Self action on MV-algebras

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We move from an aesthetic motivation, quoting from a Ball and Hagler paper: Actions on Archimedean lattice-ordered groups with strong unit: *We think of our favorite mathematical structures as being static, but perhaps that is too limited a view. Instead let us imagine that these objects are moving, pulsing vibrating.*

More explicitly they suppose that there is a fixed monoid acting on an algebra.

Following this idea we approached the concept of *Self Action* on MV-algebras, that are algebraic models of Łukasiewicz logic, very often used to model uncertainty and imprecise fenomena.

It is well known that the elements of the Lindenbaum algebra of Łukasiewicz infinite-valued logic are McNaughton functions $f : [0, 1]^n \to [0, 1]$ corresponding to propositions p. We shall be concerned here, for simplicity, with the case $n = 1$. Consider the following example of McNaughton function:

$f(x) = 2x \quad \text{if} \quad x \in [0, 1/2] \quad \text{and} \quad f(x) = 2(1 - x) \quad \text{if} \quad x \in (1/2, 1].$

This function is also known as the tent transformation and it is associated to the formula $(p \oplus p) \wedge (p^* \oplus p^*)$, where p is a propositional variable, $\oplus$ and * are the disjunction and negation operators in Łukasiewicz logic, respectively, and the operator $\wedge$ is defined as $a \wedge b := (a^* \oplus (a^* \oplus b)^*)^*$. If we switch to chaos in deterministic systems it is well known that the tent function is ”chaotic” in the following sense: a function $f : [0, 1] \to [0, 1]$ defines a dynamical system by considering for every $x \in [0, 1]$ (the initial condition) the trajectory $x, f(x), f(f(x)), \ldots, f^n(x), \ldots$. A point x is a periodic point if the trajectory having x as initial point is periodic (in an obvious sense).

The connection between the Łukasiewicz logic and chaotic deterministic systems is the chaotic behavior of the McNaughton functions. The question is if the composition of functions, essential for the considered type of dynamical systems, has any ”logical” meaning and if the chaotic behavior of the McNaughton functions has any interesting interpretation. For example, how to interpret the density of the periodic points for $(p \oplus p) \wedge (p^* \oplus p^*)$?

Composition of McNaughton functions (n=1) can be interpreted as substitution (in propositional many valued logic) so it does have logical interpretation. This was the starting point in considering the algebraic structures studied in this paper. But these structures proved to be interesting for themselves as algebras of ”operators”.

Here, though we focus on the behavior of endomorphisms of free MV-algebras, we are interested in describing, in pure algebraic terms: the action of the functional composition over the one-generated MV-algebra arising from generic logical substitution.

Actually, we are interested in:

the action of the monoid structure of the set of MV-endomorphisms over the MV-algebraic structure

Remark: The talk mainly will be based on results from the paper:

A. Di Nola, P. Flondor, B. Gerla, Composition on MV-algebras, to appear.