

**Lecture notes  
on the Stefan problem**

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## Introduction

These notes cover a portion of a course of the Doctorate *Modelli e metodi matematici per la tecnologia e la società*, given in 2001 in Rome. The title of the course was ‘Evolution equations and free boundary problems’ and its topics included, essentially, an introduction to Stefan and Hele-Shaw problems.

Here only the material concerning the Stefan problem is partially reproduced.

The present notes assume the reader has some knowledge of the elementary theory of  $L^p$  and Sobolev spaces, as well as of the basic results of existence and regularity of solutions to smooth parabolic equations.

The bibliography is minimal; only books and articles quoted in the text are referenced. See [12], [18] and [20] for further references.

I thank the audience of the course for many stimulating comments and questions, and prof. R. Ricci for interesting discussions on the subject of these notes.

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