Boundary flux identification for thermal systems with memory

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Abstract

We solve an input identification problem for a differential systems with memory which is encountered in viscoelasticity and in the theory of thermal processes with memory.

1 Introduction

In this paper we consider a thermal process with memory, described by

$$\theta_t = \left[\Delta - c^2\right] \theta(t) + \int_0^t K(t-s) \left[\Delta - c^2\right] \theta(s) \, \mathrm{d}s \,. \tag{1.1}$$

Here, $c > 0$, $\theta = \theta(x,t)$, $x \in \Omega$, a region with smooth boundary, and $t > 0$. Δ denotes the laplacian and

$$\left[\Delta - c^2\right] \theta = \Delta\theta - c^2\theta \,.$$

We assume that the initial condition is known:

$$\theta(x,0) = 0 \,;$$

an unknown boundary flux $u(t)$ acts on the system, and we assume that we can observe the temperature on the boundary of Ω. Our goal is the identification of the flux on the basis of the measured temperature. Furthermore, we would an *on-line* reconstruction algorithm; i.e. we would like that the extimate $v(t)$ of the flux at time t depends solely on previous measures.

The flux on the boundary is given by

$$\gamma_1\theta(t) + \int_0^t K(t-s)\gamma_1\theta(s) \, \mathrm{d}s$$

where γ_1 denotes the trace on the boundary of the normal derivative,

$$(\gamma_1\theta)(x,t) = \frac{\partial}{\partial n}\theta(x,t)\,, \qquad x \in \partial\Omega$$

and $\partial/\partial n$ denotes normal derivative. So, the the flux throught the boundary is given by

$$\gamma_1\theta(t) + \int_0^t K(t-s)\gamma_1\theta(s)\,\mathrm{d}s = Bu(t) \tag{1.2}$$

where $B \in \mathcal{L}(\mathbb{R}^m, L^2(\partial\Omega))$ and $u \in L^2_{\mathrm{loc}}(0,+\infty;\mathbb{R}^m)$. This is the boundary condition associated to Eq. (1.1).

The observation is

$$y(t) = B^*\theta(\cdot,t)\,, \qquad B^* \in \mathcal{L}(L^2(\partial\Omega),\mathbb{R}^m) \tag{1.3}$$

so that there are as many independent measures as the independent sources of input.

This problem has been examined in [6] using ideas developed in [5], in the absence of memory, i.e. when $K(t) = 0$. Here we shall extend the result to the case $K(t) \neq 0$.

The plan of the paper is as follows: in Section 2 we present an abstract framework which can be used to give a meaning to the mild solutions to (1.1)-(1.2). Assumptions on the kernel $K(t)$ are introduced here. The reconstruction algorithm is then studied in Section 3.

2 The abstract framework

We first need few general remarks, concerning the "concrete" problem (1.1). Let

$$X = H^1(\Omega)$$

and

$$D = \operatorname{dom} A = \{\phi \in X \;:\; \Delta\phi \in X \text{ and } \gamma_1\phi = 0\} \subseteq X\,, \quad A\phi = \Delta\phi\,. \tag{2.4}$$

Furthermore, we denote $\tilde{N}$ the (linear bounded) operator in $\mathcal{L}\left(L^2(\partial\Omega), L^2(\Omega)\right)$ such that $\phi = \tilde{N}u$ solves

$$\left[\Delta - c^2\right]\phi = 0\,, \quad \text{in } \Omega\,, \qquad \gamma_1\phi = u\,. \tag{2.5}$$

Now we proceed to give a meaning to problem (1.1)-(1.2). First of all we note that, introducing $R(t)$, the resolvent kernel of $K(t)$, the boundary condition can be written as

$$\gamma_1\theta(t) = B\left[u(t) - \int_0^t R(t-s)u(s)\,\mathrm{d}s\right]\,.$$

So, we can rename $u(t)$ the function in bracket and we can assume that the boundary condition has the simpler form

$$\gamma_1 \theta(t) = Bu(t)\,.$$

Next, we introduce

$$\xi(t) = \theta(t) - Nu(t)\,, \qquad N = \tilde{N}B$$

and we use

$$\Delta \tilde{N} = 0\,, \qquad \Delta \xi(t) = A\xi(t)\,.$$

So, Eq. (1.1) can be written as (it is convenient to consider now the general case $\theta(0) = \theta_0$, possibly not equal to 0)

$$\xi_t(t) = A\xi(t) + \int_0^t K(t-s)A\xi(s)\,\mathrm{d}s - Nu'(t)\,, \qquad \xi(0) = \theta(0) - Nu(0)$$

where

$$N = \tilde{N}B\,, \qquad .$$

This equation is an "abstract" version of Eq. (1.1) with Neumann boundary condition. From now on we shall work with this "abstract" equation and we shall impose conditions on the operator A (satisfied in the case $A = \Delta - c^2$) which insure well posedness of the problem.

The assumptions in this paper are listed in Section 2.1 below. Now we state the following result, proved in Appendix 4.1:

Theorem 2.1 *Let the assumptions in Section 2.1 hold. Then, Eq. (1.1) with boundary condition (1.2) admits a unique solution $\theta(t)$ belongs to $L^2(0,T;X)$ for every $T > 0$ and depends continuously on the initial condition $\theta_0 \in X = L^2(\Omega)$ and $u \in L^2(0,T;\mathbb{R}^m)$. Furthermore, the function $\theta(t)$ admits a Laplace transform $\hat{\theta}(\lambda)$ in $\Re e\,\lambda > 0$.*

Remark 1) The assumptions in this theorem are modelled on those in [2] where however the case of boundary control is not considered. The main difference is that we consider boundary control. The operator valued function $K(t)$ in [2] has now the special form $K(t)A$ where $K(t)$ is scalar valued.
2) Condition (2.8) implies that in the halfplane $\Re e\,\lambda > 0$ we have

$$\lim_{|\lambda| \to +\infty} \hat{K}(\lambda) = 0\,;$$

3) The condition that $K(t)$ is integrable on $(0, +\infty)$ might be weakened to $e^{-at}K(t)$ integrable, with $a \geq 0$. Analogously, exponential stability of the semigroup maight be weakened. We do not insist on this. $\square$

As suggested by (1.3) and (2.7), in the abstract framework we choose the following observation:

$$y(t) = N^*\theta(t)\,, \qquad N = \tilde{N}B\,.$$

Finally, let us compute the Laplace transform of $\theta(t)$. We proceed formally. The formal computations is justified, when the assumption in Section 2.1 hold, the computations are justified by the arguments in Appendix 4.1.

It is simpler if we compute the Laplace transform of $\xi(t)$ first:

$$\lambda\hat{\xi}(\lambda) = (I + \hat{K}(\lambda))A\hat{\xi}(\lambda) - N[\lambda\hat{u}(\lambda) - u_0] + \xi(0)$$

Using $\xi(0) = \theta(0) - Nu(0)$ we get:

$$\hat{\theta}(\lambda) = [\lambda I - Z(\lambda)A]^{-1}\,\theta_0 - Z(\lambda)A\,[\lambda I - Z(\lambda)A]^{-1}\,N\hat{u}(\lambda) \tag{2.6}$$

where

$$Z(\lambda) = 1 + \hat{K}(\lambda)\,.$$

Formula (2.6) holds in a half plane $\Re e\,\lambda > 0$.

2.1 The assumptions

The assumptions on the operator A are modelled on the properies of the Laplace operator, and of it's resolvent. The assumption on the kernel $K(t)$ is a compatibility condition between the operator A and the kernel itself. Namely we assume:

- The operator A generates a holomorphic semigroup on $H^1(\Omega)$, which is exponentially stable (since we assumed $c > 0$.
- the function $t \to Ae^{At}\tilde{N}$ belongs to $L^1(0,T)$ for every $T > 0$; moreover, there exists $\sigma \in (0,1)$ such that

  $$\|Ae^{At}\tilde{N}\| \le M\frac{e^{-at}}{t^\sigma}$$

  for suitable numbers $M > 0$ and $a > 0$ (note that $a > 0$ since $c^2 > 0$).
- Let $\phi \in D$. Then we have (γ_0 denotes the trace on $\partial\Omega$):

  $$\tilde{N}^*\phi = \gamma_0\phi\,.$$

  Of course, the adjoint is computed in the norm of $H^1(\Omega)$. This is simply seen from the definition of the weak solution of problem (2.5), which is

  $$\int_\Omega \left[\nabla\phi\nabla\psi - c^2\phi\psi\right]\,\mathrm{d}x = \int_{\partial\Omega}[\gamma_1\phi]\,\psi\,\mathrm{d}\Sigma\,.$$

Now we use $\phi = \tilde{N}u$ so that $\gamma_1\phi = u$ and the fact that the left hand side is the inner product in $H^1(\Omega)$ to obtain

$$\langle u, \tilde{N}^*\psi\rangle_{L^2(\partial\Omega)} = \langle \tilde{N}u, \psi\rangle_{H^1(\Omega)} = \langle u, \gamma_0\psi\rangle_{L^2(\partial\Omega)}$$

i.e.

$$\tilde{N}^*\psi = \gamma_0\psi \tag{2.7}$$

for every $\psi \in H^1(\Omega)$.

- the operator A generates an exponentially stable holomorphic semigroup and

$$\rho(A) \supseteq S_{\sigma_0}\,, \qquad S_{\sigma_0} = \{\lambda \in \mathbb{C}\,:\ |\arg\lambda| \le \sigma_0\}$$

and $\sigma_0 \in (\pi/2, \pi)$;

- the function $K(t)$ is integrable on $(0, +\infty)$ and for $\sigma' < \sigma_0$ we have

$$|\arg\lambda| \le \sigma' \implies |\hat{K}(\lambda)\| \le \frac{M}{|\lambda|}\,. \tag{2.8}$$

- The fact that the semigroup is holomorphic and exponentially stable implies

$$\|(\lambda I - A)^{-1}\| = \|(\lambda I - A^*)^{-1}\| \le \frac{M}{|\lambda|} \qquad \lambda \in \mathcal{S}_\sigma\,. \tag{2.9}$$

where

$$\{\lambda\,:\Re e\,\lambda > 0\} \subseteq \mathcal{S}_\sigma \subseteq \mathcal{S}_{\sigma_0}\,. \tag{2.10}$$

Our additional assumption which is satisfied by the Laplacian with Neumann boundary conditions, is that this inequality holds with

$$M = 1\,.$$

This condition is satisfied by the Laplacian with Neumann boundary condition, since it is the infinitesimal generator of a contraction semigroup, see [3, Theorem 8 p. 340 and Proposition 5 p. 347].

The previous assumptions are sufficient in order to study the properties of the solutions of Eq. (1.1) but they are not sufficient to solve the identification problem: for example, they do not prevent the case that $B = 0$, when identification is surely impossible. So, we add the following condition, which is called "minimum phase" in control theory:

- we assume:

$$\det N^*A\,(\lambda I - Z(\lambda)A)^{-1}\,N \ne 0 \qquad \text{in} \quad \Re e\,\lambda \ge 0\,. \tag{2.11}$$

Finally, let us discuss a consequence of the assumption on the kernel $K(t)$. We assumed that $K(t) \in L^1(0, +\infty)$ so that

$$|\hat{K}(\lambda)| \leq \int_0^{+\infty} e^{-(\Re e\,\lambda)t} |K(t)\, \mathrm{d}t\,.$$

Dominated convergence theorem implies in particular

$$\lim_{\Re e\,\lambda \to +\infty} \hat{K}(\lambda) = 0 \qquad \text{uniformly respect to } \mathcal{I}\mathrm{m}\,\lambda.$$

This means that for every $\epsilon > 0$ there exists γ_ϵ such that for every λ with $\Re e\,\lambda > \gamma_\epsilon$ we have

$$|\hat{K}(\lambda)| < \epsilon\,.$$

The consequences of this property that we are going to use are as follows. Let

$$Z(\lambda) = 1 + \hat{K}(\lambda)\,.$$

Then, $Z(\lambda)$ takes values in a prescribed disk of radius ϵ and center 1 if $\Re e\,\lambda$ is large enough. Note that this disk is contained in the sector $|\arg \lambda| < \arctan \epsilon$. Then:

- if $\Re e\,\lambda > \gamma$ then $Z(\lambda) \neq 0$;
- the function $\lambda / Z(\lambda)$ trasforms $\Re e\,\lambda > \gamma$ into S_σ.
- there exists $\sigma \in (0, \sigma_0)$ such that the following holds:

$$\Re e\,\lambda \geq \gamma \implies \left| \arg \frac{\lambda}{Z(\lambda)} \right| < \sigma \tag{2.12}$$

  i.e. the function $\lambda / Z(\lambda)$ transforms the half plane $\Re e\,\lambda > \gamma$ into the sector which appears in (2.9). Hence, this condition connects the operator A and the kernel $K(t)$.

The conditions on the kernel are further discussed in Appendix 4.2

3 Input reconstruction

Now we proceed with the identification problem. We recall our goal: we are going to construct an algorithm which, applied to the output, gives an estimate of the unknown input u. Furthermore, we wish an *on-line algorithm;* i.e. an algoritm which is performed at each time t of an interval $[0, T]$, and which at time t uses only information taken at previous times.

We use the same idea as in [6]. For simplicity, we disregard the fact that the observation is always corrupted by errors. Errors can be taken into account as in [6].

We associate a "model" to our system, which is described by the same boundary value problem as that of θ; i.e. we assume

$$\hat{w}(\lambda) = -Z(\lambda)A\left[\lambda I - Z(\lambda)A\right]^{-1} N\hat{v}(\lambda) . \tag{3.13}$$

This is the same equation as (2.6) with null initial condition.

In the special case that the kernel $K(t)$ is smooth then we have (see (4.21) in Appendix 4.1)

$$w(t) = \int_0^t L(t-r)w(r)\,\mathrm{d}r + \int_0^t L(t-r)Nv(r)\,\mathrm{d}r - A\int_0^t e^{A(t-r)}NBv(r)\,\mathrm{d}r .$$

Here $v(t)$ is an input, a candidate approximant of $u(t)$.

The output of the model system is

$$z(t) = N^* w(t) .$$

We use the same idea as in [6, Section 4] and we choose $v(t)$ as a high gain feedback of the discrepancy of the outputs $y(t)$ and $z(t)$:

$$v(t) = v_\alpha(t) = -\frac{1}{\alpha}[z(t) - y(t)] = -\frac{1}{\alpha}N^*[w(t) - \theta(t)] . \tag{3.14}$$

Note that $v(t)$ depends on the measured output.

The point is to understand the behavior of v when $\alpha \to 0+$. We are going to prove the following consistency results: **Theorem 3.1** *For every $T > 0$ we have*

$$\lim_{\alpha \to 0+} v_\alpha = u$$

in $L^2(0,T)$, for every $T > 0$.

In order to prove this theorem, we need several steps.

- We first prove well posedness of the closed loop.

We consider a fixed value of $\alpha > 0$ and we prove that the system of Eq. (2.6) and (3.13) wirh v given by (3.14) has a unique solution which depends continuously on u. Equation (2.6) does not involve w and it represent a well posed problem, see Theorem 2.1. So, we prove that

$$e(t) = w(t) - \theta(t)$$

is defined uniquely, $e \in L^2(0,T;X)$ for every $T > 0$ and depends continuously on $u \in L^2(0,T;U)$ for every $T > 0$.

The Laplace transform of $e(t)$ is

$$\hat{e}(\lambda) = \left\{ \left[I - \frac{Z(\lambda)}{\alpha} A \left(\lambda I - Z(\lambda) A\right)^{-1} N N^* \right]^{-1} Z(\lambda) A \left(\lambda I - Z(\lambda) A\right)^{-1} N \right\} \hat{u}(\lambda)$$
$$= \left\{ \left[I - \frac{1}{\alpha} A \left(\frac{\lambda}{Z(\lambda)} I - A \right)^{-1} N N^* \right]^{-1} A \left(\frac{\lambda}{Z(\lambda)} I - A \right)^{-1} N \right\} \hat{u}(\lambda) \, .$$

We use the fact that

$$\frac{\lambda}{Z(\lambda)} = \frac{\lambda}{1 + \hat{K}(\lambda)} \, , A \left(\frac{\lambda}{Z(\lambda)} I - A \right)^{-1} = -I + \frac{\lambda}{Z(\lambda)} \left(\frac{\lambda}{Z(\lambda)} I - A \right)^{-1}$$

and $K(\lambda) \to 0$ for $|\lambda| \to +\infty$, $\Re e \, \lambda > 0$ so that the denominator tends to 1. Hence, there exists a suitable half plane, let us say for $\Re e \, \lambda > \gamma_0$, such that

$$\Re e \, \lambda \geq \gamma_0 \implies \frac{\lambda}{Z(\lambda)} \in \mathcal{S}_\sigma$$

so that the brace is a holomorphic and bounded operator valued function. Hence, $\hat{e}(\lambda + \gamma_0)$ is an X-valued H^2-function, which depends continuously on $\hat{u}(\lambda + \gamma_0) \in H^2$. Inverse Laplace transform shows that there is a continuous dependence of $e^{-\gamma_0 t} e(t) \in L^2(0, +\infty)$ on $e^{-\gamma_0 t} u(t) \in L^2(0, +\infty)$; and so, for every T the function $e(t) \in L^2(0, T)$ depends continuously on $u(t) \in L^2(0, T)$.

Let us observe the following formula, which holds for every bounded operator K:

$$N^*(I - K N^*)^{-1} = (I - N^* K)^{-1} N^* \, .$$

Using this equality and (3.14), we have:

$$\hat{v}(\lambda) = -N^* \left[I - \frac{Z(\lambda)}{\alpha} A \left(\lambda I - Z(\lambda) A\right)^{-1} N N^* \right]^{-1} \frac{Z(\lambda)}{\alpha} A \left(\lambda I - Z(\lambda) A\right)^{-1} N \hat{u}(\lambda)$$
$$= - \left[I - \frac{Z(\lambda)}{\alpha} N^* A \left(\lambda I - Z(\lambda) A\right)^{-1} N \right]^{-1} N^* \frac{Z(\lambda)}{\alpha} A \left(\lambda I - Z(\lambda) A\right)^{-1} N \hat{u}(\lambda)$$

It then follows

$$\hat{v}(\lambda) - \hat{u}(\lambda)$$
$$= - \left\{ \left(I - \frac{Z(\lambda)}{\alpha} N^* A \left(\lambda I - Z(\lambda) A\right)^{-1} N \right)^{-1} N^* \frac{Z(\lambda)}{\alpha} A \left(\lambda I - Z(\lambda) A\right)^{-1} N + I \right\} \hat{u}(\lambda)$$
$$= - \left[I - \frac{Z(\lambda)}{\alpha} N^* A \left(\lambda I - Z(\lambda) A\right)^{-1} N \right]^{-1}$$
$$\times \left\{ \frac{Z(\lambda)}{\alpha} N^* A \left(\lambda I - Z(\lambda) A\right)^{-1} N + I - \frac{Z(\lambda)}{\alpha} N^* A \left(\lambda I - Z(\lambda) A\right)^{-1} N \right\} \hat{u}(\lambda)$$

$$= -\left[I - \frac{Z(\lambda)}{\alpha} N^* A \left(\lambda I - Z(\lambda) A\right)^{-1} N\right]^{-1} \hat{u}(\lambda)$$
$$= -\left[I - \frac{1}{\alpha} N^* A \left(\frac{\lambda}{Z(\lambda)} I - A\right)^{-1} N\right]^{-1} \hat{u}(\lambda)$$

Now we observe that **Lemma 3.1** *The following properties hold:*

Property 1) *There exists positive numbers M and γ such that the following inequality holds for every $\alpha > 0$ and $\Re e\, \lambda \geq \gamma$:*

$$\left\| \left[I - \frac{1}{\alpha} N^* A \left(\frac{\lambda}{Z(\lambda)} I - A\right)^{-1} N\right]^{-1} \right\| < M \tag{3.15}$$

Property 2) *For every fixed λ with $\Re e\, \lambda \geq \gamma$ we have:*

$$\lim_{\alpha \to 0+} \left[I - \frac{1}{\alpha} N^* A \left(\frac{\lambda}{Z(\lambda)} I - A\right)^{-1} N\right]^{-1} = 0 \tag{3.16}$$

Once that these properties have been proved, the same argument based on Paley-Wiener theorem as in [6] proves that for every $T > 0$ we have

$$\lim_{\alpha \to 0+} v_\alpha = u \qquad \text{in} \quad L^2(0, T)\,.$$

In order to sketch the proof, let us consider first the case $\gamma = 0$. The function $\hat{u}(\lambda)$ is the Laplace transform of a function $u(t) \in L^2(0, +\infty)$, so, it belongs to $H^2 = H^2(\Re e\, \lambda > 0)$. Using (3.15) we see that $\hat{v}(\lambda) - \hat{u}(\lambda) \in H^2$ i.e. $\hat{v}(\lambda) \in H^2$. Hence, $v(t)$ is square integrable and the L^2 norm of $[v(t) - u(t)]$ is equivalent to

$$\int_{-\infty}^{+\infty} \left\| \left[I - \frac{1}{\alpha} N^* A \left(\frac{i\omega}{Z(i\omega)} I - A\right)^{-1} N\right]^{-1} \hat{u}(i\omega) \right\|^2 \, \mathrm{d}\omega\,.$$

Now, the integrand converges to 0 for each $i\omega$ and furthermore it is bounded by

$$M \|\hat{u}(i\omega)\|^2\,.$$

Dominated convergence theorem now implies that $\|\hat{v}(\lambda) - \hat{u}\| \to 0$ in H^2 for $\alpha \to 0+$ i.e. that $v_\alpha \to u$ in $L^2(0, +\infty)$ for $\alpha \to 0+$.

Let now $\gamma > 0$. In this case, the previous computations make sense in a half plane $\Re e\, \lambda > \gamma$ and Paley-Wiener theorem is not applied directly to $\hat{v}(\lambda)$ but to $\hat{v}(\lambda)(\gamma + i\omega)$. This shows that $e^{-\gamma t} v(t)$ converges to $e^{-\gamma t} u(t)$ in $L^2(0, +\infty)$ and this implies $L^2(0, T)$ convergence on every bounded interval $[0, T]$.

The condition $K(t) \in L^1(0, +\infty)$ might be relaxed to $e^{-at} K(t) \in L^1(0, +\infty)$ for a suitable $a > 0$ but it is clear that this is not really needed since the reconstruction algorithm is *causal;* i.e. the value of $K(t)$ for $t > T$ does not influence $v_{\alpha,h,\epsilon}(t)$ on $[0,T]$. So, we can just replace $K(t)$ with 0 for $t > T$. This does not change $v_{\alpha,h,\epsilon}(t)$ on $[0,T]$ and now $K(t)$ is integrable.

So, in order to complete this argument, we prove Lemma 3.1.

We divide the proof in two parts, the first one having two steps.

Part 1) We first prove the existence of the inverse in (3.15).

This is is in two steps: **step 1)** the image of the operator

$$\left[I - \frac{1}{\alpha} N^* A \left(\frac{\lambda}{Z(\lambda)} I - A \right)^{-1} N \right]$$

is dense; **step 2)** the operator is injective with bounded inverse for every fixed $\alpha > 0$ and $\Re e\, \lambda > 0$. This second part is proved togheter with the existence of the bound M.

We fix $\gamma > 0$ such that

$$\Re e\, \lambda \geq \gamma \implies \frac{\lambda}{Z(\lambda)} \in \mathcal{S}_\sigma \,.$$

The existence of such γ has already been noted.

Step 1: the image is dense Let us see that the image is dense. So, let v be orthogonal to the image, i.e.

$$v = \frac{1}{\alpha} N^* A^* \left(\frac{\lambda}{Z(\lambda)} - A^* \right)^{-1} N v$$

Let use introduce the notation

$$s = \frac{\lambda}{Z(\lambda)} \,. \tag{3.17}$$

Inner product of both the sides with v gives

$$\begin{aligned} \|v\|^2 &= \frac{1}{\alpha} \langle A(sI - A)^{-1} N v, N v \rangle \\ &= \frac{1}{\alpha} \left[-\|Nv\|^2 + \langle s(sI - A)^{-1} N v, N v \rangle \right] \,. \end{aligned}$$

This equality implies that $\langle s(sI - A)^{-1} N v, N v \rangle$ is real and inequality (2.9) gives

$$\left| \langle s(sI - A)^{-1} N v, N v \rangle \right| \leq \|Nv\|^2 \,;$$

i.e., the right hand side is non positive and this shows $v = 0$. Hence, the image is dense.

Step 2) invertibility and the bound M. Now we prove in one shot that when $\alpha > 0$ is fixed then the inverse is a linear bounded operator for each λ (in $\Re e\,\lambda \geq \gamma$) and furthermore we see that inequality (3.15) holds in $\Re e\,\lambda > \gamma$ for every $\alpha > 0$, with a constant M which does not depend on λ or α. If not, then there exist sequences $\{\alpha_n\}$ $\{\lambda_n\}$ and $\{v_n\}$ with $\alpha_n > 0$, $\Re e\,\lambda_n \geq \gamma$ and $\|v_n\| = 1$ such that

$$\left[I - \frac{1}{\alpha_n} N^* A(s_n I - A)^{-1} N\right] v_n \to 0\,, \qquad s_n = \frac{\lambda_n}{Z(\lambda_n)}$$

(the sequence $\{\alpha_n\}$ is constant in the proof of Property 1.) We compute the inner product with v_n and we use again property (2.9) with $M = 1$. We get

$$1 + \frac{1}{\alpha_n}\left\{\|Nv_n\|^2 - \langle s_n(s_n I - A^*)Nv_n, Nv_n\rangle\right\}\,. \tag{3.18}$$

From here we deduce that this inner product cannot converge to zero, which is a contradiction. For this, we note that

$$\begin{aligned}&\Re e\,\left\{\|Nv_n\|^2 - \langle s_n(s_n I - A)^{-1}Nv_n, Nv_n\rangle\right\}\\ &\geq \|Nv_n\|^2 - \left|\langle s_n(s_n I - A)^{-1}Nv_n, Nv_n\rangle\right| \geq 0\,.\end{aligned}$$

So, the brace in the inner product (3.18) is nonnegative, and the inner product cannot converge to zero.

Part 2) Finally, we prove (3.16). In this part of the proof we use the fact that the input space is finite dimensional. Now $s = \lambda/Z(\lambda)$ is fixed and we prove that

$$\lim_{\alpha\to 0+} \alpha\left[\alpha I - N^*A(sI - A)^{-1}N\right]^{-1} = 0\,.$$

It is sufficient to prove that the existence of a number M such that the following inequality holds for $0 < \alpha < \epsilon$ (suitable positive ϵ):

$$\left\|\left[\alpha I - N^*A(sI - A)^{-1}N\right]^{-1}\right\| < M\,.$$

If not, there exists a sequence $\alpha_n \to 0+$ and a sequence $\{v_n\}$ in $\mathbb{R}^m$ such that

$$\|v_n\| = 1\,, \qquad \left[\alpha_n I - N^*A(sI - A)^{-1}N\right] v_n \longrightarrow 0\,.$$

A subsequence, still denoted $\{v_n\}$, converges to v_0 with $\|v_0\| = 1$ and this vector v_0 satisfies

$$N^*A(sI - A)^{-1}Nv_0 = 0$$

in contrast with assumption (2.11).

This finishes the proof.

Remark The condition $K(t) \in L^1(0, +\infty)$ might be relaxed to $e^{-at}K(t) \in L^1(0, +\infty)$ for a suitable $a > 0$ but it is clear that this is not really needed since the reconstruction algorithm is *causal;* i.e. the value of $K(t)$ for $t > T$ does not influence $v(t)$ on $[0,T]$. So, we can just replace $K(t)$ with 0 for $t > T$. This does not change $v(t)$ on $[0,T]$ and now $K(t)$ is integrable.

□

4 Appendices

4.1 Appendix: well posedness of the boundary value problem

We prove Theorem 2.1. The proof uses results from [2]. Let N be the operator $N = \hat{N}B$ and let us introduce the function $\xi(t) = \theta(t) - Nu(t)$. Formally, if $u(t)$ is "smooth" the function $\xi(t)$ solves

$$\xi'(t) = A\xi + \int_0^t K(t-s)A\xi(s)\,\mathrm{d}s - Nu'(t)\,, \qquad \xi(0) = \theta(0) - Nu(0)\,. \quad (4.19)$$

Laplace transform of both sides gives

$$\lambda\hat{\xi}(\lambda) = A\hat{\xi}(\lambda) + \hat{K}(\lambda)A\hat{\xi}(\lambda) - lambda\hat{u}(\lambda) + Nu(0) + \xi(0)$$

and $Nu(0) + \xi(0) = \theta(0)$. So,

$$\hat{\xi}(\lambda) = -\left[\lambda I - A - \hat{K}(\lambda)A\right]^{-1}\lambda N\hat{u}(\lambda) + \left[\lambda I - A - \hat{K}(\lambda)A\right]^{-1}\theta(0)\,. \quad (4.20)$$

It is known from [2] that

$$F(\lambda) = \left[\lambda I - A - \hat{K}(\lambda)A\right]^{-1} = (\lambda I - A)^{-1}\left[I - \hat{K}(\lambda)A(\lambda I - A)^{-1}\right]^{-1}\,.$$

Furthermore, in every sector $S_{\sigma'}$, with $0 < \sigma' < \sigma$, we have

$$\|F(\lambda)\| \le \frac{M(\sigma')}{|\lambda|}\,.$$

So, the function $\lambda F(\lambda)$ is holomorphic and bounded in $\Re e\,\lambda \ge 0$. It follows that

$$\lambda F(\lambda)\hat{u}(\lambda)$$

is the Laplace transform of a function which is square integrable on $(0,+\infty)$; furthermore, this function, as an element of $L^2(0,+\infty;X)$, depends continuously on $u \in L^2(0,+\infty;U)$.

As in [2],

$$F(\lambda)\theta(0)$$

is a Laplace transform of a function in $L^2(0,+\infty;X)$, which depends continuously on $\theta(0)$ (additional regularity of this function can be proved, but it is not used here).

So, equality (4.20) defines a function

$$\hat{\theta}(\lambda) = \hat{u}(\lambda) - \lambda F(\lambda)\hat{u}(\lambda) + F(\lambda)\theta(0)$$

with the properties defined in Theorem 2.1. This is by definition the mild solution of Eq.(1.1) with the prescribed initial and boundary condition.

Note that these arguments show that the solution has a Laplace transform as used in this paper. However, the solution in the time domain is not easily found along this way, since we should compute explicitly the inverse Laplace transform $F(t)$ of $F(\lambda)$, not a simple task. So, it may have an interest to see that when the kernel $K(t)$ is a bit smooth then it is possible to represent the solution as the solution of a Volterra integral equation of the second kind, which is easier to solve numerically. The starting point is the function $\xi(t)$ introduced above and the integrodifferential equation (4.19). Its solution can be written as

$$\xi(t) = e^{At}\left[\theta_0 - Nu(0)\right] + \int_0^t e^{A(t-s)}\left[\int_0^s K(s-r)A\xi(r)\,\mathrm{d}r\right]\,\mathrm{d}s$$
$$-\int_0^t e^{A(t-s)}Nu'(s)\,\mathrm{d}s\,.$$

Formal integration by parts now give the follwong Volterra integral equation for $\theta(t)$:

$$\theta(t) = e^{At}\theta_0 + \int_0^t L(t-r)\theta(r)\,\mathrm{d}r + \int_0^t L(t-r)\tilde{N}u(r)\,\mathrm{d}r - A\int_0^t e^{A(t-r)}\tilde{N}Bu(r)\,\mathrm{d}r\,. \tag{4.21}$$

Here,

$$L(t)\theta = K(t)\theta - K(0)e^{At}\theta - \int_0^t e^{A(t-s)}K'(s)\theta\,\mathrm{d}s\,.$$

In the following, we shall write $N = \tilde{N}B$.

The last integral in (4.21) defines a locally square integrable function for every locally square integrable function $u(t)$. So, the unique solution $\theta(t)$ of this equation is the mild solution of Eq. (1.1).

4.2 Appendix: discussion of the conditions

We recall the assumption, that $(\lambda I - A)^{-1}$ is holomorphic in a certain maximal sector $\mathcal{S}_{\sigma_0}$ with $\sigma_0 > \pi/2$. The assumption of the kernel implies that for every $\sigma \in (\pi/2, \sigma_0)$ we have $\lambda/Z(\lambda) \in \mathcal{S}_\sigma$ provided that $\Re e\,\lambda > \gamma$, where γ is large enough. It has a certain interest to understand when this inclusion holds with $\gamma = 0$ in the case of typical kernels. This will be the case when suitable relations hold between the kernel and the operator A. We examine two cases, the case of Abel kernels and the case of exponential kernels.

1. the example of a Abel kernel $K(t) = 1/(\Gamma(\beta)t^{1-\beta})$, $0 < \beta < 1$ so that

$$\hat{K}(\lambda) = \frac{1}{\lambda^\beta}$$

(principal determination of the power). In this case

$$\frac{\lambda}{Z(\lambda)} = \frac{\lambda^{1+\beta}}{1+\lambda^\beta} \tag{4.22}$$

and we want

$$\left|\arg \frac{\lambda}{Z(\lambda)}\right| < \sigma < \sigma_0$$

for every λ in the right half plane.

We note that $\arg Z(\lambda)$ has the same sign as $-\arg\lambda$. So we can confine ourselves to consider $0 \le \arg\lambda \le \pi/2$. If $0 \le \arg\lambda \le \pi/2$ then

$$\arg \frac{\lambda}{Z(\lambda)} = \left\{\arg\lambda^{1+\beta} - \arg\left(1+\lambda^\beta\right)\right\} \ge 0$$

and so, condition (4.22) is satisfied if

$$\frac{\pi}{2}(1+\beta) < \sigma_0 + \arg\left(1+\lambda^\beta\right).$$

These inequalities hold if

$$\beta < \frac{2}{\pi}\sigma_0 - 1 \tag{4.23}$$

(recall that $\sigma_0 > \pi/2$). This condition seems conservative, but it is not, as we see if we consider the function on the (positive) imaginary axis, i.e. $\lambda = e^{i\pi/2}y$. We have

$$\begin{aligned}\arg\left[\frac{\lambda^{1+\beta}}{1+\lambda^\beta}\right] &= \arg\left[\frac{e^{i(1+\beta)\pi/2}}{1+y^\beta e^{i\beta\pi/2}}\right] \\ &= \frac{\pi}{2}(1+\beta) - \arctan\frac{y^\beta \sin\beta\pi/2}{1+y^\beta\cos\beta\pi/2}.\end{aligned} \tag{4.24}$$

The function

$$t \to \arctan\left(\frac{ct}{1+dt}\right)$$

(with positive coefficients c and d) is increasing. Hence, the maximum of the function in (4.24) is reached at $y = 0$ and we get condition (4.23).

2. The case of an exponential kernel, $K(t) = e^{-\beta t}$, $\beta > 0$. In this case,

$$\frac{\lambda}{1+\hat{K}(\lambda)} = \frac{\lambda(\lambda+\beta)}{\lambda+\beta+1} = \lambda\left[1 - \frac{1}{\lambda+\beta+1}\right].$$

We examine the first quadrant. The fourth quadrant is studied in a similar way.

We see that for every λ in the first quadrant we have $|\arg\,(\lambda/Z(\lambda))\,| < \sigma$ when $\sigma > \pi/2$. In fact,

$$\arg \frac{\lambda}{Z(\lambda)} = \arg \lambda \frac{\lambda+\beta}{\lambda+\beta+1} \in (0, \arg\lambda) \subseteq (0,\pi/2)\,.$$

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