UNIVERSITÀ DI ROMA “LAOMA “

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Contents

Chapter 1

Introduction

We study doubly nonlinear fourth order degenerate parabolic equations whose prototype interface: The value

The second one is a class of integral inequalities, so-called “entropy estimates”, which for
p

Unfortunately, the validity of (1.2.4) and (1.2.5) is limited to non-negative functions in $C^1(\Omega)$. To overcome this difficulty, we approximate (P) with problems whose solutions are non-negative and continuously differentiable in space for almost every time, and then use (1.2.4) and (1.2.6) to turn this qualitative information into a quantitative one, which is preserved in the limit. The choice of the approximating problems will be described in Section 4.1 : It is based on adding to the right-hand side of (1.0.1) a second-order operator of the form

$$[L_\varepsilon(u_x)]_x = \varepsilon\left[u_x + |u_x|^{p-2}u_x\right]_x . \tag{1.2.7}$$

The linear diffusion term guarantees sufficient regularity of the approximating solutions, and the nonlinear term (combined with a suitable a, w

(vi) u

to call for "entropy estimates". The main merit of our method is to translate this insight into results on existence and qualitative behaviour of solutions.

The higher dimensional case remains open and deserves a comment: Besides the validity of (1.2.4) and (1.2.5) in higher space dimension, and the many technical complications whose resolution should be extended to the case $p > 2$, our approach can not be applied due to the need for *nonlinear*

which using (1.3.5) can be integrated once with respect to y, obtaining

$$y \gg u$$

such that

$$\lim_{s \,!\, 1} {}^{2p_s\; i\; 2}_{\;s} = 0$$

Chapter 2

Preliminaries

2.1 Embedding and compactness

We begin with an extended version of Gagliardo-Nirenberg's inequality which we will apply in the proof of finite speed of propagation for the support of the solutions. Gagliardo-Nirenberg's inequality was first derived by Gagliardo [36] and by Nirenberg [53]. In the formulation we provide below, some of the summability powers are allowed to be less than one; a proof of this extension may be found in [30]. The additional observation on relevant constants follows immediately from a rescaling argument.

Theorem 2.1.1 (Gagliardo-Nirenberg). *Let* $1 \le r \le \infty$, $p > 0$, $q \in (0;p)$, *In addition, there exists a positive constant c depending only upon r, m, q and N (and*

vL; ((9(onside)49(asuf 5.98 0 TD[(w)]T1.nd)-326(1)]TJBf 6.58 0 TD[(x;b7)-TJ/F7911.95 Tf 6.93 1.8 TD[(\)Farx;b7(

Let us now recall Hardy's inequalities in one space dimension([54]) for $1 \le p \le q \le \infty$ and $1 \le q < p \le \infty$.

Lemma 2.2.2. *Let $-\infty \le a < b \le \infty$ and assume that $1 \le p \le q \le \infty$. For weight functions v, w which are non-negative and measurable on (a, b), consider the quantities*

$$F_R(x) := F_R(x; a, b; w, v; q, p) = \|w^{1/q}\|_{L^q((a,x))} \cdot \|v^{-1/p}\|_{L^{p'}((x,b))}$$

Chapter 3

Modelling and formal analysis

3.1 Lubrication approximation with constant viscosity and no-slip condition

We study the evolution of an incompressible, viscous liquid over an horizontal solid surface. Let $(x; y)$ be the orthogonal system oriented such that x-axis and y-axis are respectively parallel and perpendicular to the solid surface. We have the incompressibility condition

$$\operatorname{div} \mathrm{v} = 0 \tag{3.1.1}$$

and the equation of motion

$$\mathrm{v}_t$$

This nonlinear system of partial differential equations for the velocity $\mathbf{v}$ and the pressure p can be rewritten in the form of

$$u_x + v_y = 0 \tag{3.1.4}$$

$$u_t + uu_x + vu_y = -\frac{p_x}{\rho}$$

$(H2)$

3.2 The contact-line paradox

We are interested in situations where the region wetted by the liquid is bounded, and the liquid spreads over the surface. Unfortunately, it is nowadays well known, for Newtonian fluids, that Navier-Stokes equations, together with no-slip boundary conditions at the liquid-solid interface, yield the following paradox: An infinite energy is needed to make the droplet expand; that is, ".....not even Herakles could sink a solid" (Huh, Scriven [47]).

Proposition 3.2.1. *Let* $a > 0$*,* $\gamma \geq 0$*, and let* $G \in C((0;\infty);(0;\infty))$ *such that*

$$G(H)$$

Remark 3.2.2. Let us make use of the self-similar ansatz (3.2.1) to heuristically infer that for a newtonian liquid, the no-slip condition leads to an infinite rate of viscous dissipation at advancing contact line. Indeed, by (3.1.32) we can compute

$$D = t^{i\frac{10}{7}} \int_0^a y^2$$

where ¿ = ´$(u_y + v_x)$ denotes the shear stress, A 2 f0; 1g, $\tilde{¿}$ and ´$_0$ are positive constants.

and

$$h_t + uh_x \quad = \quad v \qquad\qquad \text{on} \qquad y = h$$

3.5 Travelling wave solutions

Advancing fronts exist for $0 < q < 3$, with $(a > 0)$

$$
\begin{aligned}
f(y) &\approx a(-y)^{\frac{3}{q}} && \text{for } \tfrac{3}{2} < q < 3; \\
f(y) &\approx ay^2 |\log(-y)|^{\frac{2}{3}} && \text{for } q = \tfrac{3}{2}; \\
f(y) &\approx ay^2 + b(-y)^{5-2q} && \text{for } 0 < q <
\end{aligned}
$$

Chapter 4

Existence and qualitative behaviour of solutions for Problem (P)

4.1 Introduction

Let us consider the problem:

8
>

(P)

the other hand, such property is not guaranteed by the sign of initial data: for istance, if $0 < n < 1$ Bernis and McLeod [9] proved the existence of compactly source-type solutions of the support. Here the value $n = 2$ was crucial because only for $n < 2$ it is possible to choose an ®

Let us now recall the definition of solution and the main results.

$$fu(t) > ´g \quad = \quad fx\ 2$$

and a positive constant C (independent of u_0*)*

then a positive time $T^?$ exists such that

Estimates (4.1.21) and (4.1.22) yield in particular a uniform control on a suitable Hölder norm of u in $[0; T_\delta] \times \Omega$, where $T_\delta = \delta^{\beta - 1}$.

In Section 4.3 we show that these solutions are in fact globally defined and positive (the latter implies the former since solutions of (4.1.17) can be continued as long as they

The passage to the limit locally on the positivity set is harmless. On the other hand, thanks to the fact that $^{-} <$

4.2 Existence for non-degenerate problems

Without loss of generality, the eigenvalues are ordered so that 0 =

Collecting (4.2.10), (4.2.11), (4.2.13), (4.2.15), and using also Simon compactness criterion

In view of Lemma (4.2.2), we have

$$X_N(v) \quad \text{¸} \quad 0 \quad 8v \; 2 \; L^p((0$$

for all v 2 L^p

(ii) $u(0) = u_{0^2}$ *in* $H^1(\Omega)$;

$' = ¡ u_{xx}\hat{A}_{(0;t)}$ in (4.3.1), and after integrations by parts we obtain

$$\frac{1}{2}\int_{\Omega}$$

In view of the above, the set is not empty and $T^{*} \geq T_0$. We will now show that $T^{*} > T_{\delta}$

By Hölder and Young's inequalities, we obtain for I_3

$$|I_3| \leq \left(\iint_{\Omega_t} |G''_{\delta,\sigma}(u)|^p \, m_{\delta,\sigma}(u) \, |u_x$$

4.4 The limit ¾ # 0

The aim of this section is to let ¾ # 0 in (*P*

In addition, for almost every t 2 $(0; T_\pm)$ *and every function Á as in (4.1.15)*

$$\frac{1}{2}\int_\Omega Á^{3p}\, u_x^2(t) \quad +$$

and (4.4.1) holds true. Combining (4.4.12) with (4.4.17) we obtain (cf. Corollary 8.4 in [60])

$$u_\sigma \longrightarrow u \quad \text{in} \quad L^2((0,T_\delta);H^1(\Omega)). \tag{4.4.18}$$

• *Proof of (4.4.3)*. We claim that

$$u_{xxx} \in L^p_{loc}(\{u>0\}); \tag{4.4.19}$$

$$u_{\sigma xxx} \longrightarrow u_{xxx} \text{ in } L^p_{loc}(\{u>0\}); \tag{4.4.20}$$

which using also (4.4.11) implies (4.4.3), and using also (4.4.10) implies that

$$u_{\sigma x} \longrightarrow u_x \qquad \text{in } L^p_{loc}(\{u>0\}); \tag{4.4.21}$$

$$u_{\sigma xx} \longrightarrow u_{xx} \qquad \text{in } L^p_{loc}(\{u>0\}). \tag{4.4.22}$$

To prove (4.4.19) and (4.4.20) it is sufficient to show that

$$u_{xxx} \in L^p(J\times I); \tag{4.4.23}$$

$$u_{\sigma xxx} \longrightarrow u_{xxx} \text{ in } L^p(J\times I) \tag{4.4.24}$$

for any open rectangle $J\times I$ such that $\overline{J}\times\overline{I}$ s3 - 3Tf 1 0 . 2 3 0TD [(£) a3i J / F3 1 . 08BT / F3 1 1 . 9

(cf. Lemma 4.2.2), we see from (4.4.30) that

$$\iint C \quad \tilde{j}$$

$L^p(fu > 0g)$. Since (by (4.4.12)) u

(the last equality follows from (4.4.13)). To prove (4.4.39), for $\eta > 0$ we split as before the domain of integration. On $\{u > \eta\}$ the convergence is straightforward in view of (4.4.21), whereas on $\{u \le \eta\}$ we have

$$\iint_{\{u \le \eta\}} |A(u_{\sigma x})\, \varphi_x| \le \frac{C}{\left(\inf_{s \in (0,\eta)} G''_{\delta,\sigma}(s)\right)^{\frac{p-1}{p}}}$$

Proposition 4.5.1 (Extension of Bernis estimates). *Let ¡1 · a < b · 1, p ¸*

and a positive constant C exists, depending only on n and p, such that:

$$\int_\Omega \rho^{3p} \left| u^{\frac{n+p}{3p}} \right|^{3} dx \le C \int_{\{u>0\}} \rho^{3p} u^n |u$$

(each term is finite for $p \geq \frac{4}{3}$). Since $(n -$

Young's inequality and (4.5.15) then yield

/

Since the same argument applies to b, we conclude that $\left(u^{\frac{2(n+p)}{3p}}\right)_x = 0$ on $\partial\{u > 0\}$, which in turn implies, after two integration by parts, that

From this we easily infer (cf. also (4.4.16)) that

$$u_2 \quad i^{\,¤ha\,t}$$

To prove (*iii*) in Definition 1, we pass to the limit as $\epsilon \downarrow 0$ in (4.4.5). For a given test φ, let T such that $\operatorname{supp}(\varphi) \subset [0; T) \times \overline{\ }$

Since (again by (4.1.11)) mass is conserved, we conclude that

$$\lim_{t\to\infty} u(t;x) = ⨍_\Omega u_0.$$

It remains to prove (4.1.14). The starting point is (4.4.6): For any T (with ε so small that $T < T_\varepsilon$), almost every $t \in (0;T)$ and every function ϕ as in (4.1.15), it holds and

$$\int_{\{u_2>0\}} Á^p \, m_2(u_2)\, |u_{2x}|^p \quad \cdot$$

Estimates (4.7.6), (4.7.7) and (4.7.8) allow to select a subsequence (still denoted by u_a)

for some $^{-}$ > 0. We preliminarily observe that (arguing as in (4.7.11)) *ku*

4.8 Finite speed of propagation and waiting time

In this section we prove theorems 2 and 3. To this aim we preliminarily need a qualitative result guaranteeing boundedness of the support:

satisfies the inequality

$$G_T(½) \quad \cdot \quad \frac{C\,T}{(½ \; ¡ \; r)^{\frac{(n+7p\;¡\;6)\;(n+p)}{4}}} \; (G_T(r))^{\frac{n+p}{2}} \qquad (4.8.14)$$

for all 0

Combined with (4.8.2) and (4.8.8), this implies for arbitrary 0 < r < ½ that

$$\sup_{t2(0;T)} \mu Z_{½}^{1} u(t)$$

and therefore, by the definition of $R(T)$,

$$R(T) \le \rho_0(T) = C\,(R(T))^{\frac{\beta}{\alpha}}\,\|u_0\|^{\frac{(n+p)(\beta-1)}{}}$$

Chapter 5

Macroscopic behaviour of weakly shear-thinning liquid films

5.1 Introduction and results

We study the spreading of a thin droplet of viscous liquid on a plane surface driven by

Section 3.4):

h_x $\bar{\bar{\bar{}}}$ ${}_{@fh>0g}$ = 0; lim f3.0cD[(x)]TJ/F67.97Tf4.2370TD[(;) !TJ/F57.97Tf-18.46TD[(;)]TJ/F77.97Tf4.930T where

$$^{2} := b\,(7 \cdot t)^{i\ \frac{5}{7}}; \qquad {}^{2}p^{\,i\ 2}\ \text{¿}\ 1:$$

We may restate (5.1.10) and (5.1.11) in the original variables $(t; x)$ defining

hx

Provided (5.1.14) and (5.1.15) hold, the previous analysis therefore suggests that solutions of (5.1.1)–(5.1.3) are such that

$$h(t;x) \text{ » } \frac{3M}{2[x_c(t)]^3} \; \mathrm{i} \, [x_c(t)]^2 \;\; i$$

respect to the slip regularization, and might make the shear–thinning thin-film equation (5.1.1) more reliable in the effective modelling of spreading droplets.

Our last comment concerns the time T_0 after which we assumed the term

Hence we obtain the esplicit form:

$$u''' =$$

In particular, (5.1.9) implies that

$$\lim_{s \to \infty} F_s(y;u) = \frac{y}{u^2} \quad \text{loc. unif. for } (y;u) \in [0$$

These functions satisfy

$$
\begin{cases}
v_s''' = G_s(\xi; v_s); \; v > 0 \quad \xi \in (0; 1) \\
v_s(0) = 1 \\
v_s'(0) = 0 \\
v_s(1) = 0; \; v
\end{cases}
$$

$-$

is well defined, and in view of Theorem 5.1.2

$$\xi_s(v) \to \sqrt{1-v} \quad \text{uniformly in } [0,1]. \tag{5.4.1}$$

The functions

$$w_s(v) := (v_s'(\xi_s(v)))^2$$

are such that

$$-v_s'''(\xi_s(v)) = \frac{1}{2}\sqrt{w_s(v)}\, w_s''(v) < 0 \tag{5.4.2}$$

and solve

$$\xi_s(v) = \frac{A_s^3}{a_s^4} v^2 \left(-\frac{1}{2}\sqrt{w_s}\, w_s''\right) + \frac{{}_s^{2p_s-2} A_s^{2p_s-1}}{a_s^{3p_s-2}} v^{p_s} \left(-\frac{1}{2}\sqrt{}\right.$$

Hence we write

$$\begin{aligned}\int_0^1 vG_s(v)\,dv &= \int_0^{v_{s;\sigma}} vG_s(v)\,dv + \int_{v_{s;\sigma}}^{\sigma} vG_s(v)\,dv + \int_{\sigma}^{1} vG_s(v)\,dv \\ &= I_1 + I_2 + I_3.\end{aligned}$$

Since

that

$$v > v_{s;¾} \quad (\;) \quad µ \frac{{}^2{}_s}{¾\frac{1}{ps\;i\;2}} ¶ \; a_s »_s(v) < A_s \quad (\;) \quad a_s^4 \; {}_{ss}$$

Since by (5.4.8)

$$\lim_{s \to \infty} \frac{\log \left(\frac{\sigma}{v_{s,\sigma}} \right)}{\log \left(\frac{A_s}{\epsilon_s a_s} \right)}$$

Since by (5.4.10)

$$´^2 G_s(´) \cdot \quad a_s^4$$

Chapter 6

Existence of quasi-self-similar

and prove existence and uniqueness for the following problem

$$(P_a)\quad \begin{cases} u''' = F(r; u \\ \end{cases}$$

To this aim, we consider the approximating problem

$$(P_{\pm}) \quad \begin{cases} u''' = F(r;u \end{cases}$$

By (6.1.8),

$$u(r) = \pm + \int_0^a G(r;t)\,F(t;v(t))\,dt;$$

Since

0 1.9.975Tf-49.5244D[( ]) -978 (as)]TJ/F305425Tf-49.5Tf5.60TD[( ))]TJ/F411.95Tft.79TD[(±)]TJ/F154F111.95Tf5.60TD[( ))]TJ/

Hence $u_\pm(r) \cdot C$ independently by ±. In the same way one proves that $ju^0_\pm(r)j \cdot C$. □

Proof of Proposition 6.2.1. We wish to pass to the limit as ± # 0 in the approximating problems. By (2) of Lemma 6.2.4, there exists a subsequence (still labelled by ±) such that

$$u_\pm \;!\; u \quad \text{uniformly in} \quad [0; a] \quad \text{as} \quad \pm \# 0:$$

Since $u > 0$ in $[0; a)$ by (1) of Lemma 6.2.4, then

$$u^{000}_\pm = F(r; u_\pm) \;!\; F(r; u) \quad \text{uniformly in compact subsets of} \quad [0; a):$$

On the other hand, $u^{000}_{\pm\pm}$

Therefore the function

$$h(r) = r\,w\,w'' - w\,w' - 1$$

Then

Chapter 7

Appendix

and2

Lemma 7.0.6. *Let* $u = u_{\varepsilon,\delta,\sigma}$ *be a solution of* $(P_{\varepsilon,\delta,\sigma})$ *in the sense o*[illegible]*Defin*[illegible]*tio*[illegible]*4.3.1 a*[illegible]*d let* T *be its maximal time of existence. A*[illegible]*sitive constant,* [illegible]*ndepe*[illegible]*dent of* $\varepsilon \in (0,1)$ *and* $\sigma \in (0, \varepsilon)$ *exists such that for all* x

[F1 11.95 Tf 11.29 9.1[((0)]TJ/F3 1T of De TD[()]TJ]3 11.95 T

By (4.3.6), (4.4.11), (4.4.14) and Hölder inequality

$$I_3 \cdot \int_t$$

By contradiction, assume that there exists $\pm > 0$ and $\tilde{a} > a$ such that

$$u(x)^{n \text{ ¡ } 2p+1}$$

and in particular Z ~

so that i 3

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