

Contents

Preface	iii
I Simulation of stationary linear Hawkes processes	1
Introduction	2
1 Generalities on point processes	9
1.1 Point of view I: Random Measures	9
1.1.1 Integrals with respect to random measures	11
1.2 Point of view II: Random sets	11
1.2.1 Finite-dimensional distributions	12
1.3 Poisson Processes	14
1.3.1 Exponential formula	14
1.4 Marked point processes	14
2 Stochastic intensity	16
2.1 Preliminary detail	16

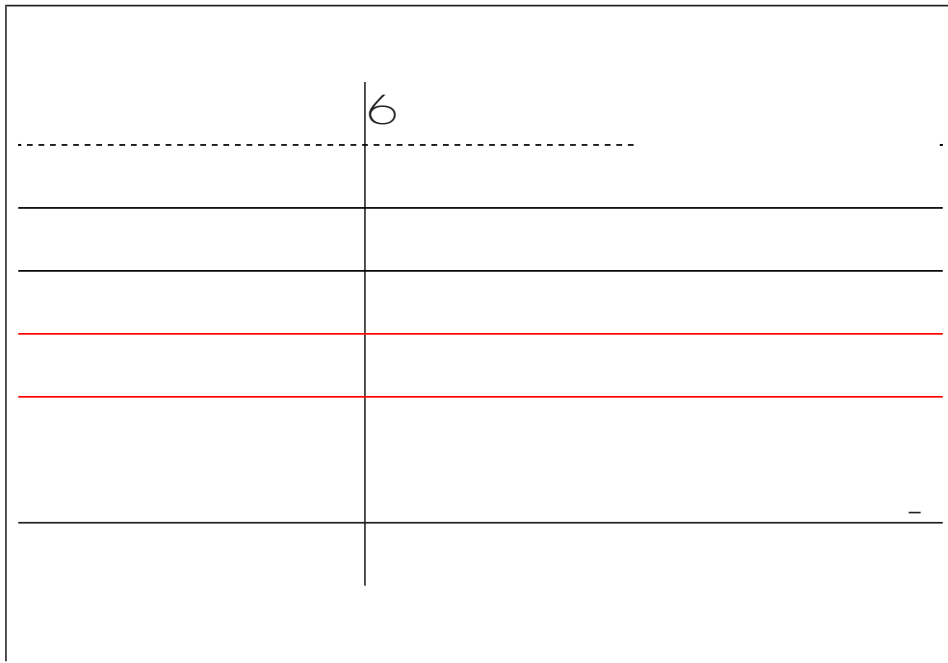
Preface

This PhD Thesis is divided into two parts:

- I Simulation of stationary linear Hawkes processes**
- II Algoritmi stocastici ACO per l'ottimizzazione combinatoria**

Acknowledgments

I wish to thank Giovanna Nappo for having guided me carefully during this work and also in the preparation of my previous thesis, the graduation one.



Recently Møller and Rasmussen (2004, [22]) have given an exact simulation algorithm. Though their algorithm is given also for marked Hawkes processes, it works only when $h(t)$ is proportional to the density function

The analytic expression of the survival function $\bar{F}_E(\cdot)$ in explicit form: strongly connected and immediately following is the problem

² Proposition 4.10:

$$P(A_a^c) = \frac{1}{2} \exp\left(-\frac{1}{2} a^2\right)$$

for $F_E(t)$ can be obtained in the light tail case, i.e. when

Chapter 1

$E \in \mathcal{E}; E) \quad 76TJ/F2410.91Tf41630TD[() -57nitioeTJ/F2210.91Tf8.38.0TD[(is)fTJ/F1010.91Tf12.1.$
 $DeA428(0)50(andom)-358(0)asureN$

CHAPTER 1. GENERALITIES ON POINT PROCESSES

10

Moreover, if for all $C \in \mathcal{E}$, $N(C)$ takes its values in $\bar{\mathbb{N}}$, N is called **point process** on $(E; \mathcal{E})$; in particular, $N(C)$ is a random variable for all $C \in \mathcal{E}$.

Definition 1.3 (Probability distribution). Let N be a random measure on $(E; \mathcal{E})$ and let P be a probability measure on $(\bar{\mathbb{N}}; \mathcal{F})$. The probability P_N on $(M(E); M(\mathcal{E}))$ defined by

$$P_N = P \circ N^{\#}$$

is called

The sequence t_n above is called the sequence of *event times*

$E_j d(x; a) < Rg$. A set $C \subset B(E)$ is *relatively compact* if and only if it is

The most important point processes are those for which $N(A)$ is Poisson distributed, and together with a condition of independence give rise to the Poisson processes, which are described in the next section.

Z

Chapter 2

Stochastic intensity

Most of the results here presented are based on Brémaud [3] and [4], whereas Paragraph is based on Brémaud, Nappo and Torrisi [6], and on the paper by Møller and Rasmussen [22].

if for each $a \in \mathbb{R}$ the mapping $(t, \omega) \mapsto X(t, \omega)$

2.2 Examples of point processes

2.2.3 Linear and nonlinear Hawkes processes

A special kind of marked point process is the *linear marked Hawkes process*, and it is defined as the process whose intensity has the form

the other points can be divided into clusters, or offspring processes. The cluster related to the offspring of the ancestor/immigrant with immigration time t_i is denoted by $C(t_i)$, and t_i is referred to as the *center* of the cluster $C(t_i)$. The relative ancestor has a mark z_i , whose distribution is Q . All the clusters, as seen from their own center, have the same distribution. Let $F = F_E$

Chapter 3

Renewal theory

This chapter studies the theory of renewal processes, already introduced in Section 2.2, but mainly the renewal equation: if G is a distribution function, then the equation is

$$H(t) =$$

a convex function.
 Moreover,

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi_n(x) &= \lim_{n \rightarrow \infty} \int_{(0;1)}^Z e^{-ny} dG(y) \\ &= \int_{(0;1)}^Z e^{-ny} dG(y) \end{aligned}$$

and, as $0 \leq e^{-ny} \leq 1$, the Dominated Convergence Theorem can be applied, and so it holds

$$\lim_{n \rightarrow \infty} \phi_n(x) = \int_{(0;1)}^Z e^{-ny} dG(y);$$

but $\int_{(0;1)}^Z e^{-ny} dG(y) = \int_{(0;1)}^Z e^{-ny} dG(y)$

it will be understood that μ and H vanish on the negative half-axis; the limits of integration may then be replaced by $-\infty$ and ∞ and the renewal equation may be written in the form of the convolution equation

$$H = \mu + G * H;$$

(A similar remark applies to all convolutions in the sequel.) The renewal equation is termed *defective* if $\rho < 1$, *proper* if $\rho = 1$ and *excessive* if $\rho > 1$.

Some examples where the renewal equation arises naturally are now following: in the first one it is shown the announced renewal equation for U_ρ , while in the second one the term

Hand

Therefore,

$$Z(t) = P(fL_1 > tg) + E \sum_{i=1}^{\infty} X_i(t_j L_1) 1_{fL_1 \cdot tg} :$$

$$\sum_{i=1}^{\infty} X_i(t L_1)_{f_1 \cdot t}$$

3.1.2 Statement of the Renewal Theorem

In the sequel G will be assumed to be such that $G(0) = 0$, so that the condition ${}^\circ G(0) <$

This last relation suggested to Feller the introduction of the functions he termed *direct Riemann integrable*⁴.

The classical Riemann integral of a function μ is defined in terms of approximating finite step functions, and it is therefore plausible that the limit relation (3.11) should hold whenever μ is Riemann integrable. This point, to be made clear, needs to recall the definition of the Riemann integral of μ over a finite interval $0 \leq x \leq a$. It suffices to consider partitions into sub-intervals of equal length $h = a/n$. Let \underline{m}_k be the largest and \overline{m}_k the smallest number such that

$$\underline{m}_k \leq \mu(x) \leq \overline{m}_k$$

$${}^2 \mathcal{H}(h) \text{ ; } \mathcal{H}(h) = h \frac{h_P}{k_1 \bar{m}}$$

The equivalence of the two assertions in Theorem 3.5 is to show now, first that (3.7) implies (3.8) and then the converse. The Definition 3.7 will be

But $\underline{H}(x)$.

$y \neq 1$; moreover, as $r \rightarrow \infty$ the mass of G^{*r} tends to be concentrated at 1. Therefore, for large r the integral in (3.14) depends essentially only on large values of y , and for such values $\phi(x_j, y)$ is close to $\phi(0)$. Letting $r \rightarrow \infty$

is equicontinuous and

$${}^3_n(x) = \gg$$

CHAPTER 3. RENEWAL THEORY

Definition 3.12 (Malthusian parameter). *The Malthusian parameter for ρ and G*

CHAPTER 3. RENEWAL THEORY

The Lemma which is following, gives, perhaps the best qualitative description of the subexponential r.v.s.: its proof needs that a definition and a theorem are preliminarily stated (see [1] about this arguments).

Definition 3.18. A positive, Lebesgue measurable function \hat{A} defined on $(0; \infty)$ is *slowly varying at ∞* (written $\hat{A} \in \mathcal{R}_0$) if

$$\lim_{x \rightarrow \infty} \frac{\hat{A}(tx)}{\hat{A}(x)} = 1; \quad t > 0;$$

Theorem 3.19 (Representation theorem for regularly varying functions). If $\hat{A} \in \mathcal{R}_0$, then

$$\hat{A}(x) = c(x) \exp \int_z^x \frac{\pm(u)}{u} du; \quad x > z;$$

for some $z > 0$ where $c(x)$ and $\pm(x)$ are measurable functions, such that $c(x) \neq 0$ and $\pm(x) \rightarrow 0$ as $x \rightarrow \infty$. The converse implication also holds.

Example 3.18 (Tij/F1311)5005f.260TD[20] where

Proof. Define

$$\mathbb{Q}_n := \sup$$

Chapter 4

Approximation theory

This chapter deals with the approximation of a stationary linear Hawkes process

$$N(C) = \sum_{n \geq 1} \mathbf{1}_C(T_n); \quad (4.1)$$

i.e. is a counting process $N(t)$ with intensity

$$\lambda(t) = \rho + \int_{t-1}^t h(t-s) N(ds) \quad (4.2)$$

where $C \in \mathcal{B}([0; +\infty))$, $\rho \in [0; +\infty)$, $h : \mathbb{R} \rightarrow \mathbb{R}_+$ is a nonnegative

has sequence of event times $\{T_n\}_{n=0}^{\infty}$ and intensity

$$e_n(t) = \lambda_n + \sum_{j=0}^{n-1} \lambda_{n-j} I_{[t_j, t_{j+1})}(t)$$

This definition has an analogous one in terms of random variables.

Definition 4.4. *Let $(\mathcal{F}_i; P_i)$*

Furthermore, if there exists a function $L(s)$ integrable in

The analytic expression of $F_E(t)$ (and then of $\bar{F}_E(t)$), however, is not known; so, other tools have to be used to calculate $P(A_a)$. The problem, at this point, is

is considered as a subexponential one. Moreover in the following X is as-

Remark 4.12 (More about the extinction time). *Inequality (4.10)*

Furthermore,

$$\lim_{t \rightarrow 1} \overline{G}^{n_i}(t_i, \bar{t})_i \overline{G}^{n_i}$$

G

$$\lim_{t \rightarrow 1} \overline{G}^{n_i}(t_i, \bar{t})_i \overline{G}^{n_i} = G$$

follows, for t ,

implies that $\int_t^R h^\alpha G^\alpha$

$\max_{i=1, \dots, m} \bar{t}_i(\lambda)$, and for all m ,

$$\frac{\int_0^R \frac{1 - \rho_0(u)}{G(u)} du}{G(t)} \leq \lambda^m$$

and therefore,

$$\begin{aligned}
 \int_a^{Z_{+1}} \bar{F}_E(s) ds \cdot \int_a^{Z_1} L(s) ds &= K \int_a^{Z_1} ds \int_a^{Z_1} g(t) dt = K \int_a^{Z_1} dt g(t) \int_a^{Z_t} ds \\
 &= K \int_a^{Z_1} dt (t_j - a) g(t) \\
 &= K E \int_a^{Z_1} (X_j - a)^+ dx ;
 \end{aligned}$$

where X has distribution G . □

Proof.

gives

$$\int_a^{\infty} P(T^E > t) dt \stackrel{(4.46)}{=} \frac{1}{\lambda} K(P)$$

Proof. The proof is based on the observation that if $\mu \geq (0; \frac{1}{2})$, then $fX_1 + X_2 > xg \mu fX_1 > (1 - \mu)xg [fX_2 > (1 - \mu)xg [fX_1 > \mu xg \setminus fX_2 > \mu xg$:

Indeed

$$fX_1 + X_2 > xg = fX_1 + X_2 > xg \setminus A_1 [A_2 [(A_1^c \setminus A_2^c)] ;$$

where

$$A_1 = fX_1 > (1 - \mu)xg; \quad A_2 = fX_2 > (1 - \mu)xg;$$

and therefore

$$A_1^c \setminus A_2^c = fX_1 \cdot (1 - \mu)xg \setminus fX_2 \cdot (1 - \mu)xg;$$

When $(1 - \mu) < 1/2$ then the set $fX_1 + X_2 > xg \setminus A_1^c \setminus A_2^c$ is empty. This is not the case when $(1 - \mu) > 1/2$. Then the following property holds

$$fX_1 + X_2 > xg \setminus fX_1 \cdot (1 - \mu)xg \mu fX_2 > \mu xg$$

together with an analogous property obtained interchanging the role of X_1 and X_2 , and therefore

$$fX_1 + X_2 > xg \setminus A_1^c \setminus A_2^c \mu fX_1 > \mu xg \setminus fX_2 > \mu xg;$$

□

The previous Lemma is the basis of the second announced result.

Proposition 4.14. *Let $G(t) = t$*

where α, β, γ are strictly positive, and $\tilde{A} = \beta^{-1}\gamma$, the function \tilde{A} can be computed explicitly⁶

$$\tilde{A}(\mu) = \frac{1}{\mu^{\alpha\gamma}};$$

so that

$$\sup_{x, T} \frac{\overline{G}^{\beta^2}(x)}{\overline{G}(x)} \leq 2 \frac{1}{(1 + \mu)^{\alpha\gamma}} + \frac{1}{\mu^{\alpha\gamma}} \frac{\tilde{A}}{1 + \frac{\mu T}{\tilde{A}}};$$

As a consequence, for any $\epsilon > 0$, and for $\beta = 1/\epsilon$, one can take

$$\mu = \mu_2 < 1/\epsilon$$

such that

$$2 \frac{1}{(1 + \mu)^{\alpha\gamma}} = 2 + \frac{\epsilon}{2};$$

and then one can take $T = T_2$ in such a way that

$$\frac{1}{\mu^{\alpha\gamma}} \overline{G}(\mu_2 T) = \frac{1}{\mu^{\alpha\gamma}} \frac{\tilde{A}}{1 + \mu_2};$$

With this choice (4.47) holds, and

$$\bar{G}(T_2) = \frac{1}{1 + \frac{T_2}{A}} = \frac{1}{1 + \frac{1}{\mu_2}} \cdot \frac{1}{i} \cdot \frac{1}{1}$$

Then one can consider the Hawkes process with $\rho > 0$, and

so that

$$P(A_a^c) \cdot$$

and $\bar{\mu}$

$$\bar{\mu} = \mu_2 = \begin{cases} \frac{8}{3} < 1; & \frac{3}{1} < 1; & \frac{1}{3} \frac{\rho_{\bar{h}}}{\bar{h}+1} \frac{1}{\bar{\mu}}; & \text{for } \bar{h} > h_0^2 \\ \mu_2 = \frac{1}{2}; & & & \text{for } \bar{h} < h_0^2; \end{cases}$$

depend only on \bar{h} and $\bar{\mu}$, actually $a = a(\pm$

20

40

60

Chapter 5

Simulations

This chapter contains a discussion of the simulations obtained according to the theory displayed in the previous chapter, [tomb\(ula0-7G-the\)-346aa0-7briefly descrip-o](#)

Then μ^{α} can be viewed as the intensity process of a locally homogeneous

² the minimum value of the intensity function is $\tau > 0$;

²

Algorithm L2: marked self-exiting process.

Algorithm L2, which is following, is a slight change of Algorithm L1: it

If a sample function of the intensity function $\lambda_s(t; T_0; Z_0; \dots; T_n; Z_n)$ is not

before time j a may contain o spring in $[0; T)$. In this sense one can say that Algorithm H1 simulate (exactly) a process without ancestors.

self-exciting process is nothing but an immigrant-birth process which is composed of an homogeneous Poisson immigrant process with rate ρ and nonhomogeneous Poisson descendant with rate $h(s)$.

In the following two examples h is a decreasing function such that $h \cdot h$

Denoting by M an arbitrary mark space equipped with a \mathbb{R} -field, such a process $X = \{(t_i, Z_i)\}_{i \in \mathbb{N}}$ is defined in the following.

Definition of a marked Hawkes process with unpredictable marks

² Each event (or time) t_i is one of two types: an immigrant or an offspring, such a

generations of o spring t_j to t_i

Remark 5.11. Consider a function $f \in C$. The condition $f \cdot \tilde{A}(f)$ or $f, \tilde{A}(f)$

Chapter 6

Simulation Programs in C++

This Programm generates the Poisson process of the ancestors and at each time of this generates a Hawkes process. It is written in C++ Language. A brief comments follows the instruction, whereas another one comes first the different parts of the program.

The aim of the code is the simulation of the process N_a , but the process N_a^θ is actually simulated, N_a^θ s.t.

$$N_a \cdot N_a^\theta \cdot$$


```
void stampa(ostream& os)
{
    for (register unsigned int i=0; i<punti.size()-1; i++)
        os << punti[i] << ' ';
    os << punti[punti.size()-1] << '\n';
}
namespace FUNZIONE_H
{
    const double pi=3.14159;
    long double x_0, rho, var1;
    double alfa, beta, sigma, mu, integrale, h_segno, acc;
    long double tau, teta;
```

These variable are used in the definition of the intensity. The declaration of

```
double valore;  
valore=0;  
valore=alfa*exp(-beta*tempo);  
return valore;  
}
```

Lognormale

Expression of $h(t_j - t_1) + h(t_j - t_2) + \dots + h(t_j - t_n)$

```
double Lognormale_integranda(double tempo, int n) { int m; acc=0;
```



```

        break;
case 4:  cout<<"Di stribuzi one di Wei bul l \n";
        cout<<"teta=";
        cin>>teta;
        cout<<"tau=";
        cin>>tau;
        cout<<"1>h_segnato=";
        cin>>h_segnato;
        var1=double Wei bul l_i ntegranda(double tempo, i nt n);
        funzi one1(long double, i nt)=var1;
        double funzi one2(double tempo)=double Wei bul l (double tempo);
        break;
    defaul t:
        cout<<"Scel ta effettuata: \n";
};
//cout<<endl <<endl ;

```

*The simulation program is starting
Inizialization of randomness*

```

i nt seme=time (NULL);
srand(seme);

```

The simulation in the interval $[j \ a;0]$ is following

```

out<<"Simul ating now between $0$ and $-a=-$" <<a<<endl ;
S=0;
rand_exp1=0;
al pha1= h_segnato*rho/x_0;
whi le(S>-a)

    {
        punti . clear();
        rand11=rand()
    }

```

Generation of a random number belonging to the interval $[0$


```

out<<" ";
out<<"Accepted " <<punti [n]+S<<"\n";
}

```

```

}

```

End of "while (sp < T1)"

```

}

```

*End of "else" and end of the procedure to generate the Hawkes process
Information about the size of the vector are following*

```

out << " The Hawkes process generated " << punti.size()-1<<" descendants"<<endl;
}

```

end of while(S > a) The simulation in [0; T] is following;

```

out<<"\n";
cout<<"Simulating now between 0 and T="<<T<<endl;
out<<"Simulating now between 0 and T="<<T<<endl;
cout<<"\n";
out<<"\n";
S=0;
rand_exp2=0;
int k=0;
while (S<T)
{
    punti positivi.clear();
    punti.clear();
    k=k+1;
    rand21=rand()%RAND_MAX;

```

Generation of a random number belonging to the interval [0; RANDMAX];

```

rand22=((double)rand21/(RAND_MAX+1));

```

Generation of a random number belonging to the interval [0;1];

```

rand_exp2=- (log(1-rand22))/nu;

```

Exponential r.v. with parameter ρ

```

S=S+rand_exp2;

```


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Part II

Introduzione

Il presente lavoro espone i risultati di uno studio svolto durante un periodo

valori possibili di q in modo decrescente all'aumentare della distanza da p , con un valore massimo per $q = p \pm 1$; è da notare che per ciascuna delle due soluzioni del problema considerato siamo sempre lungo una delle due diagonali secondarie di D i cui elementi sono tutti pari ad 1.

Per il secondo tipo si ha $\alpha = 0$ e $\beta = 1$ per cui nell'equazione (1.1) non compare la visibilità. In questo caso è stata scelta come valore iniziale per M la matrice $M_{p,q}(1) = \delta_{p,q}$ e per $A(1)$ la distribuzione uniforme. Alla prima iterazione, l'algoritmo ora considerato agisce quindi come il RS. Dalla seconda iterazione in poi, invece, le equazioni ricorrenti (1.2) e (1.3) per M permettono di incorporare nell'algoritmo la "memoria" della colonia.

Un caso particolare di entrambi i tipi di algoritmi di cui sopra è quello per cui

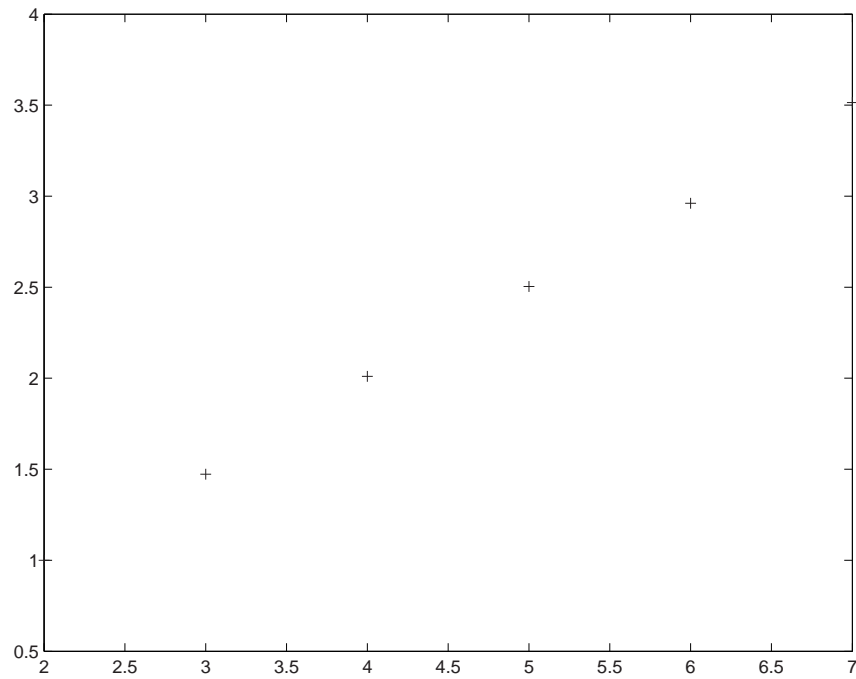


Figure 1.4: Valori di $\langle t_{min} \rangle$ in funzione dei valori di n per l'algoritmo ACO GBAS/tdev con $\theta = 0$ e n

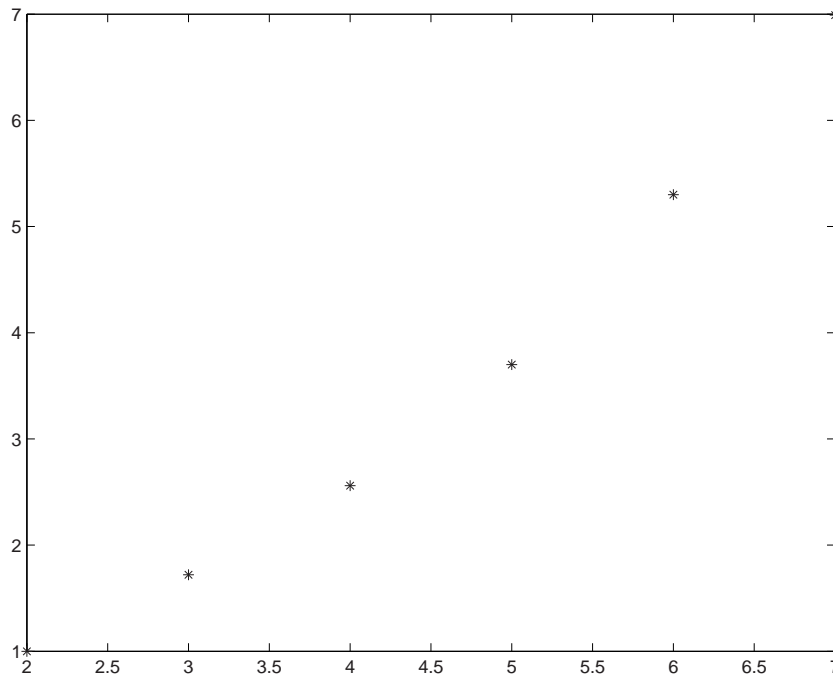


Figure 1.5: Valori di $\langle t_{min} \rangle$ in funzione dei valori di n per l'algoritmo ACO GBAS/tdev con $\alpha = 0$ e $\tau = 2$.

1.5 Programma di ottimizzazione per l'algoritmo ACO

In questa sezione viene discusso l'algoritmo ACO usato per la soluzione del problema esposto nelle sezioni precedenti.

Alla fine segue il listato del programma che è stato scritto in linguaggio Matlab.

Dato un insieme $S = 1; 2; \dots; n$, viene generata la permutazione ottima, cioè quella che rende minima la somma s

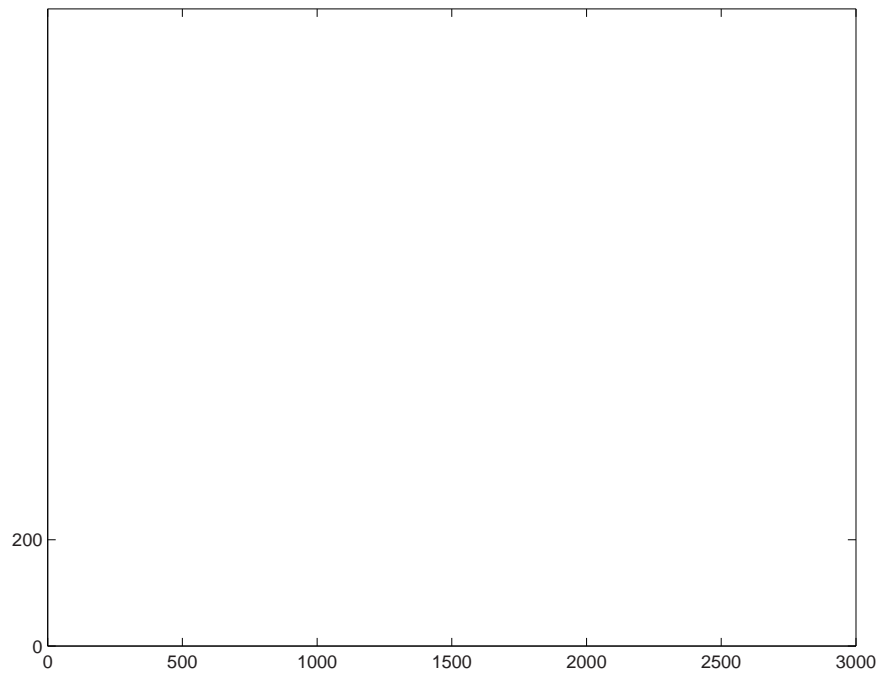
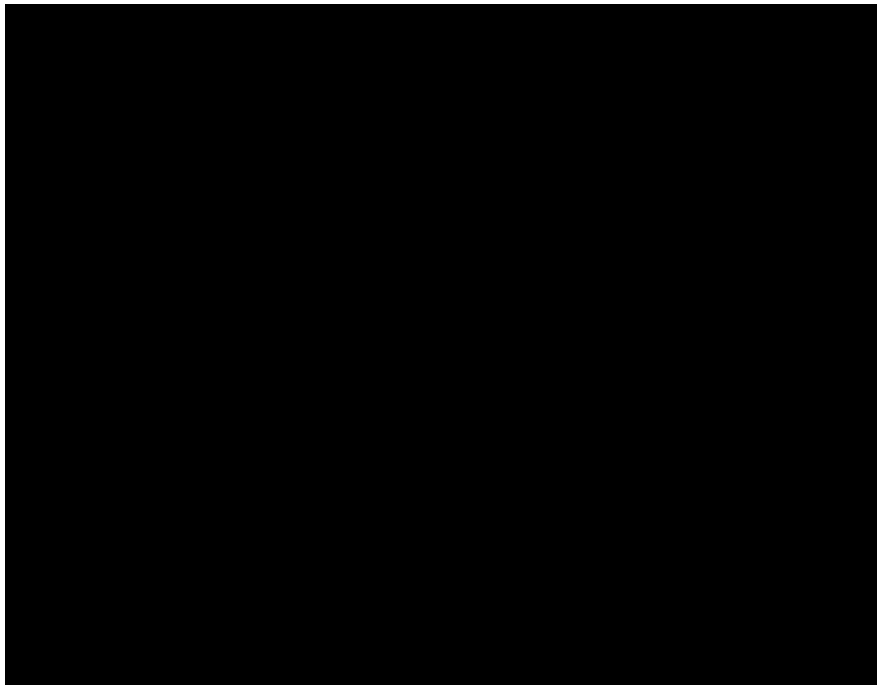


Figure 1.6: Valori di $\langle t_{min} \rangle$ in funzione dei valori di n .

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