On some nonlinear models for suspension bridges

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Abstract

In this paper we discuss some mathematical models describing the nonlinear vibrations of different kinds of single-span simply supported suspension bridges and we summarize some results about the longtime behavior of solutions to the related evolution problems. Finally, in connection with the static counterpart of a general string-beam nonlinear model, we present some original results concerning the existence of multiple buckled solutions.

1 Introduction

In recent years, an increasing attention was payed to the analysis of buckling, vibrations and post-buckling dynamics of nonlinear beam models, and many papers have been published in this field, especially in connection with industrial applications and suspension bridges (see, for instance,[20, 23] and [1, 2]). In spite of this, as far as we know, most of these papers deal with approximations and numerical simulations, and only few works are devoted to derive exact solutions, at least under stationary conditions, and to scrutinize the periodic or longtime global dynamics by analytical methods (see, for instance, [4, 12, 11]).

The first part of this paper is devoted to a brief survey on the subject. We begin by sketching out some linear and nonlinear models, and reporting earlier contributions. In the second part we scrutinize the longtime behavior of solutions to the related evolution problems. Finally, we present some original results concerning the existence of multiple buckled solutions for the static counterpart of a quite general nonlinear model.

1.1 A nonlinear equation for the bending motion of a beam

In the fifties, Woinowsky-Krieger [27] proposed to modify the theory of the dynamic Euler-Bernoulli beam, assuming a nonlinear dependence of the axial strain on the deformation gradient. The resulting motion equation,

$$m\partial_{tt}u + \delta\partial_{xxxx}u + \big(p - \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = 0, \tag{1.1}$$

where $m > 0$ is the mass per unit length of the beam, and $\delta > 0$ stands for its flexural rigidity, has been considered in the paper [5]. The parameter p summarizes the effect of an axial force acting at the ends of the beam: it is is negative when the beam is stretched, positive when compressed. For a beam of unitary length with hinged ends, some well-posedness results are proved therein and devoted to the analysis of the complex structure of equilibria. Adding an external viscous damping term $\nu\partial_t u$, $\nu > 0$, to the original conservative model, it becomes

$$m\partial_{tt}u + \delta\partial_{xxxx}u + \nu\partial_t u + \big(p - \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u = 0. \tag{1.2}$$

Stability properties of the unbuckled (trivial) and the buckled stationary states of (1.2) have been established in [5, 13] and, more formally, in [24]. In particular, if $p < p_{\mathrm{c}}$ (critical buckling load), the exponential decay of solutions to the trivial equilibrium state has been shown. The global dynamics of solutions for a general p has been first tackled in [17] where the existence of a global attractor for (1.2) was proved relying on the construction of a suitable Lyapunov functional. In [11] previous results are extended to a more general form of the nonlinear term by virtue of a suitable decomposition of the semigroup first introduced in [16] (see, also, [6, 9]).

1.2 Linear models for a suspension bridge

A different class of problems arises in the study of vibrations of a single-span suspension bridge. The dynamic response of the deck is usually analyzed by linearizing the equations of motion, in particular by neglecting the effects of its extensibility on the bending. For instance, Lazer and McKenna proposed a well-known suspension-bridge model where the coupling of the span with the main cable is taken into account (see [21, 22]). So one has the (damped) system

$$\begin{cases} m_1\partial_{tt}u + \delta_1\partial_{xxxx}u + \nu_1\partial_t u + k^2(u-v)^+ = f, \\ m_2\partial_{tt}v - \delta_2\partial_{xx}v + \nu_2\partial_t v - k^2(u-v)^+ = g, \end{cases} \tag{1.3}$$

where $m_1, m_2 > 0$ are the masses per unit length of the road bed and the cable, $\delta_1, \delta_2 > 0$ are the flexural rigidity of the deck and the coefficient of tensile strength of the cable, while the terms $\nu_1\partial_t u$, $\nu_2\partial_t v$, with $\nu_1, \nu_2 > 0$, account for an external resistant force linearly depending on the velocity. Moreover u and v represent the downward deflection of the deck midline and the suspension main cable, respectively, in the vertical plane with respect to their reference configurations (see

Fig. 1.2). The sources, f and g, represent the (given) vertical dead load distribution on the deck and the main cable, respectively, and w^+ stands for positive part of w, namely,

$$w^+ = \begin{cases} w & \text{if } w \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

The nonlinear terms $\pm\, k^2(u-v)^+$ models the restoring force due to the cable stays, which are assumed to behave as one-sided springs, obeying Hooke's law, with a restoring force proportional to their displacement if they are stretched, and with no restoring force if they are compressed.

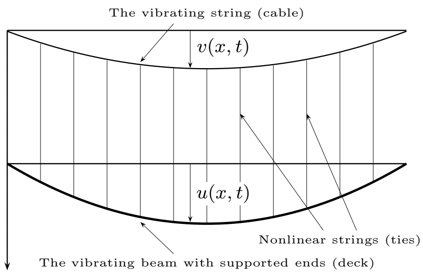

Figure 1: Center span hanged by main cable.

The first equation of (1.3) describes the vertical oscillations of a one-dimensional beam, which represents the center span of the road bed, hanging by elastic cable stays. The second equation of the system models the vertical vibrations of the main cable, whose ends are fixed to the pair of lateral piers. Since the cable stays are assumed to be flexible and linearly elastic, their restoring force is nonlinear in that it is positive when they are stretching, but vanishes when they are shortening. This model neglects the influence of the side parts and piers deformations.

A simpler model follows when the main cable holding the cable stays is replaced by a rigid and immovable frame. If this is the case, the system (1.3) reduces to the so called Lazer-McKenna damped equation,

$$m\partial_{tt}u + \delta\partial_{xxxx}u + \nu\partial_t u + k^2u^+ = f. \tag{1.5}$$

In the sequel this case will be referred to as *single beam model*. Free and forced vibrations in models of this type, both with constant and non constant load, have been scrutinized in [3] and [10]. Moreover, the existence of strong solutions and global attractors has been recently obtained in [28].

1.3 Nonlinear models for a suspension bridge

If large deflections occur, the Lazer-McKenna's model becomes inadequate and the extensibility of the deck has to be taken into account. This can be done by introducing into the model equation (1.5) a geometric nonlinear term like that appearing in (1.1). Such a nonlinear term is of some importance in the modeling of large deflections of both suspension and cable-stayed bridges (see, for instance, [19, 26]).

1.3.1 A single-beam model

Taking into account the midplane stretching of the road bed due to its elongation, the bending equation of the deck becomes

$$m\partial_{tt}u + \delta\partial_{xxxx}u + \big(p - \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u + \nu\partial_t u + k^2u^+ = f \ . \tag{1.6}$$

Equation (1.6) does not accurately describe the behavior of a real suspension bridge. Nevertheless, it correctly reflects the influence of the cable stays on the deck motion and it is reasonably simple and applicable. Accordingly, such a model has been used as a starting point for analytical and numerical studies on suspension bridges. For instance, Abdel-Ghaffar and Rubin [1, 2] presented a general theory and analyzed the nonlinear coupled vertical-torsional free vibrations of a single-span suspension bridge. If torsional vibrations are ignored, their model reduces to (1.6). Exact solutions for a related problem under stationary conditions have been recently exhibited in [15]. The longtime dynamics and the existence of a regular global attractor for the semigroup generated by (1.6) has been also proved in [7].

1.3.2 A string-beam model

A more realistic but complicate approach to the suspension-bridge modeling consists in replacing the rigid frame holding the cable stays by a movable elastic string. Thus, we have to take into account not only the motion of the deck (just like a nonlinear vibrating beam) but also the oscillation of the main cable, which behaves like a vibrating string and is coupled to the deck by means of one–sided springs (see Fig. 1.2). Thus, introducing into the coupled Lazer-McKenna's model (1.3) a geometric nonlinear term we consider jointly the suspended bridge and the cable, we have to end up with the following problem

$$\begin{cases} m_1\partial_{tt}u + \delta_1\partial_{xxxx}u + \nu_1\partial_t u \\ \qquad\qquad + (p - \|u\|^2_{L^2(0,1)})\partial_{xx}u + k^2(u-v)^+ = f, \\ m_2\partial_{tt}v - \delta_2\partial_{xx}v + \nu_2\partial_t v - k^2(u-v)^+ = g \ . \end{cases} \tag{1.7}$$

As previously stated, v measures the displacement of the main cable and u represents the bending displacement of the road bed of the bridge. The nonlinear stays connecting the beam and the string hold the road bed up and pull the cable down, so into the first equation we consider the plus sign in front of $k^2(u-v)^+$, but the minus sign in front of the same term in the second one.

1.3.3 A more general string-beam model

Other realistic models can be constructed by considering a more general form F of the restraining force experienced by both the road bed and the suspension cable as transmitted through the tie lines (stays). If this is the case, the previous system takes the form

$$\begin{cases} m_1\partial_{tt}u+\delta_1\partial_{xxxx}u+\nu_1\partial_t u \\ \qquad\qquad +(p-\|u\|^2_{L^2(0,1)})\partial_{xx}u+F(u-v)=f, \\ m_2\partial_{tt}v-\delta_2\partial_{xx}v+\nu_2\partial_t v-F(u-v)=g. \end{cases} \tag{1.8}$$

When $F(\xi)=k^2\xi^+$, then (1.7) is recovered.

On the other hand, there is some case in which F can be reasonably assumed to be linear. Indeed, let the road bed be supported by a symmetrical system of one–sided elastic ties (cable stays), each of which fastened on two symmetrically placed main (suspension) cables, one above and one below the road bed. Then, the dynamics of the resulting suspension bridge is described by the following system

$$\begin{cases} m_1\partial_{tt}u+\delta_1\partial_{xxxx}u+\nu_1\partial_t u+(p-\|u\|_1^2)\partial_{xx}u+k^2(u-v)=f, \\ m_2\partial_{tt}v-\delta_2\partial_{xx}v+\nu_2\partial_t v-k^2(u-v)=g\ . \end{cases} \tag{1.9}$$

where k^2 is the common stiffness of the ties. If one of the symmetric sets of cables stays (either above or below the road bed) is removed, then system (1.9) turns into the previous model (1.7).

In this paper we consider only one–dimensional models of a suspension bridge. Nevertheless, a more general and realistic approach can be performed by including also the width of the road bed and its torsional oscillations. Into this setting, the two–dimensional suspension bridge can be modelled as a long and narrow vibrating plate, coupled to the main cables by means of two series of non linear cable stays fixed at its lateral sides (see, for instance, [14, 18]).

1.4 Initial and boundary conditions

In all of these models, the unknown fields u, $\partial_t u$ and v, $\partial_t v$, where present, are required to satisfy the following initial conditions, respectively,

$$u(x,0)=u_0(x),\ x\in[0,1], \qquad \partial_t u(x,0)=u_1(x),\ x\in[0,1]\ ; \tag{1.10}$$

$$\begin{cases} u(x,0)=u_0(x),\ x\in[0,1], \qquad \partial_t u(x,0)=u_1(x),\ x\in[0,1] \\ v(x,0)=v_0(x),\ x\in[0,1], \qquad \partial_t v(x,0)=v_1(x),\ x\in[0,1]. \end{cases} \tag{1.11}$$

Concerning the boundary conditions, in all the bridge models the road bed is considered with both pinned ends. Namely, for every $t>0$, we assume

$$u(0,t)=u(1,t)=\partial_{xx}u(0,t)=\partial_{xx}u(1,t)=0. \tag{1.12}$$

Accordingly, for a general suspension bridge model with coupling we choose

$$\begin{cases} u(0,t) = u(1,t) = \partial_{xx}u(0,t) = \partial_{xx}u(1,t) = 0, \\ v(0,t) = v(1,t) = 0. \end{cases} \tag{1.13}$$

As a consequence, the domain of the differential operator $A = \partial_{xxxx}$ acting on $L^2(0,1)$ which appears into the motion equation of the beam is

$$\mathcal{D}(\partial_{xxxx}) = \{w \in H^4(0,1) : w(0) = w(1) = \partial_{xx}w(0) = \partial_{xx}w(1) = 0\}.$$

It is a strictly positive selfadjoint operator with compact inverse, and its discrete spectrum is given by $\lambda_n = n^4\pi^4$, $n \in \mathbb{N}$. Thus, $\lambda_1 = \pi^4$ is the smallest eigenvalue. Besides, the following peculiar relation holds true

$$A^{1/2} = -\partial_{xx} \quad , \qquad \mathcal{D}(-\partial_{xx}) = H^2(0,1) \cap H^1_0(0,1).$$

Hence, if pinned–fixed boundary conditions are considered, the initial-boundary value problems related to our models can be described by means of a single operator A which enters the equations at the powers 1 and 1/2. This fact is particularly relevant in the analysis of the *critical buckling load* which is the magnitude of the compressive axial force $p_c > 0$ at which buckled stationary states appear.

2 Functional setting and dynamical systems

In order to establish more general results, all the IBVP related to the previous models will be recast into an abstract setting.

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space, and let $A : \mathcal{D}(A) \Subset H \to H$ be a strictly positive selfadjoint operator. For $r \in \mathbb{R}$, we introduce the scale of Hilbert spaces generated by the powers of A

$$H^r = \mathcal{D}(A^{r/4}), \qquad \langle u, v\rangle_r = \langle A^{r/4}u, A^{r/4}v\rangle, \qquad \|u\|_r = \|A^{r/4}u\|.$$

When $r = 0$, the index r is omitted. The symbol $\langle\cdot,\cdot\rangle$ will also be used to denote the duality product between H^r and its dual space H^{-r}. In particular, we have the compact embeddings $H^{r+1} \Subset H^r$, along with the generalized Poincaré inequalities

$$\lambda_1\|w\|_r^4 \le \|w\|_{r+1}^4, \qquad \forall w \in H^{r+1},$$

where $\lambda_1 > 0$ is the first eigenvalue of A. From these inequalities it follows

$$\lambda\left(\|u\|_r^2 + \|v\|_r^2\right) \le \|u\|_{r+2}^2 + \|v\|_{r+1}^2\,, \quad \lambda = \min\{\lambda_1, \sqrt{\lambda_1}\}, \tag{2.14}$$

for all $u \in H^{r+2}$, $v \in H^{r+1}$.

For later convenience, we define the product Hilbert spaces

$$\mathcal{H}^\ell = H^{\ell+2} \times H^\ell\ ,$$

related to the single-beam model in the unknown fields $u(t)$, $\partial_t u(t)$, and

$$\mathcal{M}^\ell = H^{\ell+2} \times H^\ell \times H^{\ell+1} \times H^\ell,$$

connected to coupled models in the unknown fields $u(t)$, $\partial_t u(t)$, $v(t)$, $\partial_t v(t)$.

2.1 The single-beam model

When the single beam model is considered, we obtain the following Cauchy problem on $\mathcal{H}$ in the unknown variable $u = u(t)$,

$$\begin{cases} m\partial_{tt}u + \delta Au + \nu\partial_t u - \big(p - \|u\|_1^2\big)A^{1/2}u + k^2u^+ = f, \\ u(0) \,=\, u_0, \quad \partial_t u(0) \,=\, u_1 \ . \end{cases} \tag{2.15}$$

Problem (1.6), (1.10) is just a particular case of the abstract system (2.15), which is obtained by assuming $H = L^2(0,1)$ and $A = \partial_{xxxx}$ with boundary conditions (1.12).

In the sequel $z(t) = (u(t), \partial_t u(t))$ will denote the solution to (2.15). Now, recalling some results obtained in [9], we observe that the set $\mathcal{S}$ of the bridge equilibria consists of all the pairs $(u, 0) \in \mathcal{H}$ such that u is a weak solution to the equation

$$\delta Au - \big(p - \|u\|_1^2\big)A^{1/2}u + k^2u^+ = f.$$

For example, on $[0,1]$ u solves the following boundary value problem

$$\begin{cases} \delta\partial_{xxxx}u + \big(b\,\pi^2 - \|\partial_x u\|^2_{L^2(0,1)}\big)\partial_{xx}u + \kappa^2\pi^4u^+ = f, \\ u(0) = u(1) = \partial_{xx}u(0) = \partial_{xx}u(1) = 0, \end{cases} \tag{2.16}$$

where we let $k = \kappa\pi^2$, and $p = b\,\pi^2$. It is then apparent that $\mathcal{S}$ is bounded in $H^2(0,1) \cap H_0^1(0,1)$ for every $b, \kappa \in \mathbb{R}$ and $f \in L^2(0,1)$. The set of buckled solutions to problem (2.16) under a vanishing lateral load ($f = 0$) is scrutinized in [15]. When $\kappa = 0$, a general result has been established in [12] for a class of non-vanishing sources f. In [6, 9], the same strategy with minor modifications has been applied to problems close to (2.16), where the term u^+ is replaced by u (unyielding ties).

According to [7], the evolution system (2.15) admits a unique solution which continuously depends on the initial data. When $f = f(x)$ the system is autonomous and generates a strongly continuous semigroup (or dynamical system) $S(t)$ on $\mathcal{H}$. Namely, for any initial data $z \in \mathcal{H}$, $z(t) = S(t)z$ is the unique weak solution, with related $\mathcal{H}$-norm given by

$$\mathcal{E}(z(t)) \,=\, \|z(t)\|^2_{\mathcal{H}} \,=\, \delta\|u(t)\|_2^2 + m\|\partial_t u(t)\|^2.$$

where $z(t) = (u(t), \partial_t u(t)) \in \mathcal{H}$. We define the corresponding *energy* as

$$E(z(t)) = \mathcal{E}(z(t)) + \frac{1}{2}\big(\|u(t)\|_1^2 - p\big)^2 + k^2\|u^+(t)\|^2 \tag{2.17}$$

and, abusing the notation, we denote $E(z(t))$ by $E(t)$.

Proposition 2.1 *(see [7]) The flow generated by the problem* (2.15) *admits an absorbing set $\mathcal{B}_0$, that is a bounded set into which every orbit eventually enters.*

Due the joint presence of geometric and cable-response nonlinear terms in (2.15), a direct proof of the existence of the absorbing set via *explicit* energy estimates is nontrivial. Indeed, the double nonlinearity cannot be handled by means of standard arguments, as performed either in [22] or in [28].

The existence of an absorbing set $\mathcal{B}_0$ gives a first rough estimate of the dissipativity of the system and is a preliminary step to scrutinize its asymptotic dynamics (see, for instance, [25]). Indeed, it enable us to establish the existence of a global attractor $\mathcal{A}$: namely, the unique compact subset of $\mathcal{H}$ which is at the same time attracting and fully invariant.

Theorem 2.2 *(see[7]) The semigroup $S(t)$ acting on $\mathcal{H}$ possesses a connected global attractor $\mathcal{A}$ bounded in $\mathcal{H}^2$. Moreover, $\mathcal{A}$ coincides with the unstable manifold of the set $\mathcal{S}$ of the stationary points of $S(t)$, namely*

$$\mathcal{A} = \left\{ \begin{array}{c} z(0) : z \textit{ is a complete (bounded) trajectory of } S(t) : \\ \lim_{t\to\infty} \|z(-t) - \mathcal{S}\|_{\mathcal{H}} = 0 \end{array} \right\}.$$

Since problem (2.15) takes the form of a *gradient system*, this result can be achieved by using an approach which appeals to the existence of a Lyapunov functional. This fact, jointed with the boundedness of the equilibrium set $\mathcal{S}$, not only ensures that $E(t)$ is uniformly bounded for all t, but also allow us to prove Th. 2.2 by virtue of a general result due to Hale (see [17]). The key point is a suitable (exponential) asymptotic compactness property of the semigroup, which will be obtained exploiting a particular decomposition of $S(t)$ devised in [16].

2.2 The string-beam model

The abstract Cauchy problem on $\mathcal{M}$ related to system (1.7) is given by

$$\begin{cases} m_1\partial_{tt}u + \delta_1 Au + \nu_1\partial_t u - (p - \|u\|_1^2)A^{1/2}u + k^2(u-v)^+ = f, \\ m_2\partial_{tt}v + \delta_2 A^{1/2}v + \nu_2\partial_t v - k^2(u-v)^+ = g, \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1, \\ v(x,0) = v_0, \quad \partial_t v(x,0) = v_1 \, . \end{cases} \tag{2.18}$$

Problem (1.7), (1.11) is just a particular case which can be obtained by assuming $H = L^2(0,1)$ and $A = \partial_{xxxx}$ with boundary conditions (1.13).

For any $z \in \mathcal{M}$, let $z(t) = S(t)z$ and define the corresponding *energy* as

$$E(z(t)) = \mathcal{E}(z(t)) + \frac{1}{2}\big(\|u(t)\|_1^2 - p\big)^2 + k^2\|(u(t)-v(t))^+\|^2, \tag{2.19}$$

where $z(t) = (u(t), \partial_t u(t), v(t), \partial_t v(t))$ and

$$\mathcal{E}(z(t)) = \|S(t)z\|_{\mathcal{M}}^2 = \delta_1\|u(t)\|_2^2 + m_1\|\partial_t u(t)\|^2 + \delta_2\|v(t)\|_1^2 + m_2\|\partial_t v(t)\|^2 \tag{2.20}$$

is a bounded function as proved by the following

Lemma 2.2 *(see [8]) Given $f \in H^{-2}$ and $g \in H^{-1}$, for all $t > 0$ and initial data $z \in \mathcal{M}$ with $\|z\|_{\mathcal{M}} \le R$,*

$$\mathcal{E} \le Q(R).$$

For this model, the set of stationary solutions $\mathcal{S}$ under vanishing external sources is made of vectors $(u, 0, v, 0) \in \mathcal{M}$, such that $u \in H^2$ and $v \in H^1$ are weak solutions to the system

$$\begin{cases} \delta_1 Au - (p - \|u\|_1^2)A^{1/2}u + k^2(u-v)^+ = 0, \\ \delta_2 A^{1/2}v - k^2(u-v)^+ = 0. \end{cases} \tag{2.21}$$

In addition, in order to define a strongly continuous semigroup of operators on $\mathcal{M}$, the following well-posedness result holds (see [8]).

Proposition 2.3 *Assume that $f, g \in L^2(0,T;H)$. Then, for all initial data $z \in \mathcal{M}$, problem* (2.18) *with boundary conditions* (1.12)*, admits a unique solution*

$$(u(t), \partial_t u(t), v(t), \partial_t v(t)) \in \mathcal{C}(0,T;\mathcal{M}),$$

which continuously depends on the initial data.

As a consequence, this system generates a strongly continuous semigroup (or dynamical system) $S(t)$ on $\mathcal{M}$: for any initial data $z \in \mathcal{M}$, $S(t)z$ is the unique weak solution to (2.18) with related energy given by (2.19). Also in this case we are interested to the asymptotic behavior in time of the solutions. Precisely, the flow generated by problem (2.18) admits an absorbing set as proved in [8].

Theorem 2.4 *For any $p \in \mathbb{R}$, there exists an absorbing set in $\mathcal{M}$ for the dynamical system $(S(t), \mathcal{M})$.*

Moreover, (2.21) is a *gradient system* (see [8]).

Proposition 2.5 *If $f \in H^{-2}$ and $g \in H^{-1}$ are time-independent functions, the functional*

$$\mathcal{L} = E(z) - 2\langle u, f\rangle - 2\langle v, g\rangle$$

is a Lyapunov functional for $S(t)$.

The existence of a Lyapunov functional, along with the fact that $\mathcal{S}$ is a bounded set, allow us prove the following theorem:

Theorem 2.6 *The semigroup $S(t)$ acting on $\mathcal{M}$ possesses a connected global attractor $\mathcal{A}$ bounded in $\mathcal{M}^2$.*

By exploiting a particular decomposition of $S(t)$, a suitable (exponential) asymptotic compactness property of the semigroup is shown. Precisely, this result is based on a strategy devised in [16] which consists in decomposing the semigroup $S(t)$ in two suitable semigroups, one that exponentially decays and the other which is bounded in a more regular space. A detailed proof of the quoted results is presented in [8].

3 Steady states for a special string-beam model

Let $u(t)$ and $v(t)$ the unknown functions of the abstract Cauchy problem

$$\begin{cases} m_1\partial_{tt}u + \delta_1 Au + \nu_1\partial_t u - (p - \|u\|_1^2)A^{1/2}u + k^2(u-v) = f, \\ m_2\partial_{tt}v + \delta_2 A^{1/2}v + \nu_2\partial_t v - k^2(u-v) = g, \\ u(x,0) \;=\; u_0, \quad \partial_t u(x,0) \;=\; u_1, \\ v(x,0) \;=\; v_0, \quad \partial_t v(x,0) \;=\; v_1 \;. \end{cases} \tag{3.22}$$

Problem (1.9) is just a particular case of this abstract system. It is obtained by setting $H = L^2(0,1)$ and $A = \partial_{xxxx}$ with boundary conditions (1.13).

Letting $f \;=\; g = 0$, the static counterpart of problem (3.22) reads

$$\begin{cases} Au + \alpha A^{1/2}u + \kappa(u-v) = 0, \\ A^{1/2}v - \dfrac{\kappa}{\gamma}(u-v) = 0, \end{cases} \tag{3.23}$$

where

$$\gamma \;=\; \frac{\delta_2}{\delta_1} > 0, \qquad \kappa \;=\; \frac{k^2}{\delta_1} > 0, \qquad \alpha \;=\; \frac{\|u\|_1^2 - p}{\delta_1}.$$

In order to find the explicit form of regular solutions $(u,v) \in H^2 \times H^2$ to this system, it may be reduced to a single equation. If we apply $(-A^{1/2} - \alpha I)$ to $(3.23)_2$ and then add the resulting equation to $(3.23)_1$, we obtain

$$Aw + h\,A^{1/2}w + q\,w = 0 \tag{3.24}$$

where $w \;=\; u - v$, and

$$h = \frac{\kappa}{\gamma} + \alpha, \qquad q = \frac{\kappa}{\gamma}\Big(\alpha + \gamma\Big) = \frac{\kappa}{\gamma}\Big(h + \gamma - \frac{\kappa}{\gamma}\Big)\;.$$

As apparent, each regular solution w to (3.24) generates a regular solution (u,v) to (3.23). Indeed, for any given w from $(3.23)_2$ we obtain

$$v = \frac{\kappa}{\gamma}A^{-1/2}w, \qquad u = w + v = \left(I + \frac{\kappa}{\gamma}A^{-1/2}\right)w.$$

Hence, (3.24) turns out to be written in a closed form, in that

$$h = \frac{k^2}{\delta_2} + \frac{1}{\delta_1}\Big(||\big(I + \frac{k^2}{\delta_2}A^{-1/2}\big)w||_1^2 - p\Big). \tag{3.25}$$

According to this abstract setting, the multiplicity of solutions to the abstract problem (3.23) depends strongly on the spectrum of A. A further paper will be devoted to the complete analysis of this subject when a generic elliptic, selfadjoint and positive operator A is involved. In the sequel we restrict our attention to the special case $A = \partial_{xxxx}$.

3.1 A special case

Our aim is to analyze the multiplicity of solutions to the stationary problem (3.23) when $A = \partial_{xxxx}$, $x \in [0,1]$ and (1.13) hold. To this end we specialize (3.24) as

$$\begin{cases} w'''' - hw'' + qw \;=\; 0, \\ w(0) = w(1) = w''(0) = w''(1) = 0, \end{cases} \tag{3.26}$$

and then we solve it for all $p \in \mathbb{R}$. In order to prove its consistency, we first show the explicit dependence of h on w. By letting w be any solution to (3.26) and remembering $(3.23)_2$, the solution v to the boundary value problem

$$\begin{cases} v'' \;=\; -\frac{\kappa}{\gamma} w, \\ v(0) = v(1) = 0, \end{cases} \tag{3.27}$$

is given by

$$v(x) = \frac{\kappa}{\gamma} \int_0^x \left(\int_0^1 \int_0^\eta w(\zeta)\, d\zeta d\eta - \int_0^\xi w(\zeta)\, d\zeta \right) d\xi$$

Then, $||u||_1^2 = ||u'||^2 = \int_0^1 |v'(x) + w'(x)|^2 \, dx$ and finally

$$h = \frac{\kappa}{\gamma} - \frac{1}{\delta_1} \left[p - \int_0^1 \left| \frac{\kappa}{\gamma} \left[\int_0^1 \int_0^\eta w(\zeta)\, d\zeta d\eta - \int_0^x w(\zeta)\, d\zeta \right] + w'(x) \right|^2 dx \right]. \tag{3.28}$$

Notations. For any given δ_1, κ, $\gamma > 0$, let

$$\mu_n^\star = \delta_1 \Big[\frac{\gamma\kappa}{\kappa + \gamma n^2 \pi^2} + n^2\pi^2 \Big], \quad \mu_n^\dagger = \delta_1 \Big[\frac{\kappa}{\gamma} + n^2\pi^2 \Big], \quad n \in \mathbb{N},$$

and let $n_\star$ and $n_\dagger$ be the integer valued functions respectively given by

$$n_\star(p) = |\mathcal{S}_\star|,\ \mathcal{S}_\star = \{n \in \mathbb{N} : p > \mu_n^\star\}\ ; \quad n_\dagger(p) = |\mathcal{S}_\dagger|,\ \mathcal{S}_\dagger = \{n \in \mathbb{N} : p > \mu_n^\dagger\},$$

where $|\mathcal{S}|$ stands for the cardinality of the set $\mathcal{S}$. Finally, let

$$p_c^\star(\kappa,\gamma) = \min_{n\in\mathbb{N}} \{\mu_n^\star(\kappa,\gamma)\}\,, \qquad p_c^\dagger(\kappa,\gamma) = \min_{n\in\mathbb{N}} \{\mu_n^\star(\kappa,\gamma), \mu_n^\dagger(\kappa,\gamma)\}.$$

We note that there exists a nonempty set $\mathcal{R}$, called *resonant set*, such that

$$\mathcal{R} = \{(\kappa,\gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ : \mu_n^\star(\kappa,\gamma) = \mu_\ell^\star(\kappa,\gamma) \text{ for some } n, \ell \in \mathbb{N}\,,\ n \neq \ell\}.$$

When $(\kappa,\gamma)\in\mathcal{R}$, we denote the minimum non-simple $\mu_n^\star$ value as

$$r^\star(\kappa,\gamma)=\min_{n\in\mathbb{N}}\{\mu_n^\star(\kappa,\gamma):\mu_n^\star \text{ is non-simple}\}\,.$$

Theorem 3.1 *Let $\kappa\ge\gamma^2$. If $p\le p_c^\star$, then $w=0$ is the only solution to problem (3.26). Otherwise, there exist exactly $2\,n_\star(p)+1$ solutions: the straight solution $w=0$ and the buckled ones*

$$w_n^\pm(x)=\pm\,E_n\sqrt{p-\mu_n^\star}\ \sin n\pi x\,,\quad E_n=\frac{\sqrt{2}\gamma n\pi}{(\gamma n^2\pi^2+\kappa)},\quad n=1,2,...,n_\star\,.$$

Let $\kappa<\gamma^2$. If $p\le p_c^\dagger$, then $w=0$ is the only solution to problem (3.26). On the contrary, if $p>p_c^\dagger$ then

- *when $(\kappa,\gamma)\in\mathcal{R}$ and $p>r^\star(\kappa,\gamma)$, there are infinitely many solutions;*
- *when either $(\kappa,\gamma)\in\mathcal{R}$ and $p\le r^\star(\kappa,\gamma)$, or $(\kappa,\gamma)\notin\mathcal{R}$, there are exactly $2n_\star(p)+2n_\dagger(p)+1$ solutions. Namely, the straight solution $w=0$ and the buckled ones*

$$w_m^\pm(x)\;=\;C_m^\pm\,\sin m\pi x\,,\quad w_n^\pm(x)\;=\;D_n^\pm\,\sin n\pi x\,,\tag{3.29}$$

with $m=1,2,\ldots,n_\dagger$, $n=1,2,\ldots,n_\star$ and

$$C_m^\pm=\pm\,E_m\sqrt{p-\mu_m^\dagger}\,,\quad D_n^\pm=\pm\,E_n\sqrt{p-\mu_n^\star}\,.\tag{3.30}$$

Proof:

The null function $w(x)=0$ is a solution to (3.26) for all values of the involved constants. To find nontrivial solutions we have to analyze the characteristic equation

$$\lambda^4\,-\,h\,\lambda^2+q=0\,.\tag{3.31}$$

Letting $\lambda^2=\chi$, it admits solutions in the form

$$\lambda_{1,2}=\pm\sqrt{\chi_1}\;,\quad\lambda_{3,4}=\pm\sqrt{\chi_2}\;,\quad\chi_{1,2}=\frac{h\pm\sqrt{h^2-4q}}{2}\;,\tag{3.32}$$

where $h^2-4q=h^2-4\frac{\kappa}{\gamma}h+4\frac{\kappa}{\gamma}\big(\frac{\kappa}{\gamma}-\gamma\big)$ is positive if and only if

$$h<2\Big(\frac{\kappa}{\gamma}-\sqrt{\kappa}\Big)\quad\text{or}\quad h>2\Big(\frac{\kappa}{\gamma}+\sqrt{\kappa}\Big).$$

In addition $q>0$ if and only if $h>\frac{\kappa}{\gamma}-\gamma$. Please, note that $2\big(\frac{\kappa}{\gamma}-\sqrt{\kappa}\big)$ and $\frac{\kappa}{\gamma}-\gamma$ are both positive if and only if $\kappa>\gamma^2$, nevertheless

$$\frac{\kappa}{\gamma}-\gamma\le2\Big(\frac{\kappa}{\gamma}-\sqrt{\kappa}\Big)$$

where the equality holds if and only if $\kappa=\gamma^2$. Now, we separately discuss five items depending on the value of h.

1. $\dfrac{h}{2} \geq \dfrac{\kappa}{\gamma} + \sqrt{\kappa} > 0$. As a consequence,

$$q = \frac{\kappa}{\gamma}\big(h + \gamma - \frac{\kappa}{\gamma}\big) > \frac{\kappa}{\gamma}\big(\frac{\kappa}{\gamma} + 2\sqrt{\kappa} + \gamma\big) > 0.$$

Hence, $\chi_{1,2} \in \mathbb{R}^+$ so that by (3.32) all the corresponding values of λ are real. Then, taking into account the hinged boundary conditions $(3.26)_2$, we obtain $w \equiv 0$.

2. $\dfrac{\kappa}{\gamma} - \sqrt{\kappa} < \dfrac{h}{2} < \dfrac{\kappa}{\gamma} + \sqrt{\kappa}$. In this case $h^2 - 4q < 0$. Then $\chi_{1,2}$ are complex and conjugate and all the corresponding values of λ are complex with non vanishing real parts. By $(3.26)_2$ we obtain $w \equiv 0$ as in the previous case.

3. $\dfrac{\kappa}{\gamma} - \gamma < h \leq 2\Big(\dfrac{\kappa}{\gamma} - \sqrt{\kappa}\Big)$. We have the following subcases

   3.1. if $\kappa > \gamma^2$, then $0 < \frac{\kappa}{\gamma} - \gamma < h \leq 2\big(\frac{\kappa}{\gamma} - \sqrt{\kappa}\big)$, so that both h and q are positive. Then $\chi_{1,2} \in \mathbb{R}^+$ and all corresponding values of λ are real, whence $w \equiv 0$.

   3.2. if $\kappa < \gamma^2$, then $\frac{\kappa}{\gamma} - \gamma < h \leq 2\big(\frac{\kappa}{\gamma} - \sqrt{\kappa}\big) < 0$, so that $h < 0$ and $q > 0$. Then $\chi_{1,2} \in \mathbb{R}^-$ and $\lambda_{1,2} = \pm\, i\sqrt{|\chi_1|}$, $\lambda_{3,4} = \pm\, i\sqrt{|\chi_2|}$, whence

$$w = A\,\sin\omega_1\, x + B\,\sin\omega_2\, x, \tag{3.33}$$

$$\omega_1 = \sqrt{|\chi_1|} = n\pi\,, \quad \omega_2 = \sqrt{|\chi_2|} = \ell\pi\,, \qquad n, \ell \in \mathbb{N}\,,\ A, B \in \mathbb{R}.$$

4. $h = \dfrac{\kappa}{\gamma} - \gamma$. In this case $q = 0$ so that $\chi_1 = 0$, $\chi_2 = h$.

   4.1. if $\kappa > \gamma^2$, then $h > 0$, so that all corresponding values of λ are real, whence $w \equiv 0$.

   4.2. if $\kappa = \gamma^2$, then $h = q = 0$ and all corresponding values of λ vanish, whence $w \equiv 0$.

   4.3. if $\kappa < \gamma^2$, then $h < 0$, so that $\lambda_{1,2} = 0$, $\lambda_{3,4} = \pm\, i\sqrt{|h|}$ and

$$w = C\,\sin\omega\, x, \qquad \omega = \sqrt{|h|} = n\pi, \quad n \in \mathbb{N}\,,\ C \in \mathbb{R}. \tag{3.34}$$

5. $h < \dfrac{\kappa}{\gamma} - \gamma$. In this case $q < 0$ and $\chi_1 \in \mathbb{R}^-$, $\chi_2 \in \mathbb{R}^+$, whatever the sign of h may be. Two values of λ are real, $\lambda_{1,2} = \pm\sqrt{\chi_2}$, whereas the other two are purely imaginary, $\lambda_{3,4} = \pm\, i\sqrt{|\chi_1|}$, whence

$$w = D\,\sin\omega\, x, \qquad \omega = \sqrt{|\chi_1|} = n\pi, \quad n \in \mathbb{N}\,,\ D \in \mathbb{R}. \tag{3.35}$$

Now, by remembering that h is not a constant but a function of w, we have to compute explicitly the amplitudes A, B, C and D of the nontrivial solutions (3.33), (3.34) and (3.35).

- *Item 5.* From (3.32) we first have

$$|\chi_1| = \frac{1}{2}\big(-h + \sqrt{h^2 - 4q}\big) = n^2\pi^2,$$

which in turn implies

$$h = \frac{\kappa(\kappa - \gamma^2) - \gamma^2 n^4\pi^4}{\gamma(\kappa + \gamma n^2\pi^2)}\,. \tag{3.36}$$

On the other hand, from (3.28) we have

$$h = \frac{\kappa}{\gamma} - \frac{p}{\delta_1} + \frac{D^2}{2\delta_1}\left(1 + \frac{\kappa}{\gamma\, n^2\pi^2}\right)^2 n^2\pi^2\,. \tag{3.37}$$

By comparison we obtain

$$D = D_n^{\pm} = \pm\frac{\sqrt{2}\gamma n\pi}{(\gamma n^2\pi^2 + \kappa)}\sqrt{p - \mu_n^{\star}}\,.$$

and from (3.35) it follows

$$w_n^{\pm}(x) = D_n^{\pm}\,\sin n\pi x\,, \quad n = 0, 1, 2, ..., n_{\star}\,.$$

- *Item 4.3.* In this case we have $h = -n^2\pi^2$. If we compare this expression with (3.37) where D is replaced by C, we obtain

$$C = C_n^{\pm} = \pm\frac{\sqrt{2}\gamma n\pi}{(\gamma n^2\pi^2 + \kappa)}\sqrt{p - \mu_n^{\dagger}}\,.$$

From (3.35) it follows that

$$w_n^{\pm}(x) = C_n^{\pm}\,\sin n\pi x\,, \quad n = 0, 1, 2, ..., n_{\dagger}\,.$$

- *Item 3.2.* From (3.32) we first have

$$\begin{aligned} |\chi_1| &= \big(-h - \sqrt{h^2 - 4q}\big)/2 = n^2\pi^2, \\ |\chi_2| &= \big(-h + \sqrt{h^2 - 4q}\big)/2 = \ell^2\pi^2, \end{aligned}$$

which imply

$$h = h_j = \frac{\kappa(\kappa - \gamma^2) - \gamma^2 j^4\pi^4}{\gamma(\kappa + \gamma\, j^2\pi^2)}\,, \quad j = n, \ell\,. \tag{3.38}$$

Taking into account that h is uniquely determined and depends on $||u||_1^2$ through (3.25), h_n and h_ℓ must be equal, namely

$$\frac{\kappa(\kappa - \gamma^2) - \gamma^2 n^4\pi^4}{\gamma(\kappa + \gamma\, n^2\pi^2)} = \frac{\kappa(\kappa - \gamma^2) - \gamma^2 \ell^4\pi^4}{\gamma(\kappa + \gamma\, \ell^2\pi^2)}. \tag{3.39}$$

This equation is satisfied when:

– $\ell = n$. In this case, from (3.33) we obtain $w(x) = (A + B)\sin n\pi x$, so that h takes the form (3.37) where D is replaced by $A + B$. Solutions are then expressed exactly as in *Item 5*.

– $\ell \neq n$ (resonance), provided that

$$\frac{\kappa^2}{\gamma^2} + [\pi^2(\ell^2 + n^2) - \gamma]\frac{\kappa}{\gamma} + \ell^2 n^2 \pi^4 = 0. \tag{3.40}$$

This condition also implies $\mu_n^\star = \mu_\ell^\star$, so that it gives a characterization of the resonant set $\mathcal{R}$ (see Fig. 2), namely

$$\mathcal{R} = \{(\kappa, \gamma) \in \mathbb{R}^+ \times \mathbb{R}^+ : (3.40) \text{ holds for some } n, \ell \in \mathbb{N}\,,\ n \neq \ell\}.$$

When solved with respect to κ, for each pair $n, \ell \in \mathbb{N}\,,\ n \neq \ell$, (3.40) has two positive solutions

$$\kappa_{n,\ell}^\pm = \frac{\gamma}{2}[\gamma - \pi^2(\ell^2 + n^2)] \pm \frac{\gamma}{2}\sqrt{[\gamma - \pi^2(\ell - n)^2][\gamma - \pi^2(\ell + n)^2]}$$

provided that $\gamma \geq \pi^2(\ell + n)^2$. If this is the case, we cannot uniquely determine the values of the amplitudes A and B which appear into (3.33). As a consequence, infinitely many solutions may occur when $(\kappa, \gamma) \in \mathcal{R}$ and $p > r^\star(\kappa, \gamma)$.

□

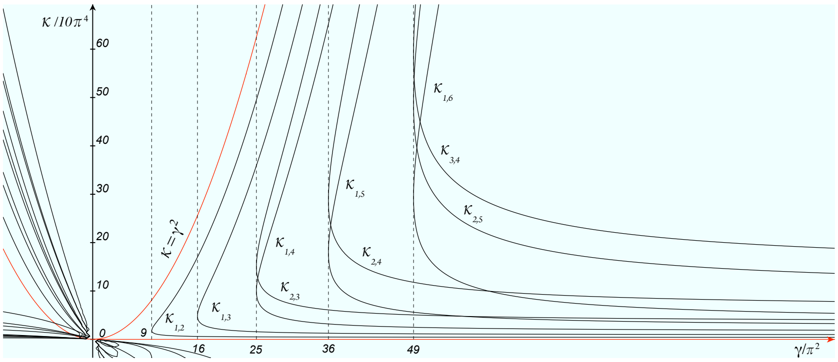

Figure 2: A sketch of the resonant set $\mathcal{R}$ in the γ–κ plane. In red the limiting curves, $\kappa = 0$ and $\kappa = \gamma^2$. The solid curves are the graphics of $\kappa_{n,\ell}$ with $n + \ell = 3, 4, 5, 6, 7$, while the dashed lines denote the bifurcation values of γ, namely $\pi^2(\ell + n)^2$.

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