) \leq 0,\ x \in \partial\Omega \setminus \Gamma, \tag{4.6}$$

for some positive constant μ_2.
Since $\varphi_\lambda(x) \geq 1$ for all $x \in \overline{\Omega}$, by virtue of Lemma 4.4 in [16], with $p = 0$, we obtain that any solution $v \in H^1((0,T); L^2(\Omega)) \cap L^2((0,T); H^2(\Omega) \cap H^1(\Omega))$ to problem (4.4) satisfies the Carleman estimate

$$\begin{aligned} & s^3 \int_{Q_T} l(t)^{-3}|v(t,x)|^2 \exp[2s\alpha_\lambda(t,x)]\, dtdx \\ & + s \int_{Q_T} l(t)^{-1}|\nabla v(t,x)|^2 \exp[2s\alpha_\lambda(t,x)]\, dtdx \\ & + s^{-1} \int_{Q_T} l(t)\varphi_\lambda(x)^{-1}\Big[|D_t v(t,x)|^2 + \sum_{i,j=1}^n |D_{x_i}D_{x_j} v(t,x)|^2\Big] \exp[2s\alpha_\lambda(t,x)]\, dtdx \\ \leq\ & 2C \int_{Q_T} |Bv(t,x)|^2 \exp[2s\alpha_\lambda(t,x)]\, dtdx \\ & + 2C \int_{Q_T} |\widetilde{f}(t,x)|^2 \exp[2s\alpha_\lambda(t,x)]\, dtdx, \quad s \geq \widehat{s}_0, \end{aligned} \tag{4.7}$$

where the positive constants C, λ and $\widehat{s}_0$ depend on μ_1, T, $\|a_0\|_{L^\infty(\Omega)}$, $\|a_{i,j}\|_{L^\infty(\Omega)}$, $\|a_j\|_{L^\infty(\Omega)}$, $i,j = 1,\ldots,n$, Ω and Γ.
We now easily deduce the estimate [5],

$$\begin{aligned} & \int_{Q_T} \exp[2s\alpha_\lambda(t,x)]\Big|\int_S |k(t,x,y)v(t,y)|\, d\sigma(y)\Big|^2 dtdx \\ & \leq K_0 \int_{(0,T)\times S} l(t)^{-1}|v(t,y)|^2\, dtd\sigma(y) \int_\Omega l(t)\exp[2s\alpha_\lambda(t,x)]|k(t,x,y)|\, dx, \end{aligned} \tag{4.8}$$

K_0 and $h_{0,s,\lambda}$ being defined, respectively, by

$$K_0 = \operatorname{ess\,sup}_{(t,x)\in Q_T} \int_S |k(t,x,y)|\, d\sigma(y) < +\infty, \tag{4.9}$$

$$h_{0,s,\lambda}(t,x,y) = l(t)\exp\{2s[\alpha_\lambda(t,x) - \alpha_\lambda(t,y)]\}. \tag{4.10}$$

Assume now that kernel k satisfies the following additional condition

$$k(t,x,y) = 0,\ t \in (0,T),\ \psi(x) > \psi(y),\ (x,y) \in \Omega \times S, \tag{4.11}$$

$$\sup_{(t,y)\in(0,T)\times S} l(t) \int_\Omega |k(t,x,y)|\, dx =: K_1, \tag{4.12}$$

[5]For the missing computations and proofs the reader is referred to [21].

for a suitable positive constant K_1. Then from (4.11) we get

$$\int_\Omega h_{0,s,\lambda}(t,x,y)|k(t,x,y)|\,dx = \int_{\{x\in\Omega:\psi(x)\le\psi(y)\}} h_{0,s,\lambda}(t,x,y)|k(t,x,y)|\,dx$$

$$\le l(t)\int_{\{x\in\Omega:\,\psi(x)\le\psi(y)\}} |k(t,x,y)|\,dx = l(t)\int_\Omega |k(t,x,y)|\,dx \le K_1. \tag{4.13}$$

Under conditions (4.9) and (4.13), from (4.8) we finally deduce the estimate

$$\int_{Q_T} |Bv(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$\le K_0K_1\int_{(0,T)\times S} l(t)^{-1}|v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtd\sigma(x). \tag{4.14}$$

Then we need the following lemma.

Lemma 4.0 *Let $\omega \subset \Omega$ be an open subset such that $\partial\omega \in C^1$. Let $w \in C^1(\overline{\omega};[0,+\infty))$ satisfy $|\nabla w(x)| \le C_0 w(x)$ for all $x \in \overline{\omega}$. Then there exist three positive constants C_1 and C_2 such that*

$$\int_{\partial\omega} w(x)|u(x)|^2\,d\sigma(x) \le (C_1+C_0)\int_\omega w(x)|u(x)|^2\,dx$$

$$+ C_2\int_\omega w(x)|\nabla u(x)|^2\,dx, \quad \forall u \in H^1(\omega). \tag{4.15}$$

In particular, if $w(t,x) = l(t)^{-1}\exp\left[2s\alpha_\lambda(t,x)\right]$, then there exists a positive constant C_3 such that

$$\int_{(0,T)\times\partial\omega} w(t,x)|u(t,x)|^2\,dtd\sigma(x)$$

$$\le \int_{(0,T)\times\omega} [C_1 + C_0C_3 s l(t)^{-1}] w(t,x)|u(t,x)|^2\,dtdx$$

$$+ C_2\int_{(0,T)\times\omega} w(t,x)|\nabla u(t,x)|^2\,dtdx, \quad u \in L^2((0,T);H^1(\omega)). \tag{4.16}$$

Consequently, from (4.14 and (4.16 we easily deduce the estimate

$$\int_{Q_T} |Bv(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$\le K_0K_1\int_{Q_T} [C_1 l(t)^2 + C_0C_3 s l(t)] l(t)^{-3}|v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$+ K_0K_1C_2\int_{Q_T} l(t)^{-1}|\nabla v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx.$$

$$\leq \frac{1}{4}T^2K_0K_1\Big[\frac{1}{4}T^2C_1 + C_0C_3s\Big]\int_{Q_T} l(t)^{-3}|v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$+ K_0K_1C_2\int_{Q_T} l(t)^{-1}|\nabla v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx. \tag{4.17}$$

We can choose now $s_0 \geq \widehat{s}_0$ so as to satisfy the inequalities

$$\frac{1}{4}T^2K_0K_1\Big[\frac{1}{4}T^2C_1 + C_0C_3s\Big] \leq \frac{1}{2}s^3, \quad C_2K_0K_1 \leq \frac{C}{2}s, \quad \forall s \in (s_0, +\infty), \tag{4.18}$$

C being the positive constant in estimate (4.7).
Then from (4.7) and (4.17) for $s \geq s_0$ we deduce the estimate

$$s^3\int_{Q_T} l(t)^{-3}|v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$+ s\int_{Q_T} l(t)^{-1}|\nabla v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$\leq C_4\int_{Q_T} |\widetilde{f}(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx, \quad s \in (s_0, +\infty). \tag{4.19}$$

So, we have proved the following theorem.

Theorem 4.1 *Let the kernel k satisfy conditions (4.9), (4.11), (4.12). Then the strong solution u to problem (IP2) satisfy the Carleman estimate (4.19), with $v = u - g$ and $s \geq s_0$. In particular, problem (IP2) admits at most one solution.*

4.1 A continuous dependence result

Since the derivation of the continuous dependence is similar to the one in Subsection 2.1, we limit ourselves here to sketching the needed procedure [6].
First we observe that the function $v_\varepsilon = \sigma_\varepsilon v$, where v is the solution to problem (4.4), solves the initial and boundary-value problem:

$$(DP2)\begin{cases} v_\varepsilon \in H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H^2(\Omega)), \\ D_t v_\varepsilon(t,x) - A(x,D)v_\varepsilon(t,x) \\ = Bv_\varepsilon(t,x) + \sigma'_\varepsilon(t)v(t,x) + \widetilde{f}_\varepsilon(t,x), & (t,x) \in (0,T)\times\Omega, \\ v_\varepsilon(0,x) = 0, & x \in \Omega, \\ D_{\nu_A} v_\varepsilon(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \end{cases} \tag{4.20}$$

[6] For the missing computations and proofs the reader is referred to [21].

where

$$\widetilde{f}_\varepsilon = \sigma_\varepsilon \widetilde{f}. \tag{4.21}$$

Assume now that k satisfies an inequality stronger than (4.13, i.e.

$$H_0 := \operatorname{ess\,sup}_{(t,x)\in Q_T} l(t)^\kappa \int_\Omega |k(t,x,y)|\,dx < +\infty, \tag{4.22}$$

where $\kappa \in (0, 1/2)$. Indeed, in this case we have $l(t) \le (T^2/4)^{1-\kappa} l(t)^\kappa$.
Then, according to (4.11) and Sobolev embedding, from Holmgren's inequality (cf., e.g., [19, Chapter 16]), for all $t \in (0,T)$, we deduce the estimate

$$\begin{aligned} \|Bv_\varepsilon(t,\cdot)\|_{L^2(\Omega)} &\le (H_0K_0)^{1/2} l(t)^{-\kappa} \|v_\varepsilon(t,\cdot)\|_{L^2(S)} \\ &\le C(H_0K_0)^{1/2} l(t)^{-\kappa} [\|v_\varepsilon(t,\cdot)\|_{L^2(\Omega)} + \|\nabla v_\varepsilon(t,\cdot)\|_{L^2(\Omega)}]. \end{aligned} \tag{4.23}$$

Proceeding as in Subsection 2.1, we can deduce the desired estimate

$$\begin{aligned} \|v_\varepsilon(\tau,\cdot)\|_{L^2(\Omega)} &\le J_1(\varepsilon, \sigma'_\varepsilon, f, g)^{1/2} \exp\Big[\frac{1}{2}\int_0^\tau \kappa_\varepsilon(r)\,dr\Big] \\ &\quad + \int_0^\tau \exp\Big[\frac{1}{2}\int_t^\tau \kappa_\varepsilon(t)\,dr\Big] \kappa_\varepsilon(t)\,dt, \quad \tau \in [0,T], \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} J_1(\varepsilon, \sigma', f, g) &= \|\sigma'_\varepsilon\|_{L^\infty(0,T)} C_3(\varepsilon, T) \|\widetilde{f}\|^2_{L^2(Q_T)}, \\ \kappa_\varepsilon(t) &= 2\mu_3 + \|\sigma'_\varepsilon\|_{L^\infty(0,T)} + 2C(H_0K_0)^{1/2} l(t)^{-\kappa} \\ &\quad + \mu_2^{-1}\big[\mu_3 + C(H_0K_0)^{1/2} l(t)^{-\kappa}\big]^4, \end{aligned}$$

for some positive constant $C_3(\varepsilon, T)$.
In particular, for all $\tau \in [2\varepsilon T, T]$ we find the desired estimate for $u = v + g$:

$$\begin{aligned} \|u(\tau,\cdot)\|_{L^2(\Omega)} &\le \|g(\tau,\cdot)\|_{L^2(\Omega)} + J_1(\varepsilon, \sigma'_\varepsilon, f, g)^{1/2} \exp\Big[\frac{1}{2}\int_0^\tau \kappa_\varepsilon(r)\,dr\Big] \\ &\quad + \int_0^\tau \exp\Big[\frac{1}{2}\int_t^\tau \kappa_\varepsilon(t)\,dr\Big] \|\widetilde{f}_\varepsilon(t,\cdot)\|_{L^2(\Omega)}\,dt, \quad \varepsilon \in (0,1/4). \end{aligned} \tag{4.25}$$

Theorem 4.2 *Let the kernel k satisfy conditions (4.11), (4.22), with $\kappa \in [0, 1/2)$. Then the strong solution u to problem (IP) satisfies the continuous dependence estimate (4.25).*

Remark If $f = g = 0$, then $J_1(\varepsilon, \sigma'_\varepsilon, f, g) = 0$ and $\kappa_\varepsilon = 0$ so that $v = 0$ in $[2\varepsilon T, T] \times \Omega$ for all $\varepsilon \in (0, 1/2)$. This implies $u = g = 0$ in $(0, T] \times \Omega$. In particular, since $u \in H^1((0,T);L^2(\Omega)) \hookrightarrow C([0,T];L^2(\Omega))$, we can conclude that $u = 0$ in Q_T, i.e. that a *unique continuation property* holds true for the solution to problem (4.1).

□

5 The second ill-posed problem with Cauchy conditions on the lateral boundary

We consider the ill-posed problem consisting in *estimating the trace* $u(t_0, \cdot)$, $t_0 \in (0,T)$, *of the solution* $u : [0,T] \times \Omega \to \mathbb{R}$ *to the problem*

$$(IP4) \quad \begin{cases} u \in H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H^2(\Omega)), \\ D_t u(t,x) - A(x,D)u(t,x) = f(t,x), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = g_0(t,x), & (t,x) \in (0,T) \times \partial\Omega, \\ D_{\nu_A} u(t,x) = g_1(t,x) + Bu(t,x), & (t,x) \in (0,T) \times \Gamma, \end{cases} \tag{5.1}$$

where operator B is defined by (4.2) [7], while $g_0 \in H^1((0,T);L^2(\Omega)) \cap L^2((0,T); H^2(\Omega))$ and $g_1 \in L^2((0,T);H^{1/2}(\Gamma))$.
Introduce the function

$$v = u - g_0, \tag{5.2}$$

where u is the solution to problem (5.1). It is a simply task to show that v solves the following boundary-value problem:

$$(IP5) \quad \begin{cases} v \in H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H^2(\Omega)), \\ D_t v(t,x) - A(x,D)v(t,x) = \widetilde{f}(t,x), & (t,x) \in (0,T) \times \Omega, \\ v(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ D_{\nu_A} v(t,x) = \widetilde{g}_1(t,x) + Bv(t,x), & (t,x) \in (0,T) \times \Gamma, \end{cases} \tag{5.3}$$

where

$$\widetilde{f} = f - D_t g + A(\cdot,D)g, \quad \widetilde{g}_1 = g_1 + Bg_0 - D_{\nu_A} g_0. \tag{5.4}$$

Owing to Theorem 2.4 in [16], with $p = 0$, since $|\nu(x) \cdot n(x)| \geq \delta > 0$ for all $x \in \partial\Omega$, we easily deduce that any solution v to problem (5.3) satisfies the Carleman estimate

$$s^3 \int_{Q_T} l(t)^{-3} |v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$+ s \int_{Q_T} l(t)^{-1} |\nabla v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

[7] For the missing computations and proofs the reader is referred to [21].

$$+ s^{-1}\int_{Q_T} l(t)\big[|D_t v(t,x)|^2+|\Delta v(t,x)|^2\big]\exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$\le 2C\int_{Q_T} |\widetilde{f}(t,x)|^2\exp\left[2s\alpha_\lambda(t,x)\right] dtdx$$

$$+ 2Cs\int_{(0,T)\times\Gamma} l(t)^{-1}|\widetilde{g}_1(t,x)|^2\exp\left[2s\alpha_\lambda(t,x)\right] dtd\sigma(x)$$

$$+ 2Cs\int_{(0,T)\times\Gamma} l(t)^{-1}|Bv(t,x)|^2\exp\left[2s\alpha_\lambda(t,x)\right] dtd\sigma(x), \quad s\ge \widehat{s}_0. \tag{5.5}$$

The positive constants C, λ and $\widehat{s}_0$ depend on μ_1, T, $\|a_0\|_{L^\infty(\Omega)}$, $\|a_{i,j}\|_{L^\infty(\Omega)}$, $\|a_j\|_{L^\infty(\Omega)}$, $i,j=1,\ldots,n$, Ω and Γ.
Consider now the estimate

$$\int_{(0,T)\times\Gamma} l(t)^{-1}\exp\left[2s\alpha_\lambda(t,x)\right]\Big[\int_S |k(t,x,y)v(t,y)|\, d\sigma(y)\Big]^2 dtd\sigma(x)$$

$$\le K_0\int_{(0,T)\times S} l(t)^{-1}|v(t,y)|^2\, dtd\sigma(y)\int_\Gamma \exp\left[2s\alpha_\lambda(t,x)\right]|k(t,x,y)|\, d\sigma(x), \tag{5.6}$$

K_0 being defined by

$$K_0 = \operatorname{ess\,sup}_{(t,x)\in(0,T)\times\Gamma}\int_S |k(t,x,y)|\, d\sigma(y) < +\infty. \tag{5.7}$$

Assume now that function ψ satisfies, in addition to properties (2.3), also the following

$$\psi(x) = const, \quad x\in\partial\Omega. \tag{5.8}$$

Then the kernel $h_{0,s,\lambda}$ defined by

$$h_{0,s,\lambda}(t,x,y) = \exp\left\{2s[\alpha_\lambda(t,x)-\alpha_\lambda(t,y)]\right\} \tag{5.9}$$

satisfies

$$h_{0,s,\lambda}(t,x,y) = 1, \quad x,y\in\partial\Omega \tag{5.10}$$

Moreover, assume that k satisfies

$$\operatorname{ess\,sup}_{(t,y)\in(0,T)\times\Gamma}\int_\Gamma |k(t,x,y)|\, d\sigma(x) =: K_1. \tag{5.11}$$

Consequently, we get

$$\operatorname{ess\,sup}_{(t,y)\in(0,T)\times S}\int_\Gamma h_{0,s,\lambda}(t,x,y)|k(t,x,y)|\, d\sigma(x)$$

$$= \operatorname{ess\,sup}_{(t,y)\in(0,T)\times S}\int_\Gamma |k(t,x,y)|\, d\sigma(x) \le K_1. \tag{5.12}$$

Under conditions (5.7) and (5.11) we easily deduce the estimate

$$\int_{(0,T)\times\Gamma} l(t)^{-1}|Bv(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dt d\sigma(x)$$

$$\leq K_0 K_1 \int_{(0,T)\times S} l(t)^{-1}|v(t,x)|^2 \exp\left[2s\alpha_\lambda(t,x)\right] dt d\sigma(x). \tag{5.13}$$

From Lemma 4.0 we deduce estimate (4.19). Consequently, we have proved the following theorem.

Theorem 5.1 *Let the kernel k satisfy conditions (5.7) and (5.11). Then the strong solution u to problem (IP4) satisfies the Carleman estimate (5.5), with $v = u - g_0$ and $s \geq s_0$, the last integral being dropped out. In particular, problem (IP4) admits at most one solution.*

Remark Though it is possible to give a specific procedure to construct function ψ in dimension n satisfying properties (4.6) and (5.8), we omit it due to its length,

□

For lack of space we limit ourselves to stating our continuous dependence result [8].

Theorem 5.3 *Let the kernel k satisfy conditions (5.7) and (5.11). Then the strong solution u to problem (IP4) satisfy the Carleman estimate (5.5), with $v = u - g$ and $s \geq s_0$, the last integral being dropped out. In particular, problem (IP4) admits at most one solution.*

6 Final remarks

As the reader will have already noted, treating nowhere non-vanishing kernels in the integrodifferential case by the Carleman estimates is an open problem, also in the case where the kernel does not vanish on a subset, with positive measure, of the sets in (2.12) and (4.11). So, also constructing (simple?) counter-examples would be an interesting task to understand more clearly which is the situation in a quite new field of investigation. The very general problem exposed in Subsection 1.3 is, at present, a general framework where only very specific results are available. Of course, in addition to the questions highlighted here, several other results for integrodifferential problems without initial conditions are ready and will be sent to mathematical journals, including also some results for parabolic differential ill-posed problems with deviating arguments and with Cauchy boundary conditions on a open subset of the boundary of the open set Ω.

[8] For the missing computations and proofs the reader is referred to [21].

This paper intends to focus the interest of mathematicians in this research area.

Acknowledgments

The author is a member of G.N.A.M.P.A. of the Italian Istituto di Alta Matematica. The present paper was partially financed by the the project PRIN 2008 "Analisi Matematica nei Problemi Inversi per le Applicazioni" of the Italian Ministero dell'Istruzione, dell'Università e della Ricerca (M.I.U.R.).

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