DOUBLY NONLINEAR THIN-FILM EQUATIONS IN ONE SPACE DIMENSION

LIDIA ANSINI AND LORENZO GIACOMELLI

Abstract. We consider a free boundary problem for a class of fourth order nonlinear parabolic equations which are degenerate both with respect to the unknown and to its third derivative. The problem is relevant in the description of the surface tension driven evolution of a thin film of non-Newtonian liquid over a solid surface in the "complete wetting" regime. Relying solely on global and local energy estimates and on Bernis inequalities, we prove existence of solutions to this problem, and obtain sharp bounds for the propagation of their free boundary. A necessary condition for the occurrence of waiting time phenomena is also derived.

Contents

1. Introduction

This paper is concerned with doubly nonlinear fourth order degenerate parabolic equations of the form

$$u_t + \left[|u|^n \, |u_{xxx}|^{p-2} \, u_{xxx}\right]_x \;=\; 0 \tag{1.1}$$

where $n > 0$ and $p \geq 2$ are real constants.

1.1. **The physics behind the equation.** It is nowadays well-known that for $p = 2$ (1.1) describes the surface tension driven evolution of the height $u(t,x)$ of a thin layer of Newtonian liquid on a solid surface (see the review paper [31]). The exponent n accounts of the condition imposed at liquid-solid interface: The value $n = 3$ corresponds to assuming the *no-slip* condition, whereas different *slip* conditions lead to operators which at leading order as $u \downarrow 0$ have the form (1.1) with $p = 2$, $n \in (0,3)$.

The introduction of slip conditions in the modelling of spreading phenomena is essentially motivated by a paradox, which was first discovered by Huh and Scriven [28] and Dussan V. and Davis [20]: The combination of constant viscosity and no-slip condition leads to an infinite rate of viscous dissipation at advancing contact lines. Besides slip conditions, different proposals for removing the paradox have been put forward (among them, we mention a recent revisiting of lubrication theory by Barenblatt, Beretta and Bertsch [3]). Here we are concerned with the one (to our best knowledge first investigated by Weidner and Schwarz [34]) which consists in keeping the no-slip condition and assuming instead a shear-dependent viscosity. Different constitutive law may be assumed: For "power-law fluids", cf. [29], the lubrication approximation renders operators of the form (1.1) with $n = p + 1$, whereas the modified power-law proposed in [34] leads to slightly different operators, whose leading part as $u \downarrow 0$ can be nevertheless formally seen to be of the same type, again with $n = p + 1$, cf. [2]. Viscosity decreases with shear stress for $p > 2$; these are *shear-thinning* fluids. The combination of shear-dependent viscosity and slip conditions leads to operators of the form (1.1) with $n < p + 1$.

The problem of the spreading of a thin liquid film is in fact a free-boundary problem: Equation (1.1) is to be satisfied on the positivity set $\{u > 0\}$, which is also unknown, and the free boundary is given by $\partial\{u > 0\}$, i.e. the triple junctions where liquid, solid and air meet. At the free boundary, in addition to the trivial defining condition $u = 0$ and the condition of zero mass-flux

$$\lim_{x \to \partial\{u(t)>0\}} u^n(t,x) \, |u_{xxx}(t,x)|^{p-1} \;=\; 0, \tag{1.2}$$

since the equation is of fourth order, a third condition is to be imposed. Here we will be concerned with the so-called *complete wetting regime* (cf. [18]), which prescribes a *zero-contact angle* condition:

$$u_x\Big|_{\partial\{u>0\}} = 0. \tag{1.3}$$

Hence, the problem we will consider in the sequel is the following:

$$(P)\begin{cases} u_t = -\left[u^n\,|u_{xxx}|^{p-2}\,u_{xxx}\right]_x & \text{in } \{u>0\}\cap(\mathbf{R}^+\times\Omega) \\ u_x = u^n\,|u_{xxx}|^{p-1} = 0 & \text{at } \partial\{u>0\}\cup(\mathbf{R}^+\times\partial\Omega) \\ u(0,x) = u_0(x)\geq 0 & x\in\Omega, \end{cases}$$

where

$$\Omega = \mathbf{R} \quad \text{or} \quad \Omega = (-a,a), \quad a\in\mathbf{R}^+ \tag{1.4}$$

(in the latter case, (P) is complemented with Neumann boundary conditions). Formal asymptotic expansions near the contact line [12, 29] show, for $p>2$, that (P) admits both travelling wave solutions and compactly supported self-similar source type solutions of the form

$$u(t,x) = t^{-\frac{1}{n+4(p-1)}}\,f(xt^{-\frac{1}{n+4(p-1)}}) \tag{1.5}$$

if and only if $n<2p-1$; in particular, if $n=p+1$. These facts support the conjecture that shear-thinning rheology indeed remove the aforementioned paradox of a force singularity at advancing contact lines. The aim of this work is to give rigorous bases to this conjecture, proving that for $p>2$ and $n<2p-1$ (P) admits solutions whose support is compact for all times and fills the whole real line as time tends to infinity.

1.2. **The analytical theory in the Newtonian case.** The analytical theory for (P) in the case $p=2$ started in 1990 with the paper of Bernis and Friedman [6], and is after a decade of studies sufficiently well established, though many questions yet remain open: Among them, uniqueness of solutions, existence of solutions in space dimension N higher than three, existence of solutions with non zero contact angle (solved only in the case $n=1$, $N=1$ by Otto [32]), regularity properties (such as continuity and even boundedness) in higher space dimension.

The basics in the development of the theory consist essentially of two estimates. The first one is an energy estimate (the Dirichlet integral being the leading order expansion of surface energy in lubrication approximation with zero contact-angle, cf. [27]), which can be formally obtained multiplying (1.1) by $-u_{xx}$:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega u_x^2 \;\le\; -\int_{\{u>0\}} u^n\, u_{xxx}^2. \tag{1.6}$$

The second one is a class of integral inequalities, so-called "entropy estimates", which for $p=2$ can be formally obtained multiplying (1.1) by $\frac{1}{\alpha}u^\alpha$: In short they read as

$$\frac{d}{dt}\int_\Omega \frac{u^{\alpha+1}}{\alpha(\alpha+1)} \;\lesssim\; -\int_\Omega |(u^{\frac{\alpha+n+1}{2}})_{xx}|^2, \quad \alpha\in\left[\tfrac12-n, 2-n\right]. \tag{1.7}$$

It is not difficult to see from (1.6) and (1.7) that for $n\ge\frac72$ (i.e. $\alpha+1\le-2$) an initially positive solution will stay positive for all times. This observation was used by Bertozzi and Pugh [10] and independently by Beretta, Bertsch, Dal Passo [4] to build up an approximating procedure and construct non-negative solutions to (P). In particular, (1.7) guarantees sufficient regularity to ensure that the solution is C^1 in space for almost every time, which means that the zero-contact angle condition is satisfied.

Extending the existence results to higher space dimensions turned out to be highly non-trivial. For $0<n<2$, $N\le3$ it has been done by Dal Passo, Garcke and Grün [14] and Bertsch, Dal Passo, Garcke and Grün [11]. For $n\ge2$ the structure of the operator changes in some respects (as indicated for instance by the signs in (1.7); cf. [13] for an enlightening example): To treat this case, a completely different method, based on approximation via solutions of an obstacle problem, has been recently introduced by Grün [22] Existence with measure initial data has been considered by Dal Passo and Garcke [13].

Propagation of support of solutions of (P) is both mathematically and physically among the fundamental issues. Due to the lack of comparison principle, its study has been based on energy/entropy methods, starting from localized variants of (1.7) for $n<2$, and of (1.6) for $n \gtrapprox 2$, in the latter case using also Bernis' inequalities [7]: These are interpolation inequalities of the form

$$\int_\Omega |(u^{\frac{n+2}{6}})_x|^6 \;\lesssim\; \int_{\{u>0\}} u^n\, u_{xxx}^2, \quad n\in\left(\tfrac12,3\right),$$

$$\int_\Omega |(u^{\frac{n+2}{3}})_{xx}|^3 \;\lesssim\; \int_{\{u>0\}} u^n\, u_{xxx}^2, \quad n\in\left(\tfrac12,3\right),$$

which are known to hold for non-negative $u\in C^1(\Omega)$ (cf. [27]). They have been recently extended by Grün [23] to higher space dimensions for $2 \lessapprox n<3$ and positive $u\in H^2_*(\Omega)$. Sharp estimates from above of the support for generic initial data are due to Bernis [7, 8] and Hulshov and Shishkov [28] in one space dimension, and to Bertsch, Dal Passo, Garcke and Grün [11] and Grün [24] in higher space dimensions. Necessary conditions for the occurrence of waiting time phenomena have been obtained by Dal Passo, Giacomelli and Grün [15] and Grün [25].

1.3. **The doubly nonlinear case: A "second-order smoothing" approach.** A dramatic change in the case $p \neq 2$ is the sudden disappearance of entropy estimates (1.7). Such loss compels to concentrate on the essential structure of the operator, which is that given by the energy estimate:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega u_x^2 \;\le\; -\int_{\{u>0\}} u^n\,|u_{xxx}|^p. \tag{1.8}$$

The energy estimate by itself is not sufficient to characterize a solution of (P) — for instance, overturned parabolas are steady states for (1.1) which satisfy (1.8), i.e. local minima for the mass-constrainted energy functional. An additional information, which no longer can follow from entropy estimates, is needed to infer the zero-contact angle condition. The crucial observation is that the following extension of Bernis inequalities does provide such information:

$$\int_\Omega |(u^{\frac{n+p}{3p}})_x|^{3p} \;\lesssim\; \int_{\{u>0\}} u^n\,|u_{xxx}|^p, \quad n\in\left(\tfrac{p-1}{2},2p-1\right), \tag{1.9}$$

$$\int_\Omega |(u^{\frac{2(n+p)}{3p}})_{xx}|^{\frac{3p}{2}} \;\lesssim\; \int_{\{u>0\}} u^n\,|u_{xxx}|^p, \quad n\in\left(\tfrac{p-1}{2},2p-1\right). \tag{1.10}$$

Indeed, if $u(x_0)=0$ then

$$\left|\frac{u(x)-u(x_0)}{x-x_0}\right| \;\le\; \left(\int_\Omega |(u^{\frac{n+p}{3p}})_x|^{3p}\right)^{\frac{1}{n+p}} |x-x_0|^{\frac{2p-1-n}{n+p}}. \tag{1.11}$$

Unfortunately, the validity of (1.9) and (1.10) is limited to non-negative C^1 functions. To overcome this difficulty, we approximate (P) with problems whose solutions are non-negative and continuously differentiable for almost every time, and then use (1.9) and (1.11) to turn this qualitative information into a quantitative one, which is preserved in the limit. The choice of the approximating problems will be described in Sect. 1.5: It is based on adding to the right-hand side of (1.1) a second-order operator of the form

$$[L_\epsilon(u_x)]_x \;=\; \epsilon\left[u_x+|u_x|^{p-2}\,u_x\right]_x. \tag{1.12}$$

The linear diffusion term guarantees sufficient regularity of the approximating solutions, and the lonlinear term (combined with a suitable approximation of the mobility u^n) yields non-negativity. In other words, the smoothing effect of (1.12) rules out that the evolution gets stuck in the aforementioned local minima.

1.4. **The main results.** Motivated by (1.9)-(1.10), we assume throughout the paper

$$n \in \left(\tfrac{p-1}{2}, 2p-1\right), \qquad p \in (2,\infty), \tag{1.13}$$

and let

$$A(s) \;=\; |s|^{p-2}\, s. \tag{1.14}$$

We shall throughout use the shortenings:

$$\begin{aligned}
\{u(t) > \eta\} \;&=\; \{x \in \Omega :\; u(x,t) > \eta\}\,,\\
\{u > \eta\} \;&=\; \cup_{t \in \mathrm{Dom}(u)} \{u(t) > \eta\}\,,\\
\{u > \eta\}_T \;&=\; \cup_{t \in \mathrm{Dom}(u)\cap[0,T]} \{u(t) > \eta\}\\
\Omega_T \;&=\; (0,T)\times\Omega, \quad S_T \;=\; (0,T)\times\partial\Omega.
\end{aligned}$$

A solution of (P) is defined as follows:

Definition 1. *Assume (1.4), (1.13), and let $u_0 \in H^1(\Omega)\cap L^1(\Omega)$ be non-negative. A non-negative function $u \in C([0,\infty)\times\overline{\Omega}) \cap L^\infty(\mathbf{R}^+; H^1(\Omega))$ is called a solution of (P) with initial datum u_0 if:*

(i) $u_t \;\in\; L^{\frac{p}{p-1}}(\mathbf{R}^+; (W^{1,p}(\Omega))')$;
(ii) $u_{xxx} \;\in\; L^p_{loc}(\{u>0\})$ *and* $u^{\frac{n}{p}}\, u_{xxx} \;\in\; L^p(\{u>0\})$;
(iii) for all $\varphi \;\in\; C_c^\infty([0,\infty)\times\overline{\Omega}))$

$$\int_0^\infty <u_t, \varphi>\, dt \;=\; \iint_{\{u>0\}} u^n\, A(u_{xxx})\, \varphi_x\,; \tag{1.15}$$

(iv) $u(0,x) \;=\; u_0(x)$;
(v) $u^{\frac{n+p}{3p}} \;\in\; L^{3p}_{loc}([0,\infty); W^{1,3p}(\Omega))$;
(vi) $u_x \;=\; 0$ *in* $L^p(\mathbf{R}^+\times\partial\Omega)$.

As from (1.11), (v) implies in particular that the zero-contact angle condition (1.3) is attained for almost every t; the condition (1.2) of zero mass-flux follows easily from (ii) and the continuity of u. Our first main result is the following:

Theorem 1 (Existence and long-time behaviour of solutions). *Assume (1.4) and (1.13). For every non-negative $u_0 \in H^1(\Omega)$ such that $|x|^{3p}\, u_{0x}^2 \in L^1(\Omega)$, a solution u of (P) with initial datum u_0 exists in the sense of Definition 1. In addition*

$$\|u(t)\|_{L^1(\Omega)} \;=\; \|u_0\|_{L^1(\Omega)} \quad \textit{for all } t > 0, \tag{1.16}$$

$$\lim_{t\to\infty} u(t,x) \;=\; \left\{ \begin{array}{ll} ⨍_\Omega u_0 & \textit{if } \Omega \textit{ is bounded} \\ 0 & \textit{if } \Omega = \mathbf{R} \end{array} \right\} \quad \textit{uniformly in } \Omega \tag{1.17}$$

and a positive constant C (independent of u_0) exists such that

$$\int_\Omega \phi^{3p}\, u_x^2(t) + C^{-1} \iint_{\{u>0\}_t} \phi^{3p}\, u^n\, |u_{xxx}|^p \;\le\; \int_\Omega \phi^{3p}\, u_{0x}^2 + C \iint_{\{\phi_x>0\}_t} u^{n+p} \tag{1.18}$$

for almost every $t \in (0,\infty)$ and for all ϕ such that

$$\phi \;\equiv\; 1 \quad \textit{or} \quad \phi \;=\; (x-r)_+,\; r \in \mathbf{R}, \tag{1.19}$$

with $C = \frac{1}{2}$ if $\phi \equiv 1$.

Remark 1.1. *Note, if $\Omega = \mathbf{R}$, that u_0 is allowed to have unbounded support. In addition, the assumptions on u_0 imply that $u_0 \in L^1(\mathbf{R})$ (cf. Lemma 6.1).*

The long time behaviour given by (1.17) positively answers to the conjectured role of shear-thinning rheology in removing the paradox outlined in Sect. 1.1. A more complete picture of the evolution should also rule out the possibility of instantaneous complete wetting, i.e., that the free boundary does not exist at all. The next result provides this information in the case that $\Omega = \mathbf{R}$:

Theorem 2 (Finite speed of propagation). *Let $\Omega = \mathbf{R}$. A positive constant C, depending only on n and p, exists such that for any solution u of (P) obtained in Theorem 1 with*

$$\operatorname{supp}(u(0)) \;\subset\; (-\infty, R),$$

for all $t > 0$ it holds that

$$\operatorname{supp}(u(t)) \;\subset\; \left(-\infty, R + C\,\|u_0\|_1^{\frac{n+p-2}{n+4(p-1)}}\, t^{\frac{1}{n+4(p-1)}}\right).$$

The exponent $1/(n+4(p-1))$ coincides with that of the self-similar scaling (1.5), whence it is sharp for generic initial data. On the other hand, the speed of propagation can very well be slower, and even zero or negative during a certain time. This is the "waiting time" phenomenon. In our last main result we provide a sufficient condition on the growth of the initial data near the interface for the occurrence of such phenomenon:

Theorem 3 (Waiting time). *Let $\Omega = \mathbf{R}$, and let u be a solution of (P) with initial datum u_0 obtained in Theorem 1. If* $\operatorname{supp}(u_0) \subseteq (-\infty, x_0]$, *and*

$$\limsup_{r\to 0} r^{-2(\frac{3p-2}{n+p-2}-1)} \fint_{B(x_0,r)} u_{0x}^2 \;<\; \infty, \tag{1.20}$$

then a positive time $T^\star$ exists such that

$$\operatorname{supp}(u(t)) \;\subseteq\; (-\infty, x_0] \quad \text{for all} \quad t \in (0, T^*).$$

The critical exponent $\frac{3p-2}{n+p-2}$ coincides, for $p = 2$, with that obtained in [15].

These are to our best knowledge the first results on existence and qualitative behaviour of solutions for doubly nonlinear parabolic equations of higher order without variational structure. The second order relaxation applies (actually in a fairly simpler way) also to the case $p = 2$. It shows that the energy estimate, together with an additional information on regularity (which must follow from a careful choice of the approximating scheme), is already sufficient both to characterize a solution of (1.1) as a zero-contact angle solution, i.e. as a solution of (P), and to prove that this solution has finite speed of propagation. There is no need to call for "entropy estimates". This was already quite clear from the gradient flow perspective (cf. [1, 26]), the contact angle condition being a natural (Neumann) rather than an essential (Dirichlet) boundary condition. The main merit of our method is to translate this insight into results on existence and qualitative behaviour of solutions.

The higher dimensional case remains open and deserves a comment: Save the validity of (1.9) and (1.10) in higher space dimension, and the many technical complications whose resolution should be extended to the case $p > 2$, our approach can not be applied due to the need for *nonlinear* regularizing terms. On the other hand, the technique introduced by Grün in [21] seems to rely upon entropy estimates, too. A combination of the two approaches might lead to a solution of this open question.

1.5. **Plan of the proofs.** We first consider (P) in a bounded domain. The existence result is based on a multi-step approximating procedure. We replace u^n by nonlinearities which are bounded and strongly degenerate at $u = 0$, and add artificial regularizing terms in the equation:

$$u_t \;=\; -\left[m_{\delta,\sigma}(u)\, A(u_{xxx})\right]_x + \left[L_\epsilon(u_x)\right]_x + \delta\, (m_{\delta,\sigma}(u))^\beta\, A(u_{xx}), \tag{1.21}$$

where

$$m_{\delta,\sigma}(s) \;=\; \frac{|s|^{n+4(p-1)}}{\sigma\,|s|^n + |s|^{4(p-1)} + \delta\,|s|^{n+4(p-1)}}, \tag{1.22}$$

$$\beta \;\in\; \left(1 - \tfrac{p-2}{4(p-1)}, 1\right), \tag{1.23}$$

and L_ϵ is defined through (1.12). We also raise the initial datum of a height ϵ:

$$u_{0\epsilon}(x) \;=\; u_0(x) + \epsilon. \tag{1.24}$$

Local existence of positive solutions of the Neumann problem for (1.21) is proved in the appendix. It follows from a Galerkin type argument which relies solely on the energy balance

$$\begin{aligned} \tfrac{1}{2}\int_\Omega u_x^2(t) \;&=\; \tfrac{1}{2}\int_\Omega u_x^2(0) - \iint_{\Omega_t} m_{\delta,\sigma}(u)\,|u_{xxx}|^p \\ &-\; \epsilon \iint_{\Omega_t} \left[1 + (p-1)\,|u_x|^{p-2}\right]\,u_{xx}^2 - \delta \iint_{\Omega_t} m_{\delta,\sigma}^\beta(u)\,|u_{xx}|^p \end{aligned} \tag{1.25}$$

and on the following control on the mass:

$$\left|\int_\Omega u(t) - \int_\Omega u(0)\right| \;=\; \left|-\delta\iint_{\Omega_t} m_{\delta,\sigma}^\beta(u)\,A(u_{xx})\right| \;\overset{(1.25),(1.22)}{\lesssim}\; \delta^{\frac{1-\beta}{p}}\,t^{\frac{1}{p}}. \tag{1.26}$$

Estimates (1.25) and (1.26) yield in particular a uniform control on a suitable Hölder norm of u in $[0, T_\delta]\times\Omega$, where $T_\delta = \delta^{\beta-1}$.

In **Section 2** we show that these solutions are in fact globally defined and positive (the latter implies the former since solutions of (1.21) can be continued as long as they are positive). A sketch of the formal argument yielding positivity in the model case where

$$m_{\delta,\sigma}(u) \;=\; u^{4(p-1)}$$

might be useful to enlighten the choice of $m_{\delta,\sigma}$ itself, and to see how the lack of entropy estimates can be overcome. Multiplying by $-u^{-3}/3$ and integrating by parts we obtain:

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{1}{u^2} \;&=\; \int_\Omega u^{4p-8}\,u_x\,A(u_{xxx}) \\ &-\; \epsilon\int_\Omega u^{-4}\,\left[u_x^2 + |u_x|^p\right] - \frac{\delta}{3}\int_\Omega u^{4\beta(p-1)-3}\,A(u_{xx}). \end{aligned}$$

The first integral on the right-hand side is estimated by the second one and (1.25):

$$\int_\Omega u^{4p-8}\,u_x\,A(u_{xxx}) \;\le\; \left(\int_\Omega u^{4(p-1)}\,|u_{xxx}|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega u^{-4}\,|u_x|^p\right)^{\frac{1}{p}}.$$

The last one is estimated by the integral on left-hand side and by (1.25):

$$\int_\Omega u^{4\beta(p-1)-3} A(u_{xx}) \le \left(\int_\Omega u^{4\beta(p-1)} |u_{xx}|^p\right)^{\frac{p-1}{p}} \left(\int_\Omega u^{4\beta(p-1)-3p}\right)^{\frac{1}{p}}.$$

Together with the boundedness of u and the choice of β, a Gronwall argument implies that the integral on the left hand side can not blow-up in finite time; combined with (1.25), this implies that u can not touch down to zero in finite time.

In **Section 3** we pass to the limit as $\sigma \downarrow 0$, obtaining a non-negative solution of the Neumann problem for (1.21) with $\sigma = 0$. Here we face two mainly technical difficulties. The first one is the identification of $A(u_{xxx})$: In order to obtain strong convergence for the third derivative, we have to stay safely inside connected components of the positivity set, where the energy estimate does not degenerate. The second issue is the convergence of lower order terms in local energy estimates. A particularly delicate one, which motivates the introduction of the "δ" term in the approximating procedure, has the form

$$I_\sigma = \iint_{\Omega_T} (\phi^{3p})_x \, m_{\delta,\sigma}(u_\sigma) \, u_{\sigma xx} \, A(u_{\sigma xxx}).$$

The passage to the limit locally on the positivity set is harmless. On the other hand, thanks to the fact that $\beta < 1$ and to the "δ" term in (1.25), I_σ are uniformly small where the u_σ's are.

Remark 1.2. *Let us observe that if $p = 2$ one could set $\delta = 0$ throughout. Moreover, if $p \neq 2$, one could as well set $\delta = 0$ as far as local energy estimates are not concerned — in particular, to prove existence of solutions.*

In **Section 4** we prove the extension (1.9)-(1.10) of Bernis estimates. Here the proof appreciably differs from the corresponding one in [7, 27] only in the case $\frac{p}{2} \le n \le 2p - 2$ (which is of course where the nonlinearity with respect to the third derivative affects integration by parts).

In **Section 5** we pass to the limit as $\delta \downarrow 0$ and $\epsilon \downarrow 0$, proving Theorem 1 in the case of bounded Ω. In fact, it turns out that we may choose $\delta = \epsilon$ (we nevertheless preferred to keep separate notations as for the different role played by the two terms, cf. Remark 1.2). The key observation here is that each approximating solution is non-negative and, thanks to the regularizing terms in (1.25), of class C^1 for almost every t. Hence Bernis estimates hold, and turn this qualitative information into a quantitative one, which is preserved in the limit. As is nowadays well-known, they also provide sufficient control on lower order terms for the passage to the limit in local energy estimates.

In **Section 6** we prove Theorem 1 in the case $\Omega = \mathbf{R}$. The solution is obtained as limit, as $a \uparrow \infty$, of solutions of (P) in $\Omega = (-a, a)$.

In **Section 7** we prove Theorem 2 and Theorem 3. Here we use a technique, based on iterative procedures and on Stampacchia's Lemma, which was developed by Dal Passo, Giacomelli and Grün (cf. [15, 16, 24, 25]). It turns out that such technique is sufficiently versatile to cover the case into consideration with only a few modifications.

In the **Appendix** we state and prove an existence result for non-degenerate problems (which serves as starting point of this work), and a simple but essential calculus lemma.

2. Positive solutions of strongly degenerate problems

Given a positive initial datum, thin-film type operators are known to preserve positivity provided the nonlinearity is sufficiently strong near zero. In this section we show that this property continues to hold in the doubly nonlinear case, and we use it to construct positive approximating solutions. To guarantee initial positivity, we also raise the initial datum of a height ϵ. We thus consider the following problems:

$$(P_{\epsilon,\delta,\sigma}) \quad \begin{cases} u_t = [-m_{\delta,\sigma}(u)\, A(u_{xxx}) + L_\epsilon(u_x)]_x + \delta\, m^\beta_{\delta,\sigma}(u)\, A(u_{xx}) & \text{in } \Omega_T \\ u_x = u_{xxx} = 0 & \text{on } S_T \\ u(0,x) = u_{0\epsilon}(x) = u_0(x) + \epsilon & x \in \Omega, \end{cases}$$

with $T > 0$, $m_{\delta,\sigma}$, L_ϵ and β defined respectively by (1.22), (1.12) and (1.23). A solution of $(P_{\epsilon,\delta,\sigma})$ is defined as follows.

Definition 2.1. *A function $u \in L^p((0,T); W^{3,p}(\Omega)) \cap C([0,T]; H^1(\Omega))$ with $u_t \in L^{\frac{p}{p-1}}((0,T); (W^{1,p}(\Omega))')$ is called a solution of $(P_{\epsilon,\delta,\sigma})$ in Ω_T if:*

(i) for all $\tau < T$ and all $\varphi \in L^p((0,\tau); W^{1,p}(\Omega))$

$$\int_0^\tau < u_t, \varphi > dt = \iint_{\Omega_t} [m_{\delta,\sigma}(u)\, A(u_{xxx}) - L_\epsilon(u_x)]\, \varphi_x + \iint_{\Omega_t} \delta\, m^\beta_{\delta,\sigma}(u)\, A(u_{xx})\, \varphi; \tag{2.1}$$

(ii) $u(0) = u_{0\epsilon}$ in $H^1(\Omega)$;
(iii) $u_x = 0$ in $L^p(S_T)$.

The main result of the section is:

Proposition 2.2. *Let ϵ, $\delta \in (0,1)$, $\sigma \in (0,\epsilon)$, $T_\delta = \delta^{\beta-1}$. For any non-negative $u_0 \in H^1(\Omega)$, a solution u of $(P_{\epsilon,\delta,\sigma})$ in Ω_{T_δ} exists in the sense of Definition 2.1. In addition*

$$u > 0 \quad in \ \overline{\Omega}_{T_\delta}, \tag{2.2}$$

$$\iint_{\Omega_{T_\delta}} G''_{\delta,\sigma}(u)\,\left(|u_x|^p + u_x^2\right) \quad \text{is bounded unif. w.r.t. } \sigma \tag{2.3}$$

(with $G_{\delta,\sigma}$ defined by (2.11) below), and a constant $C > 0$ independent of σ, ϵ and δ exists such that for all x, x_1, $x_2 \in \Omega$ and all t, $t_1\, t_2 \in [0, T_\delta]$:

$$|u(t,x_1) - u(t,x_2)| \ \leq \ C\,|x_1 - x_2|^{\frac{1}{2}} \tag{2.4}$$

$$|u(t_1,x) - u(t_2,x)| \ \leq \ C\,|t_1 - t_2|^{\frac{1}{5p-2}} \quad \text{if } |t_2 - t_1| \leq 1, \tag{2.5}$$

$$u(t,x) \ \leq \ C \tag{2.6}$$

Proof. The first step is to show that $(P_{\epsilon,\delta,\sigma})$ admits a positive and Hölder continuous *local* solution. To this aim, we introduce the auxiliary problem

$$(\tilde{P}_{\epsilon,\delta,\sigma}) \quad \begin{cases} u_t \ = \ [-\tilde{m}_{\delta,\sigma}(u)\,A(u_{xxx}) + L_\epsilon(u_x)]_x + \delta\,\tilde{m}^\beta_{\delta,\sigma}(u)\,A(u_{xx}) & \text{in } \Omega_T \\ u_x \ = \ u_{xxx} \ = \ 0 & \text{on } S_T \\ u(0,x) \ = \ u_{0\epsilon}(x) & x \in \Omega, \end{cases}$$

where

$$\tilde{m}_{\delta,\sigma}(s) \ = \ \min\left\{\tfrac{1}{\sigma}, \max\left\{m_{\delta,\sigma}(s), m_{\delta,\sigma}\left(\tfrac{\sigma}{2}\right)\right\}\right\}.$$

Proposition 8.2 in the appendix guarantees that $(\tilde{P}_{\epsilon,\delta,\sigma})$ admits a global solution $u = \tilde{u}_{\epsilon,\delta,\sigma}$ in the sense of Definition 2.1 (with $m_{\delta,\sigma}$ replaced by $\tilde{m}_{\delta,\sigma}$). The regularity of u allows to choose $\varphi = -u_{xx}\chi_{(0,t)}$ in (2.1), and after integrations by parts we obtain

$$\tfrac{1}{2}\int_\Omega u_x^2(t) \ + \ \iint_{\Omega_t} \Big[\tilde{m}_{\delta,\sigma}(u)\,|u_{xxx}|^p + \epsilon\,(p-1)\,|u_x|^{p-2}u_{xx}^2 + \epsilon\,u_{xx}^2$$
$$+ \ \delta\,\tilde{m}^\beta_{\delta,\sigma}(u)\,|u_{xx}|^p\Big] \ = \ \tfrac{1}{2}\int_\Omega u_{0\epsilon x}^2 \tag{2.7}$$

for all $t \geq 0$. It follows immediately from (2.7) that

$$|u(t,x_1) - u(t,x_2)| \ \leq \ C\,|x_1 - x_2|^{\frac{1}{2}} \quad \forall\, x_1, x_2 \in \Omega, \ t \in [0,\infty) \tag{2.8}$$

(in the course of the proof, C denotes a generic constant independent of ϵ, δ, σ and time). By (2.1) and Hölder inequality

$$\left|\int_\Omega u(t) - \int_\Omega u_{0\epsilon}\right| \ = \ \left|\delta \iint_{\Omega_t} \tilde{m}^\beta_{\delta,\sigma}(u)\,A(u_{xx})\right| \ \stackrel{(2.7),(1.22)}{\leq} \ \delta^{\frac{1-\beta}{p}}\,t^{\frac{1}{p}}.$$

Choosing $T_\delta = \delta^{-(1-\beta)}$ we obtain

$$\int_\Omega u(t) \;\le\; C \quad \forall\, t \in [0, T_\delta],$$

which together with (2.7) implies

$$u(t,x) \;\le\; C \quad \forall\, x \in \Omega, t \in [0, T_\delta]. \tag{2.9}$$

We also see, arguing as in [6, proof of Lemma 2.1], that for all $x \in \Omega$ and all t_1, $t_2 \in [0, T_\delta]$:

$$|u(t_1,x) - u(t_2,x)| \;\le\; C\,|t_1 - t_2|^{\frac{1}{5p-2}} \quad \text{if } |t_1 - t_2| \le 1. \tag{2.10}$$

In particular u is uniformly continuous, and since $u_{0\epsilon} \ge \sigma$, a positive T_0 exists such that $\frac{\sigma}{2} \le u \le \frac{1}{\sigma}$ in $\overline{\Omega}_{T_0}$. Hence $\tilde m_{\delta,\sigma}(u(t,x)) \equiv m_{\delta,\sigma}(u(t,x))$ in $\overline{\Omega}_{T_0}$, which implies that u is a positive solution of $(P_{\epsilon,\delta,\sigma})$ in Ω_{T_0}.

We may then introduce the maximal time of existence

$$T^* \;=\; \sup\left\{ T > \frac{T_0}{2} : \begin{array}{l} u \text{ is a solution of } (P_{\epsilon,\delta,\sigma}) \text{ in } \Omega_T \\ u > 0 \text{ in } [0,T) \times \overline{\Omega} \end{array} \right\}.$$

In view of the above, the set is not empty and $T^* \ge T_0$. We will now show that $T^* > T_\delta$, which is all we need to complete the proof. Assume by contradiction that $T^* \le T_\delta$. In view of (2.8) and (2.10):

$$\lim_{t \to T^*} u(t,x) \;=\; u_*(x) \in H^1(\Omega).$$

Our goal is to show that in fact $u_* > 0$ in $\overline{\Omega}$. To this aim, let

$$G_{\delta,\sigma}(s) = \frac{\sigma^{\frac{1}{p-1}}\, s^{-2}}{6} + H_{\delta,\sigma}(s) + \frac{1}{2}\,\delta^{\frac{1}{p-1}}\, s^2 \tag{2.11}$$

where

$$H_{\delta,\sigma}(s) = \begin{cases} \frac{(p-1)^2}{(2p-2-n)\,(p-1-n)}\, s^{\frac{2p-2-n}{p-1}} & \text{if } n \ne p-1,\ 2(p-1) \\ s\,(\log s - 1) & \text{if } n = p-1 \\ -\log s & \text{if } n = 2(p-1), \end{cases}$$

With this choice we have

$$G''_{\delta,\sigma}(s) \;=\; \sigma^{\frac{1}{p-1}}\, s^{-4} + s^{-\frac{n}{p-1}} + \delta^{\frac{1}{p-1}},$$

so that we obtain immediately

$$\left|G''_{\delta,\sigma}(u)\right|^{p-1} m_{\delta,\sigma}(u) \;\le\; C. \tag{2.12}$$

In addition, the following inequality holds (cf. Lemma 8.3):

$$|G'_{\delta,\sigma}(s)|^p\, m^{\beta}_{\delta,\sigma}(s) \;\le\; C + C\,[G_{\delta,\sigma}(s)]_+ . \tag{2.13}$$

Since u is positive in $[0,T^*)\times\overline{\Omega}$, we may choose $\varphi = G'_{\delta,\sigma}(u)\chi_{(0,t)}$, $t\in(0,T^*)$, as a test function in (2.1). Integrating by parts:

$$\begin{aligned}
&\int_\Omega [G_{\delta,\sigma}(u(t))]_+ + \epsilon \iint_{\Omega_t} G''_{\delta,\sigma}(u)\left(|u_x|^p + u_x^2\right)\\
&\quad= \int_\Omega G_{\delta,\sigma}(u_{0\epsilon}) + \int_\Omega [G_{\delta,\sigma}(u(t))]_- + \iint_{\Omega_t} G''_{\delta,\sigma}(u)\, m_{\delta,\sigma}(u)\, u_x\, A(u_{xxx})\\
&\qquad+ \delta \iint_{\Omega_t} G'_{\delta,\sigma}(u)\, m^{\beta}_{\delta,\sigma}(u)\, A(u_{xx}) \;=\; I_1 + I_2 + I_3 + I_4.
\end{aligned}\tag{2.14}$$

By Hölder and Young's inequalities, we obtain for I_3

$$\begin{aligned}
|I_3| \;&\le\; \frac{\epsilon}{2}\iint_{\Omega_t} |G''_{\delta,\sigma}(u)|^p\, m_{\delta,\sigma}(u)\,|u_x|^p + C_\epsilon \iint_{\Omega_t} m_{\delta,\sigma}(u)\,|u_{xxx}|^p\\
&\overset{(2.12),(2.7)}{\le}\; \frac{\epsilon}{2}\iint_{\Omega_t} G''_{\delta,\sigma}(u)\,|u_x|^p + C_\epsilon,
\end{aligned}\tag{2.15}$$

and for I_4

$$\begin{aligned}
|I_4| \;&\le\; \frac{\delta p}{p-1}\iint_{\Omega_t} m^{\beta}_{\delta,\sigma}(u)\,|u_{xx}|^p + \frac{\delta}{p}\iint_{\Omega_t} m^{\beta}_{\delta,\sigma}(u)\,|G'_{\delta,\sigma}(u)|^p\\
&\overset{(2.13),(2.7)}{\le}\; C + C\iint_{\Omega_t} [G_{\delta,\sigma}(u)]_+ .
\end{aligned}\tag{2.16}$$

In view of (1.24), resp. (2.9), we also have

$$|I_1| \;\le\; C_{\epsilon,\delta}, \quad |I_2| \;\le\; C. \tag{2.17}$$

Gathering (2.15)-(2.17) into (2.14), a Gronwall argument implies that

$$\sup_{t\in(0,T^*)}\int_\Omega [G_{\delta,\sigma}(u(t))]_+ + \epsilon\iint_{\Omega_t} G''_{\delta,\sigma}(u)\left(|u_x|^p + u_x^2\right) \;\le\; C_{\epsilon,\delta}. \tag{2.18}$$

Passing to the limit as $t\uparrow T^*$, we obtain in particular

$$\sigma^{\frac{1}{p-1}}\int_\Omega u_*^{-2} \;\le\; C_{\epsilon,\delta}.$$

Since $u_*\in H^1(\Omega)$, this implies that $u_*(x)>0$ for all $x\in\overline{\Omega}$. Let then $u_* = \min_{x\in\Omega} u_*(x)$. Choosing $u_*(x)$ as initial datum and

$$\tilde m_{\delta,\sigma}(s) \;=\; \min\left\{\tfrac{1}{\sigma}, \max\left\{m_{\delta,\sigma}(s), m_{\delta,\sigma}\left(\tfrac{u_*}{2}\right)\right\}\right\},$$

the positivity of u_* would allow, arguing as before, to extend u to a positive solution beyond T^*. This contradicts the definition of T^*, whence $T^* > T_\delta$. Since (2.3), (2.4), (2.5), (2.6) follow respectively from (2.18), (2.8), (2.10) and (2.9), the proof of Proposition 2.2 is complete. □

3. The limit $\sigma \downarrow 0$

The aim of this section is to let $\sigma \downarrow 0$ in $(P_{\epsilon,\delta,\sigma})$ and thus obtain existence of non-negative solutions for:

$$(P_{\epsilon,\delta}) \quad \begin{cases} u_t = [-m_\delta(u)\, A(u_{xxx}) + L_\epsilon(u_x)]_x + \delta\, m_\delta^\beta(u)\, A(u_{xx}) & \text{in } \Omega_T \\ u_x = u_{xxx} = 0 & \text{on } S_T \\ u(0,x) = u_0(x) \geq 0 & x \in \Omega, \end{cases}$$

where $m_\delta(s) = m_{\delta,0}(s)$. Here we face two main technical difficulties: One is to identify $A(u_{xxx})$ on the positivity set (as opposite to the standard thin-film equation, Schauder type estimates are not known even in one space dimension). The second one is to get sufficiently robust controls to handle the passage to the limit in local energy estimates. The second order regularization play an essential role in this respect.

Proposition 3.1. *Let $\epsilon, \delta \in (0,1)$. For any non-negative $u_0 \in H^1(\Omega)$, a non-negative function $u \in C([0,T_\delta] \times \overline{\Omega}) \cap L^\infty((0,T_\delta); H^1(\Omega))$ exists such that*

$$u_t \in L_{loc}^{\frac{p}{p-1}}((0,T_\delta); (W^{1,p}(\Omega))'), \tag{3.1}$$

$$u \in L^2((0,T_\delta); H^2(\Omega)) \cap L^p((0,T_\delta); W^{1,p}(\Omega)), \tag{3.2}$$

$$u_{xxx} \in L_{loc}^p(\{u>0\}) \quad \text{and} \quad (m_\delta(u))^{\frac{1}{p}}\, u_{xxx} \in L^p(\{u>0\}), \tag{3.3}$$

$$m_\delta^{\frac{\beta}{p}}(u)\, u_{xx} \in L^p(\Omega_{T_\delta}) \quad \text{and} \quad |u_x|^{\frac{p-2}{2}}\, u_{xx} \in L^2(\Omega_{T_\delta}), \tag{3.4}$$

which solves $(P_{\epsilon,\delta})$ in the following sense:

(i) for all $\varphi \in L^p((0,T_\delta); W^{1,p}(\Omega))$

$$\int_0^{T_\delta} < u_t, \varphi > dt = \iint_{\{u>0\}} m_\delta(u)\, A(u_{xxx})\, \varphi_x - \iint_{\Omega_{T_\delta}} L_\epsilon(u_x)\, \varphi_x + \delta \iint_{\Omega_{T_\delta}} m_\delta^\beta(u)\, A(u_{xx})\, \varphi; \tag{3.5}$$

(ii) $u(0,x) = u_{0\epsilon}(x)$;

(iii) $u_x = 0$ in $L^2(S_{T_\delta})$.

In addition, for almost every $t \in [0,T_\delta)$ and every function ϕ as in (1.19)

$$\begin{aligned} \tfrac{1}{2}\int_\Omega \phi^{3p}\, u_x^2(t) \;&+\; \iint_{\{u>0\}_t} \phi^{3p}\, m_\delta(u)\,|u_{xxx}|^p \\ &+\; \iint_{\Omega_t} \phi^{3p}\,\Big[\epsilon\,(1+(p-1)\,|u_x|^{p-2})\,u_{xx}^2 + \delta\, m_\delta^\beta(u)\,|u_{xx}|^p\Big] \\ &\le\; \tfrac{1}{2}\int_\Omega \phi^{3p}\, u_{0\epsilon x}^2 - 2\iint_{\{u>0\}_t} (\phi^{3p})_x\, m_\delta(u)\, u_{xx}\, A(u_{xxx}) \\ &-\; \iint_{\{u>0\}_t} (\phi^{3p})_{xx}\, m_\delta(u)\, u_x\, A(u_{xxx}) - \epsilon\iint_{\Omega_t} (\phi^{3p})_x\, u_x\, u_{xx} \\ &-\; \epsilon(p-1)\iint_{\Omega_t} (\phi^{3p})_x\, A(u_x)\, u_{xx} - \delta\iint_{\Omega_t} (\phi^{3p})_x\, m_\delta^\beta(u)\, u_x\, A(u_{xx}), \end{aligned} \tag{3.6}$$

and a positive constant C independent of ϵ and δ exists such that

$$|u(t,x_1)-u(t,x_2)| \le C\,|x_1-x_2|^{\frac{1}{2}} \qquad \forall\, x_1,x_2\in\Omega,\; t\in[0,T_\delta], \tag{3.7}$$

$$|u(t_1,x)-u(t_2,x)| \le C\,|t_1-t_2|^{\frac{1}{5p-2}} \qquad \forall\, x\in\Omega,\; t_1\, t_2\in[0,T_\delta],\; |t_1-t_2|\le 1 \tag{3.8}$$

$$u(t,x) \le C \qquad \forall\, x\in\Omega,\; t\in[0,T_\delta]. \tag{3.9}$$

Proof. Let u_σ be the positive solution of $(P_{\epsilon,\delta,\sigma})$ obtained in Proposition 2.2. In the sequel, C denotes a generic positive constant independent of σ. In view of (2.4)-(2.6), by Ascoli-Arzelá's Theorem, a subsequence (still denoted by u_σ) exists such that

$$u_\sigma \overset{\sigma\downarrow 0}{\longrightarrow} u \qquad \text{in } C([0,T_\delta]\times\overline{\Omega}), \tag{3.10}$$

and u satisfies (3.7)-(3.9) and (ii).

• *Energy estimate and direct consequences.* The regularity of u_σ allows to choose $\varphi = -u_{\sigma xx}\chi_{(0,t)}$ in (2.1), obtaining (after integrations by parts)

$$\begin{aligned} \sup_{t\in[0,T_\delta]}\int_\Omega u_{\sigma x}^2(t) \;&+\; \iint_{\Omega_{T_\delta}} \big[m_{\delta,\sigma}(u_\sigma)\,|u_{\sigma xxx}|^p + \epsilon(p-1)\,|u_{\sigma x}|^{p-2}\,u_{\sigma xx}^2 + \epsilon u_{\sigma xx}^2\big] + \\ &+\; \delta\iint_{\Omega_{T_\delta}} m_{\delta,\sigma}^\beta(u_\sigma)\,|u_{\sigma xx}|^p = \int_\Omega u_{0\epsilon x}^2, \end{aligned} \tag{3.11}$$

which immediately implies

$$\begin{aligned} u_\sigma &\overset{*}{\rightharpoonup} u \quad \text{in} \quad L^\infty((0,T_\delta);H^1(\Omega)), \\ u_\sigma &\rightharpoonup u \quad \text{in} \quad L^2((0,T_\delta);H^2(\Omega)). \end{aligned} \tag{3.12}$$

(3.12) yields (iii) (since $u_{\sigma x} = 0$ on S_{T_δ} for all $\sigma > 0$), and implies that $u(t) \in C^1(\overline{\Omega})$ for almost every t, whence

$$u_x(t) = 0 \quad \text{on} \quad \overline{\{u(t) = 0\}} \quad \text{for a.e. } t > 0. \tag{3.13}$$

By Poincaré inequality

$$\begin{aligned} \epsilon \iint_{\Omega_{T_\delta}} |u_{\sigma x}|^p &= \epsilon \iint_{\Omega_{T_\delta}} \left| |u_{\sigma x}|^{\frac{p-2}{2}} u_{\sigma x} \right|^2 \quad \leq \quad C\epsilon \iint_{\Omega_{T_\delta}} |(|u_{\sigma x}|^{\frac{p-2}{2}} u_{\sigma x})_x|^2 \\ &= C\epsilon \iint_{\Omega_{T_\delta}} |u_{\sigma x}|^{p-2} u_{\sigma xx}^2 \overset{(3.11)}{\leq} C. \end{aligned} \tag{3.14}$$

Hence

$$u_\sigma \rightharpoonup u \quad \text{in} \quad L^p((0, T_\delta); W^{1,p}(\Omega)), \tag{3.15}$$

which together with (3.12) implies (3.2). Estimating the right hand side of (2.1) with the help of Hölder inequality, we see that

$$\begin{aligned} \left| \int_0^{T_\delta} < u_{\sigma t}, \varphi > dt \right| \leq & \left(\iint_{\Omega_{T_\delta}} m_{\delta,\sigma}(u_\sigma) |u_{\sigma xxx}|^p \right)^{\frac{p-1}{p}} \left(\iint_{\Omega_{T_\delta}} u_\sigma^n |\varphi_x|^p \right)^{\frac{1}{p}} \\ & + \epsilon \left(\iint_{\Omega_{T_\delta}} |u_{\sigma x}|^2 \right)^{\frac{1}{2}} \left(\iint_{\Omega_{T_\delta}} |\varphi_x|^2 \right)^{\frac{1}{2}} \\ & + \left(\epsilon \iint_{\Omega_{T_\delta}} |u_{\sigma x}|^p \right)^{\frac{p-1}{p}} \left(\epsilon \iint_{\Omega_{T_\delta}} |\varphi_x|^p \right)^{\frac{1}{p}} \\ & + \left(\delta \iint_{\Omega_{T_\delta}} m_{\delta,\sigma}^\beta(u_\sigma) |u_{\sigma xx}|^p \right)^{\frac{p-1}{p}} \left(\delta \iint_{\Omega_{T_\delta}} u_\sigma^{n\beta} |\varphi|^p \right)^{\frac{1}{p}}, \end{aligned} \tag{3.16}$$

which in view of (2.6), (3.11) and (3.14) implies that

$$u_{\sigma t} \rightharpoonup u_t \ \text{in} \ L^{\frac{p}{p-1}}((0, T_\delta); (W^{1,p}(\Omega))'), \tag{3.17}$$

and (3.1) holds true. Combining (3.12) with (3.17) we obtain (cf. Corollary 8.4 in [33])

$$u_\sigma \longrightarrow u \quad \text{in} \quad L^2((0, T_\delta); H^1(\Omega)). \tag{3.18}$$

• *Proof of (3.3).* We claim that

$$u_{xxx} \in L^p_{loc}(\{u > 0\}), \tag{3.19}$$

$$u_{\sigma xxx} \longrightarrow u_{xxx} \text{ in } L^p_{loc}(\{u > 0\}), \tag{3.20}$$

which using also (3.11) implies (3.3), and using also (3.10) implies that

$$u_{\sigma x} \longrightarrow u_x \qquad \text{in } L^p_{loc}(\{u>0\}), \tag{3.21}$$

$$u_{\sigma xx} \longrightarrow u_{xx} \qquad \text{in } L^p_{loc}(\{u>0\}). \tag{3.22}$$

To prove (3.19) and (3.20) it is sufficient to show that

$$u_{xxx} \in L^p(J\times I), \tag{3.23}$$

$$u_{\sigma xxx} \longrightarrow u_{xxx} \text{ in } L^p(J\times I) \tag{3.24}$$

for any open rectangle $J\times I$ such that $\overline{J}\times\overline{I}\subset\{u>0\}$. Since u is continuous, an open rectangle $J_1\times I_1$ exists such that $\overline{J}\times\overline{I}\subset J_1\times I_1$ and $\min_{J_1\times I_1} u=\mu>0$. By (3.10), $u_\sigma\ge\frac{\mu}{2}$ in $J_1\times I_1$ for σ sufficiently small. Therefore, it follows from (3.11) that

$$\int_{J_1\times I_1} |u_{\sigma xxx}|^p \le C_\mu, \tag{3.25}$$

which allows to select a subsequence such that $u_{\sigma xxx}\rightharpoonup f$ in $L^p(J_1\times I_1)$. Since for all $\varphi\in C_0^\infty(J_1\times I_1)$ and all $\sigma>0$

$$\iint_{J_1\times I_1} \varphi_{xxx}\, u_\sigma = -\iint_{J_1\times I_1} \varphi\, u_{\sigma xxx}, \tag{3.26}$$

passing to the limit as $\sigma\to 0$ in (3.26) identifies $f=u_{xxx}$ in $L^p(J_1\times I_1)$, and proves (3.23). To prove (3.24), we argue by contradiction and assume that a positive constant C and a subsequence (still denoted by u_σ) exist such that

$$\iint_{J\times I} |u_{\sigma xxx}-u_{xxx}|^p \ge C > 0. \tag{3.27}$$

By (3.25) and (3.17), we have (possibly up to a subsequence)

$$u_\sigma \longrightarrow u \ \text{ in } \ L^p\left(J_1; W^{2,p}(I_1)\right) \tag{3.28}$$

and, in turn,

$$\lim_{\sigma\to 0}\int_{I_1} |u_{\sigma x}(t)-u_x(t)|^p = 0 \quad \text{for a.e. } t\in J_1. \tag{3.29}$$

Let now $\tilde{J}=(t_1,t_2)$ such that $J\subset\tilde{J}\subset J_1$ and (3.29) holds at $t=t_1$, and let $\varphi\in C^\infty(\mathbf{R})$ be a cut-off function such that $\varphi\equiv 1$ in I and $\varphi\equiv 0$ outside of I_1. Using

$[\varphi^2\,(u_\sigma - u_\eta)_x]_x\,\chi_{\tilde J}$ as a test function in (2.1) with $u = u_\sigma$ and $u = u_\eta$, we obtain

$$
\begin{aligned}
\tfrac12 \int_{I_1} \varphi^2\,(u_\sigma - u_\eta)_x^2 \Big|_{t_1}^{t_2} &= -\iint_{\tilde J\times I_1} [\varphi^2\,(u_\sigma - u_\eta)_x]_x\,(u_\sigma - u_\eta)_t \\
&= -\iint_{\tilde J\times I_1} [\varphi^2\,(u_\sigma - u_\eta)_x]_{xx}\,[m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx}) - m_{\delta,\eta}(u_\eta)\,A(u_{\eta xxx})] \\
&+ \epsilon \iint_{\tilde J\times I_1} [\varphi^2\,(u_\sigma - u_\eta)_x]_{xx}\,[A(u_{\sigma x}) - A(u_{\eta x}) + u_{\sigma x} - u_{\eta x}] \\
&- \delta \iint_{\tilde J\times I_1} [\varphi^2\,(u_\sigma - u_\eta)_x]_x\,[m^\beta_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xx}) - m^\beta_{\delta,\sigma}(u_\sigma)\,A(u_{\eta xx})] \\
&= R_1 + R_2 + R_3. \qquad (3.30)
\end{aligned}
$$

We rewrite R_1 as:

$$
\begin{aligned}
R_1 &= -\iint_{\tilde J\times I_1} \varphi^2\,(u_\sigma - u_\eta)_{xxx}\,m_{\delta,\sigma}(u_\sigma)\,[A(u_{\sigma xxx}) - A(u_{\eta xxx})] \\
&- \iint_{\tilde J\times I_1} \varphi^2\,(u_\sigma - u_\eta)_{xxx}\,A(u_{\eta xxx})\,[m_{\delta,\sigma}(u_\sigma) - m_{\delta,\eta}(u_\eta)] \\
&- 4\iint_{\tilde J\times I_1} \varphi\,\varphi_x\,(u_\sigma - u_\eta)_{xx}\,[m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx}) - m_{\delta,\eta}(u_\eta)\,A(u_{\eta xxx})] \\
&- 2\iint_{\tilde J\times I_1} (\varphi_x^2 + \varphi\,\varphi_{xx})\,(u_\sigma - u_\eta)_x\,[m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx}) - m_{\delta,\eta}(u_\eta)\,A(u_{\eta xxx})] \\
&= R_{1,1} + R_{1,2} + R_{1,3} + R_{1,4}.
\end{aligned}
$$

Since $m_{\delta,\sigma}(u_\sigma)$ is bounded from below on $J_1 \times I_1$ and

$$
(A(s) - A(t))\,(s-t) \;\geq\; C_p|s-t|^p \quad \forall\, s,\, t \in \mathbf{R} \qquad (3.31)
$$

(cf. [19, Lemma 4.4]), we see from (3.30) that

$$
\begin{aligned}
C\iint_{\tilde J\times I_1} |(u_\sigma - u_\eta)_{xxx}|^p \;\leq\; -R_{1,1} \;&\leq\; \tfrac12 \int_{I_1} \varphi^2\,(u_\sigma(t_1) - u_\eta(t_1))_x^2 \\
&+\; R_{1,2} + R_{1,3} + R_{1,4} + R_2 + R_3. \qquad (3.32)
\end{aligned}
$$

We now show that the right-hand side of (3.32) vanishes as $\sigma, \eta \downarrow 0$. For the first integral we simply recall (3.29) and the choice of t_1:

$$
\int_{I_1} (u_{\sigma x}(t_1) - u_{\eta x}(t_1))^2 \;=\; o_{\sigma,\eta}(1). \qquad (3.33)
$$

To handle $R_{1,2}$ we write

$$\begin{aligned} |R_{1,2}| &\leq C\,\|m_{\delta,\sigma}(u_\sigma) - m_{\delta,\eta}(u_\eta)\|_\infty \iint_{J_1\times I_1} (|u_{\sigma xxx}|^p + |u_{\eta xxx}|^p) \\ &\overset{(3.10),(3.25)}{=} o_{\sigma,\eta}(1). \end{aligned} \tag{3.34}$$

By (3.11) and (3.25), we obtain for $R_{1,3}$, $R_{1,4}$ and R_3

$$|R_{1,3}| + |R_{1,4}| + |R_3| \leq C\,\|u_\sigma - u_\eta\|_{L^p(J_1;W^{2,p}(I_1))} \overset{(3.28)}{=} o_{\sigma,\eta}(1). \tag{3.35}$$

Using also (3.14) and the inequality

$$|A(s) - A(t)| \leq C_p(|s|^{p-2} + |t|^{p-2})\,|s-t|$$

we obtain for R_2

$$|R_2| \leq C\,\|u_\sigma - u_\eta\|_{L^p(J_1;W^{2,p}(I_1))} \overset{(3.28)}{=} o_{\sigma,\eta}(1). \tag{3.36}$$

Collecting (3.33)-(3.36) into (3.32) we obtain

$$C\iint_{J\times I} |(u_\sigma - u_\eta)_{xxx}|^p \leq o_{\sigma,\eta}(1),$$

in contradiction with (3.27). This completes the proof of (3.20).

• *Proof of (3.4).* It follows from (3.11) that $m_{\delta,\sigma}^{\beta/p}(u_\sigma)\,u_{\sigma xx} \rightharpoonup f$ in $L^p(\Omega_{T_\delta})$, with $f = m_\delta^{\beta/p}(u)\,u_{xx}$ a.e. on $\{u>0\}$ in view of (3.10) and (3.22). Hence $m_\delta^{\beta/p}(u)\,u_{xx} \in L^p(\{u>0\})$. Since (by (3.12)) u_{xx} is defined a.e. in Ω_{T_δ} and $m_\delta(0)=0$, we obtain the first part of (3.4). The second part follows by the same argument, taking (3.13) into account.

• *Proof of (3.5).* We prove (3.5) passing to the limit as $\sigma \to 0$ in (2.1). First we show that

$$I_\sigma = \iint_{\Omega_{T_\delta}} m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx})\,\varphi_x \overset{\sigma\downarrow 0}{\longrightarrow} \iint_{\{u>0\}} m_\delta(u)\,A(u_{xxx})\,\varphi_x. \tag{3.37}$$

For any $\eta>0$, we split I_σ as follows:

$$I_\sigma = \iint_{\{u>\eta\}} m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx})\,\varphi_x + \iint_{\{u\leq\eta\}} m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx})\,\varphi_x.$$

From (3.10) and (3.20), we have for the first integral

$$\iint_{\{u>\eta\}} m_{\delta,\sigma}(u_\sigma)\,A(u_{\sigma xxx})\,\varphi_x \overset{\sigma\downarrow 0}{\longrightarrow} \iint_{\{u>\eta\}} m_\delta(u)\,A(u_{xxx})\,\varphi_x,$$

whereas the second one is small, uniformly w.r.t. σ:

$$\iint_{\{u\le\eta\}} m_{\delta,\sigma}(u_\sigma)\, A(u_{\sigma xxx})\, \varphi_x \;\le\; C\,\eta^{\frac{n}{p}} \left(\iint_{\Omega_{T_\delta}} m_{\delta,\sigma}(u_\sigma)\, |u_{\sigma xxx}|^p\right)^{\frac{p-1}{p}} \overset{(3.11)}{=} o_\eta(1).$$

Dominated convergence with respect to η yields (3.37). By the same argument (recalling (3.22))

$$\iint_{\Omega_{T_\delta}} m^\beta_{\delta,\sigma}(u_\sigma)\, A(u_{\sigma xx})\, \varphi \;\overset{\sigma\downarrow 0}{\longrightarrow}\; \iint_{\{u>0\}} m^\beta_\delta(u)\, A(u_{xx})\, \varphi \;=\; \iint_{\Omega_{T_\delta}} m^\beta_\delta(u)\, A(u_{xx})\, \varphi \tag{3.38}$$

(in the last equality we used the fact that u_{xx} is defined a.e. in Ω_{T_δ}). The convergence

$$\iint_{\Omega_{T_\delta}} u_{\sigma x}\, \varphi_x \;\overset{\sigma\downarrow 0}{\longrightarrow}\; \iint_{\Omega_{T_\delta}} u_x\, \varphi_x \tag{3.39}$$

follows immediately from (3.12). Finally, we claim that

$$\iint_{\Omega_{T_\delta}} A(u_{\sigma x})\, \varphi_x \;\overset{\sigma\downarrow 0}{\longrightarrow}\; \iint_{\{u>0\}} A(u_x)\, \varphi_x \;=\; \iint_{\Omega_{T_\delta}} A(u_x)\, \varphi_x \tag{3.40}$$

(the last equality follows from (3.13)). To prove (3.40), for $\eta > 0$ we split as before the domain of integration. On $\{u > \eta\}$ the convergence is straightforward in view of (3.21), whereas on $\{u \le \eta\}$ we have

$$\iint_{\{u\le\eta\}} |A(u_{\sigma x})\, \varphi_x| \;\le\; \frac{C}{\left(\inf_{s\in(0,\eta)} G''_{\delta,\sigma}(s)\right)^{\frac{p-1}{p}}} \left(\iint_{\Omega_{T_\delta}} G''_{\delta,\sigma}(u_\sigma)\, |u_{\sigma x}|^p\right)^{\frac{p-1}{p}} \overset{(2.3)}{=} o_\eta(1)$$

since

$$\inf_{s\in(0,\eta)} G''_{\delta,\sigma}(s) \;\ge\; \eta^{-\frac{n}{p-1}}. \tag{3.41}$$

Collecting (3.17), (3.37)-(3.40) and passing to the limit as $\sigma \downarrow 0$ in (2.1) we obtain (3.5).

• *Proof of (3.6).* Choosing $\varphi = \chi_{(0,t)}\,[\phi^{3p}\,u_{\sigma x}]_x$ as a test function in (2.1), we obtain

$$\begin{aligned}
\tfrac{1}{2}\int_\Omega \phi^{3p}\,u_{\sigma x}^2(t) \;&+\; \iint_{\Omega_t} \phi^{3p}\,\left[m_{\delta,\sigma}(u_\sigma)\,|u_{\sigma xxx}|^p + \epsilon\,(p-1)\,|u_{\sigma x}|^{p-2}\,u_{\sigma xx}^2 + \epsilon\,u_{\sigma xx}^2\right]\\
&+\; \delta \iint_{\Omega_t} \phi^{3p}\,m_{\delta,\sigma}^{\beta}(u_\sigma)\,|u_{\sigma xx}|^p\\
&=\; \tfrac{1}{2}\int_\Omega \phi^{3p}\,u_{0\epsilon x}^2 - 2\iint_{\Omega_t} (\phi^{3p})_x\,m_{\delta,\sigma}(u_\sigma)\,u_{\sigma xx}\,A(u_{\sigma xxx})\\
&-\; \iint_{\Omega_t} (\phi^{3p})_{xx}\,m_{\delta,\sigma}(u_\sigma)\,u_{\sigma x}\,A(u_{\sigma xxx}) - \epsilon \iint_{\Omega_t} (\phi^{3p})_x\,u_{\sigma x}\,u_{\sigma xx}\\
&-\; \epsilon\,(p-1)\iint_{\Omega_t} (\phi^{3p})_x\,A(u_{\sigma x})\,u_{\sigma xx} - \delta \iint_{\Omega_t} (\phi^{3p})_x\,m_{\delta,\sigma}^{\beta}(u_\sigma)\,u_{\sigma x}\,A(u_{\sigma xx})\\
&=\; \tfrac{1}{2}\int_\Omega \phi^{3p}\,u_{0\epsilon x}^2 + I_{1\sigma} + I_{2\sigma} + I_{3\sigma} + I_{4\sigma} + I_{5\sigma} \qquad (3.42)
\end{aligned}$$

for all $t \in [0, T_\delta]$. The passage to the limit as $\sigma \to 0$ on the left-hand side of (3.42) follows by lower semi-continuity arguing as in the proof of (3.4). To handle $I_{1\sigma}$, we fix $\eta > 0$ and split it as usual. On the one hand, by (3.20)

$$\lim_{\sigma\to 0} \iint_{\{u\geq\eta\}_t} (\phi^{3p})_x\,m_{\delta,\sigma}(u_\sigma)\,u_{\sigma xx}\,A(u_{\sigma xxx}) \;=\; \iint_{\{u\geq\eta\}_t} (\phi^{3p})_x\,m_\delta(u)\,u_{xx}\,A(u_{xxx}),$$

and on the other hand

$$\begin{aligned}
\left|\iint_{\{u<\eta\}_t} (\phi^{3p})_x\,m_{\delta,\sigma}(u_\sigma)\,u_{\sigma xx}\,A(u_{\sigma xxx})\right| \;&\leq\; \frac{\eta^{\frac{n(1-\beta)}{p}}}{\delta^{\frac{1}{p}}} \left(\iint_{\Omega_t} m_{\delta,\sigma}(u_\sigma)\;|u_{\sigma xxx}|^p\right)^{\frac{p-1}{p}}\\
&\quad\times \left(\delta \iint_{\Omega_t} m_{\delta,\sigma}^{\beta}(u_\sigma)\;|u_{\sigma xx}|^p\right)^{\frac{1}{p}}\\
&\stackrel{(3.11)}{=}\; o_\eta(1).
\end{aligned}$$

The arbitrariness of η then gives

$$\lim_{\sigma\to 0} I_{1\sigma} \;=\; \iint_{\{u>0\}_t} (\phi^{3p})_x\,m_\delta(u)\,u_{xx}\,A(u_{xxx}).$$

The passage to the limit of $I_{2\sigma}$ is similar (using (3.14)). The one in $I_{3\sigma}$ is straightforward in view of (3.12) and (3.18). Concerning $I_{4\sigma}$, we have

$$\begin{aligned}\left|\iint_{\{u<\eta\}_t} (\phi^{3p})_x\, A(u_{\sigma x})\, u_{\sigma xx}\right| \quad &\le \quad \frac{C}{\inf_{s\in(0,\eta)}(G''_{\delta,\sigma}(s))^{\frac12}} \left(\iint_{\Omega_t} |u_{\sigma x}|^p\; G''_{\delta,\sigma}(u_\sigma)\right)^{\frac12} \\ &\qquad \times \left(\iint_{\Omega_t} |u_{\sigma x}|^{p-2}\; |u_{\sigma xx}|^2\right)^{\frac12} \\ &\overset{(2.3),(3.11)}{\le} \quad \frac{C}{\inf_{s\in(0,\eta)}(G''_{\delta,\sigma}(s))^{\frac12}} \overset{(3.41)}{\le} o_\eta(1)\end{aligned}$$

so that

$$\lim_{\sigma\to 0} I_{4\sigma} \;=\; -\epsilon\,(p-1)\iint_{\{u>0\}_t} (\phi^{3p})_x\, A(u_x)\, u_{xx} \overset{(3.13)}{=} -\epsilon\,(p-1)\iint_{\Omega_t} (\phi^{3p})_x\, A(u_x)\, u_{xx}.$$

The passage to the limit in $I_{5\sigma}$ proceeds analogously. □

4. Further regularity: An extension of Bernis estimates

In this section we prove the following result:

Proposition 4.1 (Extension of Bernis estimates). *Let $-\infty \le a < b \le \infty$, $p \ge \frac43$, $n \in \left(\frac{p-1}{2}, 2p-1\right)$. A positive constant C, depending only on p and n, exists such that the inequalities*

$$\int_a^b \phi^{3p}\, u^{n-2p}\, |u_x|^{3p} \;\le\; C\int_a^b \phi^{3p}\, u^n\, |u_{xxx}|^p + C\int_a^b |\phi_x|^{3p}\, u^{n+p} \tag{4.1}$$

$$\int_a^b \phi^{3p}\, u^{n-\frac p2}\, |u_{xx}|^{\frac{3p}{2}} \;\le\; C\int_a^b \phi^{3p}\, u^n\, |u_{xxx}|^p + C\int_a^b |\phi_x|^{3p}\, u^{n+p} \tag{4.2}$$

hold for any non-negative function $\phi \in W^{1,\infty}((a,b))$ and any function u such that

(A) *$u \in C([a,b];[0,\infty)) \cap W^{3,p}_{loc}((a,b))$, $u > 0$ in (a,b), $u_x \in L^2((a,b))$;*

(B) *sequences $a_m \searrow a$, $b_m \nearrow b$ exist such that $u_x(a_m) \to 0$, $u_x(b_m) \to 0$;*

(C) *$\int_a^b u^n\, |u_{xxx}|^p < \infty$, $\int_a^b |\phi_x|^{3p}\, u^{n+p} < \infty$.*

Estimates (4.1) and (4.2) have been first proved by Bernis [7] for smooth positive functions in one space dimension, either for $p = 2$, $n \in \left(\frac12, 3\right)$, and for $p \ge \frac43$, $n \in \left(\frac{p-1}{2}, \frac{p+1}{2} - \frac{1}{3p}\right)$. In one space dimension and for $p = 2$, the assumptions on u have been later relaxed to those in Proposition 4.1 by Giacomelli and Otto in [27]. The higher dimensional version has been obtained by Grün [23] for positive functions in $H^2(\Omega)$ with zero normal derivative at the boundary provided $p = 2$, $n \in \left[2 - \sqrt{1 - \frac{N}{N+8}}, 3\right)$.

A simple consequence of Proposition 4.1 is the following:

Corollary 4.2. *Let $\Omega \subseteq \mathbf{R}$, $p \geq \frac{4}{3}$, $n \in \left(\frac{p-1}{2}, 2p-1\right)$, and $u \in C^1(\overline{\Omega}) \cap W^{3,p}_{loc}(\{u > 0\})$ such that*

$$u \geq 0, \quad u_x\Big|_{\partial\Omega} = 0, \quad u_x \in L^2(\Omega), \quad \int_{\{u>0\}} u^n \, |u_{xxx}|^p \, dx < \infty.$$

Then

$$u^{\frac{n+p}{3p}} \in W^{1,3p}(\Omega), \tag{4.3}$$

$$u^{\frac{2(n+p)}{3p}} \in W^{2,\frac{3p}{2}}(\Omega), \tag{4.4}$$

$$u^{\frac{n+p}{p}} \in W^{3,p}(\Omega), \tag{4.5}$$

and a positive constant C exists, depending only on n and p, such that:

$$\int_\Omega \phi^{3p} \left| \left(u^{\frac{n+p}{3p}}\right)_x \right|^{3p} \leq C \int_{\{u>0\}} \phi^{3p} \, u^n \, |u_{xxx}|^p + C \int_\Omega |\phi_x|^{3p} \, u^{n+p} \tag{4.6}$$

$$\int_\Omega \phi^{3p} \left| \left(u^{\frac{2(n+p)}{3p}}\right)_{xx} \right|^{\frac{3p}{2}} \leq C \int_{\{u>0\}} \phi^{3p} \, u^n \, |u_{xxx}|^p + C \int_\Omega |\phi_x|^{3p} \, u^{n+p}, \tag{4.7}$$

$$\int_\Omega \phi^{3p} \left| \left(u^{\frac{n+p}{p}}\right)_{xxx} \right|^{p} \leq C \int_{\{u>0\}} \phi^{3p} \, u^n \, |u_{xxx}|^p + C \int_\Omega |\phi_x|^{3p} \, u^{n+p} \tag{4.8}$$

for any non-negative function $\phi \in W^{1,\infty}(\Omega)$ such that $\int_\Omega |\phi_x|^{3p} \, u^{n+p} < \infty$.

The rest of the section is concerned with the proof of Proposition 4.1 and Corollary 4.2. In what follows, C indicates a generic constant which depends only on n and p.

Proof of Proposition 4.1. The starting point is the following lemma, which permits to control — either by sign or by magnitude — the different boundary terms which appear when integrating by parts on (a_m, b_m). Since its proof is identical to that of [27, Lemma 5.4], we omit it here.

Lemma 4.3. *Under the assumptions of Proposition 4.1, new sequences $a_m \searrow a$, $b_m \nearrow b$ exist such that*

$$u(a_m)^{\frac{n-2p+1}{3p-1}} \, u_x(a_m) \to 0, \quad u(b_m)^{\frac{n-2p+1}{3p-1}} \, u_x(b_m) \to 0, \tag{4.9}$$

$$u_x(a_m) \, u_{xx}(a_m) \geq 0, \quad u_x(b_m) \, u_{xx}(b_m) \leq 0. \tag{4.10}$$

For notational convenience, we write $\int$ for $\int_{a_m}^{b_m}$, and ϵ_m for a generic quantity which vanishes as $m \to \infty$. For $\sigma \in \left[0, \frac{3p}{2}\right]$ we define:

$$
\begin{aligned}
I_\sigma &:= \int \phi^{3p}\, u^{n+\sigma-2p}\, |u_x|^{3p-2\sigma}\, |u_{xx}|^{\sigma}, \\
D &:= \int \phi^{3p}\, u^{n}\, |u_{xxx}|^{p}, \\
E &:= \int |\phi_x|^{3p}\, u^{n+p}.
\end{aligned}
$$

We wish to estimate I_0 and $I_{\frac{3p}{2}}$ in terms of D and E. Integrating by parts, we have

$$
\begin{aligned}
I_0 = \int \phi^{3p}\, u^{n-2p}\, |u_x|^{3p-2}\, u_x\, u_x \overset{(4.9)}{=} & \ \epsilon_m - \tfrac{3p-1}{n-2p+1} \int \phi^{3p}\, u^{n-2p+1}\, |u_x|^{3p-2}\, u_{xx} \\
& - \tfrac{3p}{n-2p+1} \int \phi^{3p-1}\, \phi_x\, u^{n-2p+1}\, |u_x|^{3p-2}\, u_x \qquad (4.11)
\end{aligned}
$$

and, for $\sigma \in \left[2, \frac{3p}{2}\right]$,

$$
\begin{aligned}
I_\sigma &= \int \phi^{3p}\, u^{n+\sigma-2p}\, |u_x|^{3p-2\sigma}\, |u_{xx}|^{\sigma-2}\, u_{xx}\, u_{xx} \\
&\overset{(4.10)}{\leq} -\tfrac{n+\sigma-2p}{3p-2\sigma+1} \int \phi^{3p}\, u^{n+\sigma-2p-1}\, |u_x|^{3p-2\sigma+2}\, |u_{xx}|^{\sigma-2}\, u_{xx} \\
&\quad - \tfrac{\sigma-1}{3p-2\sigma+1} \int \phi^{3p}\, u^{n+\sigma-2p}\, |u_x|^{3p-2\sigma}\, u_x\, |u_{xx}|^{\sigma-2}\, u_{xxx} \\
&\quad - \tfrac{3p}{3p-2\sigma+1} \int \phi^{3p-1}\, \phi_x\, u^{n+\sigma-2p}\, |u_x|^{3p-2\sigma}\, u_x\, |u_{xx}|^{\sigma-2}\, u_{xx}. \qquad (4.12)
\end{aligned}
$$

In particular, for $\sigma = \frac{3p}{2}$, an application of Hölder inequality yields

$$
I_{\frac{3p}{2}} \leq C\, (I_0)^{\frac{2}{3p}} \left(I_{\frac{3p}{2}}\right)^{\frac{3p-2}{3p}} + C\, (I_0)^{\frac{1}{3p}} \left(I_{\frac{3p}{2}}\right)^{\frac{3p-4}{3p}} D^{\frac{1}{p}} + C\, (I_0)^{\frac{1}{3p}} \left(I_{\frac{3p}{2}}\right)^{\frac{3p-2}{3p}} E^{\frac{1}{3p}}. \qquad (4.13)
$$

The rest of the proof is split into three cases.

• *Case* $2p-2 < n < 2p-1$. Integrating by parts the first integral at the right-hand side of (4.11), we see that

$$\begin{aligned} I_0 \overset{(4.10)}{\leq} \; & \epsilon_m + \tfrac{3(p-1)\,(3p-1)}{(n-2p+1)\,(n-2p+2)} \int \phi^{3p}\, u^{n-2p+2}\, |u_x|^{3p-4}\, |u_{xx}|^2 \\ & + \tfrac{(3p-1)}{(n-2p+1)\,(n-2p+2)} \int \phi^{3p}\, u^{n-2p+2}\, |u_x|^{3p-4}\, u_x\, u_{xxx} \\ & + \tfrac{3p\,(3p-1)}{(n-2p+1)\,(n-2p+2)} \int \phi^{3p-1}\, \phi_x\, u^{n-2p+2}\, |u_x|^{3p-4}\, u_x\, u_{xx} \\ & + \tfrac{3p}{(n-2p+1)} \int \phi^{3p-1}\, \phi_x\, u^{n-2p+1}\, |u_x|^{3p-2}\, u_x \\ =: \; & \epsilon_m + \tfrac{3\,(p-1)\,(3p-1)}{(n-2p+1)\,(n-2p+2)}\, I_2 + J_2 + J_3 + J_4. \end{aligned}$$

(each term is finite for $p \geq \frac{4}{3}$). Since $(n-2p+1)(n-2p+2) < 0$, the first term on the right-hand side has negative sign. The others can be estimated via Hölder and Young inequalities:

$$\begin{aligned} J_2 &\leq \tfrac{1}{4}\, I_0 + C\, D, \\ J_3 &\leq \tfrac{1}{4}\, I_0 + \tfrac{1}{4}\, I_2 + C\, E, \\ J_4 &\leq \tfrac{1}{4}\, I_0 + C\, E. \end{aligned}$$

Hence we obtain

$$I_0 \leq I_0 + I_2 \leq \epsilon_m + C\,(D+E) \tag{4.14}$$

and (4.1) follows immediately passing to the limit as $m \to \infty$. Substituting (4.14) into (4.13) we see that

$$I_{\frac{3p}{2}} \leq C\,(\epsilon_m + D + E)^{\frac{2}{3p}} \left(I_{\frac{3p}{2}}\right)^{\frac{3p-2}{3p}} + C\,(\epsilon_m + D + E)^{\frac{4}{3p}} \left(I_{\frac{3p}{2}}\right)^{\frac{3p-4}{3p}}.$$

Hence, by Young's inequality,

$$I_{\frac{3p}{2}} \leq \epsilon_m + C\,(D+E),$$

and (4.2) follows passing to the limit as $m \to \infty$.

• *Case* $\frac{p}{2} \leq n \leq 2p-2$. From (4.11), using Hölder inequality we obtain for $\sigma > 1$

$$I_0 \leq \epsilon_m + C\,(I_\sigma)^{\frac{1}{\sigma}}\,(I_0)^{\frac{\sigma-1}{\sigma}} + C\,(E)^{\frac{1}{3p}}\,(I_0)^{\frac{3p-1}{3p}}$$

so that, by Young's inequality,

$$I_0 \leq \epsilon_m + C\,(I_\sigma + E). \tag{4.15}$$

In the range $n \in [\frac{p}{2}, 2p-2]$ we may choose $\sigma = 2p - n \in \left[2, \frac{3p}{2}\right]$ in (4.12). With this choice the first integral on the right-hand side of (4.12) vanishes, and using Hölder inequality we obtain

$$I_\sigma \leq C\, D^{\frac{1}{p}}\, (I_\sigma)^{\frac{\sigma-2}{\sigma}}\, (I_0)^{\frac{2p-\sigma}{p\sigma}} + C\, E^{\frac{1}{3p}}\, (I_\sigma)^{\frac{\sigma-1}{\sigma}}\, (I_0)^{\frac{3p-\sigma}{3p\sigma}}.$$

Young's inequality and (4.15) then yield

$$I_\sigma \leq \epsilon_m + C\,(D+E),$$

and using again (4.15) we also infer

$$I_0 \leq \epsilon_m + C\,(D+E), \tag{4.16}$$

which gives (4.1) passing to the limit as $m \to \infty$. The proof of (4.2) now proceed as in the previous case, substituting (4.16) into (4.13).

• *Case* $\frac{p-1}{2} < n < \frac{p}{2}$. In this range of values of n, Proposition 4.1 with $\phi = 1$ has already been proved in [7] for positive smooth functions. The extension to non-negative functions satisfying assumptions (A), (B) and (C) follows by means of Lemma 4.3, and the additional terms coming from a non-constant ϕ can be controlled as in the previous two cases. □

Proof of Corollary 4.2. Since $u \in C^1(\Omega)$ and $u_x = 0$ on $\partial\Omega$, we may apply Proposition 4.1 on each connected component (a,b) of $\{u > 0\}$, obtaining:

$$\int_{\{u>0\}} \phi^{3p}\, u^{n-2p}\, |u_x|^{3p} \leq C \int_{\{u>0\}} \phi^{3p}\, u^n\, |u_{xxx}|^p + C \int_\Omega |\phi_x|^{3p}\, u^{n+p}, \tag{4.17}$$

$$\int_{\{u>0\}} \phi^{3p}\, u^{n-\frac{p}{2}}\, |u_{xx}|^{\frac{3p}{2}} \leq C \int_{\{u>0\}} \phi^{3p}\, u^n\, |u_{xxx}|^p + C \int_\Omega |\phi_x|^{3p}\, u^{n+p}. \tag{4.18}$$

Inequality (4.17) immediately implies (4.3) and (4.6). To prove (4.4) and (4.7), let (a,b) be any connected component of $\{u > 0\}$. We preliminarily observe that by (4.3)

$$u(x) \leq C\, |x-a|^{\frac{3p-1}{n+p}} \quad \forall\, x \in (a,b). \tag{4.19}$$

This implies that

$$\frac{1}{\delta}\int_a^{a+\delta} \left|\left(u^{\frac{2(n+p)}{3p}}\right)_x\right| = \frac{1}{\delta}\int_a^{a+\delta} \left|u^{\frac{2n-p}{3p}}\, u_x\right| \overset{(4.3)}{\leq} \frac{C}{\delta}\left(\int_a^{a+\delta} u^{\frac{n+p}{3p-1}}\right)^{\frac{3p-1}{3p}}$$
$$\overset{(4.19)}{\leq} C\, \delta^{\frac{3p-2}{3p}} \downarrow 0$$

Since the same argument applies to b, we conclude that $\left(u^{\frac{2(n+p)}{3p}}\right)_x = 0$ on $\partial\{u>0\}$, which in turn implies, after two integration by parts, that

$$\int_\Omega u^{\frac{2(n+p)}{3p}}\,\psi_{xx} \;=\; \int_{\{u>0\}} u^{\frac{2(n+p)}{3p}}\,\psi_{xx} \;=\; \frac{2(n+p)}{3p}\int_{\{u>0\}}\left(\frac{2n-p}{3p}\,u^{\frac{2(n-2p)}{3p}}\,u_x^2 + u^{\frac{2n-p}{3p}}\,u_{xx}\right)\psi$$

for any test function $\psi \in C^2_{loc}(\overline{\Omega})$. Hence, by (4.17) and (4.18),

$$\left(u^{\frac{2(n+p)}{3p}}\right)_{xx} \;=\; \frac{2(n+p)}{3p}\left(\frac{2n-p}{3p}\,u^{\frac{2(n-2p)}{3p}}\,u_x^2 + u^{\frac{2n-p}{3p}}\,u_{xx}\right)\chi_{\{u>0\}} \quad \text{in } L^{\frac{3p}{2}}(\Omega), \tag{4.20}$$

which in particular implies (4.4). Now (4.7) follows at once from (4.17) and (4.18), raising (4.20) to the power $\frac{3p}{2}$, multiplying by ϕ^{3p} and integrating over Ω. The proof of (4.5) and (4.8) is analogous, and we omit it. □

5. Solutions in bounded domains

In this section we pass to the limit as $\delta \downarrow 0$, $\epsilon \downarrow 0$ and obtain a solution of (P) in a bounded domain. The arguments are in many respects similar to those of section 3. The main difference is that some of the controls given by the second-order relaxation are no longer uniform, and have to be replaced by those given by Bernis' estimates. It turns out that one can choose $\delta = \epsilon$ and perform the two limiting procedures at once.

Proof of Theorem 1 for bounded Ω. Let u_ϵ be the solution of $(P_{\epsilon,\epsilon})$ obtained in Proposition 3.1. In the sequel, C denotes a generic constant independent of ϵ and time. In view of (3.7)-(3.9), we obtain for a subsequence

$$u_\epsilon \overset{\epsilon\downarrow 0}{\longrightarrow} u \qquad \text{in } C([0,T]\times\overline{\Omega}) \qquad \forall\, T<\infty \tag{5.1}$$

which in particular implies (iv) in Def. 1. It follows from (3.6) with $\phi \equiv 1$ that for all T and all $\epsilon < T^{\frac{1}{\beta-1}}$

$$\begin{aligned}\sup_{t\in[0,T]}\frac{1}{2}\int_\Omega u_{\epsilon x}^2(t) \;&+\; \iint_{\{u_\epsilon>0\}_T} m_\epsilon(u_\epsilon)\,|u_{\epsilon xxx}|^p + \epsilon\iint_{\Omega_T}\left(1+(p-1)\,|u_{\epsilon x}|^{p-2}\right)u_{\epsilon xx}^2\\ &+\;\epsilon\iint_{\Omega_T} m_\epsilon^\beta(u_\epsilon)\,|u_{\epsilon xx}|^p \;\le\; \frac{1}{2}\int_\Omega u_{0\epsilon x}^2.\end{aligned} \tag{5.2}$$

From this we easily infer (cf. also (3.16)) that

$$\begin{aligned} u_\epsilon &\overset{*}{\rightharpoonup} u \quad &&\text{in} \quad L^\infty_{loc}([0,\infty);H^1(\Omega)),\\ u_{\epsilon t} &\rightharpoonup u_t \quad &&\text{in} \quad L^{\frac{p}{p-1}}_{loc}([0,\infty);(W^{1,p}(\Omega))'),\end{aligned} \tag{5.3}$$

and that in fact $u \in L^\infty([0,\infty); H^1(\Omega))$. Strong convergence locally on the positivity set is obtained as in Proposition 3.1 (cf. the proof of (3.3)):

$$u_{\epsilon xxx} \longrightarrow u_{xxx} \quad \text{in} \quad L^p_{loc}(\{u>0\}), \tag{5.4}$$

which together with (5.2) implies (ii) in Def. 1, and together with (5.1) implies that

$$u_{\epsilon x} \longrightarrow u_x \quad \text{in} \quad L^p_{loc}(\{u>0\}), \tag{5.5}$$

$$u_{\epsilon xx} \longrightarrow u_{xx} \quad \text{in} \quad L^p_{loc}(\{u>0\}). \tag{5.6}$$

In turn (5.5) and (5.6), together with (iii) in Proposition 3.1, yield

$$u_x \;=\; 0 \quad \text{in } L^p_{loc}(\{u>0\} \cap (\mathbf{R}^+ \times \partial\Omega)). \tag{5.7}$$

The regularity of u_ϵ given by (3.2) guarantees that $u_\epsilon(t) \in C^1(\Omega)$ for almost every $t \in [0, T_\epsilon]$. In addition, by (3.9)

$$m_\epsilon(u_\epsilon) \;\geq\; \frac{1}{2} u_\epsilon^n \tag{5.8}$$

for all $\epsilon \leq \epsilon_0$. Then, (5.2) and Corollary 4.2 imply for all T and ϵ sufficiently small

$$\iint_{\Omega_T} |(u_\epsilon^{\frac{n+p}{3p}})_x|^{3p} + \iint_{\Omega_T} |(u_\epsilon^{\frac{2(n+p)}{3p}})_{xx}|^{\frac{3p}{2}} + \iint_{\Omega_T} |(u_\epsilon^{\frac{n+p}{p}})_{xxx}|^p \;\leq\; C, \tag{5.9}$$

so that (v) in Def. 1 holds, and in addition

$$\int_0^\infty \int_\Omega |(u^{\frac{n+p}{3p}})_x|^{3p} \;\leq\; C. \tag{5.10}$$

As already pointed out in the Introduction (cf. (1.11)), (5.10) guarantees that $u_x(t) = 0$ on $\overline{\{u(t)=0\}}$ for a.e. $t > 0$, and recalling (5.7) we infer (vi) in Def. 1.

To prove (iii) in Def. 1, we pass to the limit as $\epsilon \downarrow 0$ in (3.5). For a given test φ, let T such that $\operatorname{supp}(\varphi) \subset [0,T) \times \overline{\Omega}$. Arguing as in the proof of Proposition 3.1 we obtain

$$I_\epsilon = \iint_{\{u_\epsilon > 0\}} m_\epsilon(u_\epsilon)\, A(u_{\epsilon xxx})\, \varphi_x \;\overset{\epsilon\downarrow 0}{\longrightarrow}\; \iint_{\{u>0\}} u^n\, A(u_{xxx})\, \varphi_x, \tag{5.11}$$

whereas the lower order terms vanish in the limit:

$$\epsilon \iint_{\Omega_T} \left[|u_{\epsilon x}|^{p-2}\, u_{\epsilon x} + u_{\epsilon x}\right]\, \varphi_x + \epsilon \iint_{\Omega_T} m_\epsilon{}^\beta(u_\epsilon)\, A(u_{\epsilon xx})\, \varphi \;\overset{\epsilon\downarrow 0}{\longrightarrow}\; 0. \tag{5.12}$$

Indeed, by (5.9) and (3.9), we have for ϵ sufficiently small

$$\iint_{\Omega_T} |u_{\epsilon x}|^p \;\leq\; \left(\iint_{\Omega_T} |(u_\epsilon^{\frac{n+p}{3p}})_x|^{3p}\right)^{\frac{1}{3}} \left(\iint_{\Omega_T} u_\epsilon^{\frac{2p-n}{2}}\right)^{\frac{1}{3}} \;\leq\; C, \tag{5.13}$$

and by (5.2)

$$\epsilon \iint_{\Omega_T} |m_\epsilon^\beta(u_\epsilon)\, A(u_{\epsilon xx})\, \varphi| \;\le\; C\, \epsilon^{\frac{1}{p}}.$$

Therefore (5.12) holds, and together with (5.11) and (5.3) implies (iii) in Definition 1. It is now easy to infer a-posteriori that the regularity of u_t given by (5.3) slightly improves to $u_t \in L^{\frac{p}{p-1}}([0,\infty);(W^{1,p}(\Omega))')$. Hence we have shown that u is a solution of (P) in the sense of Definition 1.

The proof of (1.17) is analogous to that in [4, proof of Theorem 3.1], and we reproduce it for completeness. Arguing as in [6, proof of Lemma 2.1], one easily sees from (1.15) that u is uniformly Hölder continuous in time:

$$|u(t_1,x) - u(t_2,x)| \;\le\; C\,|t_1 - t_2|^{\frac{1}{5p-2}} \quad \forall\, x \in \Omega,\; t_1,\, t_2 \in [0,\infty). \tag{5.14}$$

Since

$$f(t) \;=\; \left(\max_{x\in\Omega} u^{\frac{n+p}{3p}}(t,x) - \min_{x\in\Omega} u^{\frac{n+p}{3p}}(t,x) \right)^{3p} \;\le\; |\Omega|^{3p-1} \int_\Omega |(u^{\frac{n+p}{3p}})_x|^{3p},$$

it follows from (5.10) that $f(t) \in L^1(\mathbf{R}^+)$. Hence, in view of (5.14), $f(t) \to 0$ as $t \to \infty$, which means that

$$\max_{x\in\Omega} u(t,x) - \min_{x\in\Omega} u(t,x) \;\stackrel{t\uparrow\infty}{\longrightarrow}\; 0.$$

Since (again by (1.15)) mass is conserved, we conclude that

$$\lim_{t\to\infty} u(t,x) \;=\; ⨍_\Omega u_0.$$

It remains to prove (1.18). The starting point is (3.6): For any T, almost every $t \in [0,T)$ (with ϵ so small that $T < T_\epsilon$) and every function ϕ as in (1.19), it holds

$$\begin{aligned}
\tfrac{1}{2}\int_\Omega \phi^{3p}\, u_{\epsilon x}^2(t) \;&+\; \iint_{\{u_\epsilon>0\}_t} \phi^{3p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xxx}|^p \\
&\le\; \tfrac{1}{2}\int_\Omega \phi^{3p}\, u_{0\epsilon x}^2 - 2\iint_{\{u_\epsilon>0\}_t} (\phi^{3p})_x\, m_\epsilon(u_\epsilon)\, u_{\epsilon xx}\, A(u_{\epsilon xxx}) \\
&-\; \iint_{\{u_\epsilon>0\}_t} (\phi^{3p})_{xx}\, m_\epsilon(u_\epsilon)\, u_{\epsilon x}\, A(u_{\epsilon xxx}) - \epsilon \iint_{\Omega_t} (\phi^{3p})_x\, u_{\epsilon x}\, u_{\epsilon xx} \\
&-\; \epsilon\,(p-1) \iint_{\Omega_t} (\phi^{3p})_x\, A(u_{\epsilon x})\, u_{\epsilon xx} - \epsilon \iint_{\Omega_t} (\phi^{3p})_x\, m_\epsilon^\beta(u_\epsilon)\, u_{\epsilon x}\, A(u_{\epsilon xx}) \\
&=\; \tfrac{1}{2}\int_\Omega \phi^{3p}\, u_{0\epsilon x}^2 + I_{1\epsilon} + I_{2\epsilon} + I_{3\epsilon} + I_{4\epsilon} + I_{5\epsilon}.
\end{aligned} \tag{5.15}$$

We will recover (1.18) from (5.15) in the limit $\epsilon \downarrow 0$. If $\phi = 1$ this is straightforward. Else, by Hölder and Young inequality

$$\begin{aligned} |I_{1\epsilon}| + |I_{2\epsilon}| &\le \tfrac{1}{4} \iint_{\{u_\epsilon>0\}_t} \phi^{3p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xxx}|^p + C \iint_{\{u_\epsilon>0\}_t} \phi^{2p}\, |\phi_x|^p\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xx}|^p \\ &+ C \iint_{\{u_\epsilon>0\}_t} \phi^p\, |\phi_x|^p\, m_\epsilon(u_\epsilon)\, |u_{\epsilon x}|^p. \end{aligned} \tag{5.16}$$

Estimating the last two integrals on the right-hand side of (5.16) with the help of Bernis' estimates (4.6)-(4.7), and of (5.8), we find that for almost every $\tau \in (0,t)$ (omitting dependence on τ for notational convenience)

$$\begin{aligned} \int_{\{u_\epsilon>0\}} \phi^{2p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xx}|^p &\le \left(\int_{\{u_\epsilon>0\}} \phi^{3p}\, u_\epsilon^{n-\frac{p}{2}}\, |u_{\epsilon xx}|^{\frac{3p}{2}}\right)^{\frac{2}{3}} \left(\int_{\{\phi_x>0\}} u_\epsilon^{n+p}\right)^{\frac{1}{3}} \\ &\le \eta \int_{\{u_\epsilon>0\}} \phi^{3p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xxx}|^p + C_\eta \int_{\{\phi_x>0\}} u_\epsilon^{n+p}, \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} \int_{\{u_\epsilon>0\}} \phi^{p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon x}|^p &\le \left(\int_{\{u_\epsilon>0\}} \phi^{3p}\, u_\epsilon^{n-2p}\, |u_{\epsilon x}|^{3p}\right)^{\frac{1}{3}} \left(\int_{\{\phi_x>0\}} u_\epsilon^{n+p}\right)^{\frac{2}{3}} \\ &\le \eta \int_{\{u_\epsilon>0\}} \phi^{3p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xxx}|^p + C_\eta \int_{\{\phi_x>0\}} u_\epsilon^{n+p}. \end{aligned} \tag{5.18}$$

Choosing η sufficiently small and integrating in time (5.17) and (5.18), (5.16)-(5.18) yield

$$|I_{1\epsilon}| + |I_{2\epsilon}| \le \tfrac{1}{2} \iint_{\{u_\epsilon>0\}_t} \phi^{3p}\, m_\epsilon(u_\epsilon)\, |u_{\epsilon xxx}|^p + C \int_0^t \int_{\{\phi_x>0\}} u_\epsilon^{n+p}. \tag{5.19}$$

The remaining terms in (5.15) are controlled uniformly in ϵ:

$$|I_{3\epsilon}| \le C\,\epsilon^{\frac{1}{2}} \left(\epsilon \iint_{\Omega_T} |u_{\epsilon xx}|^2\right)^{\frac{1}{2}} \left(\iint_{\Omega_T} |u_{\epsilon x}|^2\right)^{\frac{1}{2}} \overset{(5.2)}{\le} C\,\epsilon^{\frac{1}{2}}, \tag{5.20}$$

$$|I_{4\epsilon}| \le C\,\epsilon^{\frac{1}{2}} \left(\epsilon \iint_{\Omega_T} |u_{\epsilon x}|^{p-2}\, u_{\epsilon xx}^2\right)^{\frac{1}{2}} \left(\iint_{\Omega_T} |u_{\epsilon x}|^p\right)^{\frac{1}{2}} \overset{(5.2),(5.13)}{\le} C\,\epsilon^{\frac{1}{2}}, \tag{5.21}$$

$$|I_{5\epsilon}| \overset{(3.9)}{\le} C\,\epsilon^{\frac{1}{p}} \left(\epsilon \iint_{\Omega_T} m_\epsilon^\beta(u_\epsilon)\, |u_{\epsilon xx}|^p\right)^{\frac{p-1}{p}} \left(\iint_{\Omega_T} |u_{\epsilon x}|^p\right)^{\frac{1}{p}} \overset{(5.2),(5.13)}{\le} C\,\epsilon^{\frac{1}{p}}. \tag{5.22}$$

Collecting (5.19)-(5.22) into (5.15) and passing to the limit as $\epsilon \downarrow 0$ we recover (1.18), and the proof is complete. $\square$

6. Solutions in R

This section is devoted to the proof of Theorem 1 in the case $\Omega = \mathbf{R}$. We premit the following lemma.

Lemma 6.1. *Assume that a function $f : \mathbf{R} \to \mathbf{R}$ is such that*

$$\int_{\mathbf{R}} f^2 + \int_{\mathbf{R}} (1+|x|^{3p}) f_x^2 \ < \ \infty. \tag{6.1}$$

Then $f \in L^1(\mathbf{R})$, and a universal constant C exists such that

$$\int_{\mathbf{R}} |x|^{3p-2} f^2 \ \le \ C \int_{\mathbf{R}} |x|^{3p} f_x^2, \tag{6.2}$$

$$\int_{\mathbf{R}} |x|^{\frac{3(p-1)}{2}} |f| \ \le \ C \left(\int_{\mathbf{R}} |x|^{3p} f_x^2 \right)^{\frac{1}{2}}. \tag{6.3}$$

Proof. Writing (w.l.o.g. for $x > 0$)

$$x f^2(x) \ \le \ \int_0^x \left| f^2(\xi) + \xi\, f(\xi)\, f_x(\xi) \right| \, d\xi, \tag{6.4}$$

it follows immediately from (6.1) that $f(x) \to 0$ as $|x| \to \infty$. Then Hardy inequality yields (6.2) and (6.3), and the combination (6.1) and (6.3) yields $f \in L^1(\mathbf{R})$. □

For $a \ge 1$, we denote by u_a the solution of (P) in Ω_a with $u_0 = u_0|_{\Omega_a}$ obtained in the previous section, and with C a generic positive constant independent of a, t. It follows from (1.18) and (1.15) that

$$\tfrac{1}{2} \int_{\Omega_a} u_{ax}^2(t) + \iint_{\{u_a > 0\}_t} u_a^n \, |u_{axxx}|^p \ \le \ \tfrac{1}{2} \int_{\Omega_a} u_{0x}^2, \tag{6.5}$$

$$\int_{\Omega_a} u_a(t) \ = \ \int_{\Omega_a} u_0 \ \le \ C. \tag{6.6}$$

In particular

$$|u_a(t,x_1) - u_a(t,x_2)| \ \le \ C\, |x_1 - x_2|^{\frac{1}{2}} \quad \forall\, x_1,\, x_2 \in \Omega_a,\ t \in [0,\infty) \tag{6.7}$$

and

$$\sup_{x \in \Omega_a} u_a(t,x) \ \le \ C \left(\int_{\Omega_a} u_a \int_{\Omega_a} u_{ax}^2 \right)^{\frac{1}{3}} + \frac{C}{a} \int_{\Omega_a} u_a \ \le C \quad \forall\, t > 0. \tag{6.8}$$

Thanks to (6.5) and (6.8), arguing as in [6, proof of Lemma 2.1] one easily finds that

$$|u_a(t_1,x) - u_a(t_2,x)| \ \le \ C\, |t_1 - t_2|^{\frac{1}{5p-2}} \quad \forall\, t_1,\, t_2 \in [0,T],\ x \in \Omega_a. \tag{6.9}$$

Estimates (6.7), (6.8) and (6.9) allow to select a subsequence (still denoted by u_a) such that

$$u_a \longrightarrow u \quad \text{in } C_{loc}([0,\infty)\times\mathbf{R}) \quad \text{as } a\to\infty.$$

Concerning the time derivative, we have from (1.15)

$$\begin{aligned}\left|\int_0^T < u_{at},\varphi > dt\right| &\le \left(\iint_{\{u_a>0\}} u_a^n\,|u_{axxx}|^p\right)^{\frac{p-1}{p}}\left(\iint_{\Omega_{a,T}} u_a^n\,|\varphi_x|^p\right)^{\frac{1}{p}}\\ &\overset{(6.5),(6.8)}{\le} C\left(\iint_{\Omega_{a,T}}|\varphi_x|^p\right)^{\frac{1}{p}},\end{aligned} \tag{6.10}$$

which implies

$$u_{at} \rightharpoonup u_t \quad \text{in } L^{\frac{p}{p-1}}(\mathbf{R}^+;(W^{1,p}_{loc}(\mathbf{R}))') \quad \text{as } a\to\infty,$$

and in fact $u_t\in L^{\frac{p}{p-1}}(\mathbf{R}^+;(W^{1,p}(\mathbf{R}))')$ since the estimate in (6.10) is independent of a and T. Items (ii), (v) and (vi) in Definition 1 guarantee the applicability of Corollary 4.2 to each u_a, and passing to the limit as $a\uparrow\infty$ we obtain

$$\int_0^\infty\int_{\Omega_a}\left[\left|\left(u^{\frac{n+p}{3p}}\right)_x\right|^{3p}+\left|\left(u^{\frac{2(n+p)}{3p}}\right)_{xx}\right|^{\frac{3p}{2}}+\left|\left(u^{\frac{n+p}{p}}\right)_{xxx}\right|^{p}\right] \le C. \tag{6.11}$$

In addition, by Gagliardo-Nirenberg inequality (cf. Theorem 8.1),

$$\begin{aligned}\int_{\Omega_a} u_a^{n+p}(t) &\le C\left(\int_{\Omega_a}u_{ax}^2(t)\right)^{\frac{\theta(n+p)}{2}}\left(\int_{\Omega_a}u_a(t)\right)^{(1-\theta)(n+p)}\\ &\quad + a^{-(n+p-1)}\left(\int_{\Omega_a}u_a(t)\right)^{n+p} \overset{(6.5),(6.6)}{\le} C.\end{aligned} \tag{6.12}$$

Therefore

$$\iint_{\Omega_{a,T}} u_a^{n+p} \le C\,T,$$

which together with (6.11) implies (v) in Definition 1. The proof of the other properties in Definition 1 goes through as in section 5.

Let us turn to the additional properties. To prove that mass is conserved, it is enough to show that

$$\int_{\Omega_a}(1+|x|)^\beta u_a(t) \le C$$

for some $\beta > 0$. We preliminarily observe that (arguing as in (6.12)) $\|u_a(t)\|_{L^2(\Omega_a)} \leq C$, and (from (1.18)) that $\|(1+|x|)^{3p/2}u_{ax}(t)\|_{L^2(\Omega_a)} \leq C$. Then (arguing as in (6.4)) $u_a(t,x) \leq C/\sqrt{x}$. Choosing $\varphi = (1+|x|)^\beta$, $\beta > 0$ in (6.10), we obtain

$$\int_{\Omega_a} (1+|x|)^\beta u_a(t) \leq \int_{\mathbf{R}} (1+|x|)^\beta u_0 + C\,T^{\frac{1}{p}} \int_{\Omega_a} (1+|x|)^{p(\beta-1)-\frac{n}{2}}.$$

The first integral is finite in view of Lemma 6.1 for $\beta \leq \frac{3(p-1)}{2}$, whereas the second one is uniformly bounded for $\beta < \frac{2p+n-2}{2p}$. Concerning the passage to the limit in the weighted energy estimates (1.18), the only additional difficulty is to show that

$$\lim_{a\to\infty} \int_0^t \int_r^a u_a^{n+p} \;=\; \int_0^t \int_r^\infty u^{n+p} \tag{6.13}$$

for all $t \in [0,\infty)$ and all $r \in \mathbf{R}$. The proof of (6.13) is identical to that in [9, proof of Lemma 4.2], and therefore we omit it. Our last task is to prove (1.17). Let $\gamma > 1$. By Gagliardo-Nirenberg inequality:

$$\int_0^\infty \int_{\mathbf{R}} u^\gamma \;\leq\; C\,\|u_0\|_1^{\gamma(1-\theta)} \int_0^\infty \left(\int_{\mathbf{R}} \left|\left(u^{\frac{n+p}{3p}}\right)_x\right|^{3p}\right)^{\frac{\theta\gamma}{n+p}}, \quad \theta = \frac{(n+p)(\gamma-1)}{\gamma(n+4p-1)}.$$

Requiring $\frac{\theta\gamma}{n+p} = 1$ fixes $\gamma = n+4p$. Hence, by (6.11),

$$\int_0^\infty \int_{\mathbf{R}} u^{n+4p} \;\leq\; C.$$

On the other hand, we may write for almost every t

$$\begin{aligned} f(t) \;&=\; \left(\sup_{x\in\mathbf{R}} u^{\frac{n+4p+2}{2}}(t,x) - \inf_{x\in\mathbf{R}} u^{\frac{n+4p+2}{2}}(t,x)\right)^2 \;\leq\; C\left(\int_{\mathbf{R}} |u^{\frac{n+4p}{2}}(t)\,u_x(t)|\right)^2 \\ &\leq\; C\left(\int_{\mathbf{R}} u_{0x}^2\right) \int_{\mathbf{R}} u^{n+4p}(t), \end{aligned}$$

and therefore $f \in L^1(\mathbf{R}^+)$. Since by (6.9) f is uniformly continuous in $\mathbf{R}^+$, we infer

$$\left(\sup_{x\in\mathbf{R}} u(t,x) - \inf_{x\in\mathbf{R}} u(t,x)\right) \stackrel{t\uparrow\infty}{\longrightarrow} 0,$$

and since mass is conserved we obtain (1.17). □

7. Finite speed of propagation and waiting time

In this section we prove theorems 2 and 3. To this aim we preliminarily need a qualitative result guaranteeing boundedness of the support:

Proposition 7.1. *Under the assumptions of Theorem 2, a positive constant C exists such that*

$$\operatorname{supp}(u_0) \subset (-\infty, R) \implies \operatorname{supp}(u(t)) \subset \left(-\infty, R + C\, t^{\frac{2}{n+7p-6}}\right) \quad \forall\, t > 0.$$

Both Theorem 3 and Proposition 7.1 are based on the following inequality:

Lemma 7.2. *Under the assumptions of Theorem 2, a positive constant C exists such that for any $r < \rho$:*

$$\begin{aligned} \int_0^T \left(\int_\rho^\infty u^2\right)^{\frac{n+p}{2}} &\le \frac{C\,T}{(\rho - r)^{\frac{(n+7p-6)\,(n+p)}{4}}} \times \\ &\times \left[(\rho - r)^{\frac{p+n-2}{2}} \int_r^\infty (x-r)^{3p}\, u_{0x}^2 + \int_0^T \left(\int_r^\infty u^2\right)^{\frac{n+p}{2}}\right]^{\frac{n+p}{2}} \end{aligned} \tag{7.1}$$

Our starting point is the following variant of the weighted energy estimate, which is easily obtained combining (1.18) and (4.8):

$$\begin{aligned} \sup_{t\in(0,T)} \int_r^\infty (x-r)^{3p}\, u_x^2(t) \;+\; C^{-1} \int_0^T \int_r^\infty (x-r)^{3p} \left|\left(u^{\frac{n+p}{p}}\right)_{xxx}\right|^p \\ \le \int_r^\infty (x-r)^{3p}\, u_{0x}^2 + C \int_0^T \int_r^\infty u^{n+p}. \end{aligned} \tag{7.2}$$

We shall use two main tools in the proof. The first one is an extension of Stampacchia's Lemma obtained in [15, Lemma 3.1] (the very minor modifications in the statement below are left to the reader):

Lemma 7.3 (An Extension of Stampacchia's Lemma)**.** *Assume that a given nonnegative, non-increasing function $G : (0, \rho_0) \to \mathbf{R}$ satisfies:*

$$G(\xi) \le \frac{c_0}{(\xi - \eta)^\alpha} \left(G(\eta) + A\,(\rho_0 - \eta)^\sigma\right)^\beta \tag{7.3}$$

for $0 \le \eta < \xi \le \rho_0$, $A \ge 0$ and positive numbers c_0, α, β and σ such that

$$\beta > 1 \qquad \text{and} \qquad \sigma \ge \frac{\alpha}{\beta - 1}. \tag{7.4}$$

Assume further that

$$\rho_0^\alpha \ge 2^{\frac{\alpha\beta}{\beta-1}} \left(1 + 2^{\frac{\alpha}{\beta-1} - \sigma}\right)^\beta \cdot c_0 \cdot \left(G(0) + A\,\rho_0^\sigma\right)^{\beta-1}. \tag{7.5}$$

Then

$$G(\rho_0) = 0.$$

The second one is an iteration lemma [15, section 6]:

Lemma 7.4. *Assume that*

$$V(\rho') + (r'-\rho')^2\, U(\rho') \;\le\; \epsilon(r'-\rho')^2\, U(r') + F_\epsilon(\rho', r') \tag{7.6}$$

for arbitrary $0 < \rho < \rho' < r' < r$ *and* $\epsilon > 0$ *sufficiently small. Then a positive constant* K_ϵ *exists such that*

$$V(\rho) + \frac{(r-\rho)^2}{4} U(\rho) \;\le\; K_\epsilon \cdot F_\epsilon(\rho, r). \tag{7.7}$$

Proof of Lemma 7.2. Combining the weighted energy estimate (7.2) with Hardy inequality

$$\int_r^\infty (x-r)^{3p-2}\, u^2 \;\le\; C \int_r^\infty (x-r)^{3p}\, u_x^2$$

and the inequality

$$(x-r) \;\ge\; (\rho - r) \quad \forall\, x \ge \rho, \tag{7.8}$$

we obtain for arbitrary numbers $r < \rho$:

$$\begin{aligned} \sup_{t\in(0,T)} \int_\rho^\infty u^2(t) \;\; + \;\; (\rho-r)^2 \int_0^T \int_\rho^\infty \left| \left(u^{\frac{n+p}{p}} \right)_{xxx} \right|^p & \\ \le \;\; \frac{C}{(\rho-r)^{3p-2}} \left[\int_r^\infty (x-r)^{3p}\, u_{0x}^2 + \int_0^T \int_r^\infty u^{n+p} \right]. & \end{aligned} \tag{7.9}$$

We estimate the last integral on the right-hand side using Gagliardo-Nirenberg (cf. Theorem 8.1) and Hölder inequalities:

$$\int_0^T \int_r^\infty u^{n+p} \;\le\; C \left(\int_0^T \int_r^\infty \left| \left(u^{\frac{n+p}{p}} \right)_{xxx} \right|^p \right)^\theta \left(\int_0^T \left(\int_r^\infty u^2 \right)^{\frac{n+p}{2}} \right)^{1-\theta}, \tag{7.10}$$

where

$$\theta = \tfrac{n+p-2}{n+7p-2}.$$

Young's inequality implies that

$$\begin{aligned} \frac{1}{(\rho-r)^{3p-2}} \int_0^T \int_r^\infty u^{n+p} \;\; \le \;\; & \epsilon\, (r-\rho)^2 \int_0^T \int_r^\infty \left| \left(u^{\frac{n+p}{p}} \right)_{xxx} \right|^p \\ & + C_\epsilon\, (\rho - r)^{-\frac{n+7p-6}{2}} \int_0^T \left(\int_r^\infty u^2 \right)^{\frac{n+p}{2}}. \end{aligned} \tag{7.11}$$

Combining (7.11) with (7.9) we obtain, by Lemma 7.4,

$$\sup_{t\in(0,T)} \int_\rho^\infty u^2 \;+\; (\rho-r)^2 \int_0^T \int_\rho^\infty \left| \left(u^{\frac{n+p}{p}} \right)_{xxx} \right|^p \tag{7.12}$$

$$\leq \frac{C}{(\rho-r)^{\frac{n+7p-6}{2}}} \left[(\rho-r)^{\frac{n+p-2}{2}} \int_r^\infty (x-r)^{3p}\, u_{0x}^2 + \int_0^T \left(\int_r^\infty u^2 \right)^{\frac{n+p}{2}} \right],$$

and (7.1) follows. □

Proof of Proposition 7.1. We will equivalently show that if $u_0(x) = 0$ in $(0,\infty)$, then

$$u(t,x) \;=\; 0 \;\text{ in }\; (C\, t^{\frac{2}{n+7p-6}}, \infty) \quad \forall\, t > 0. \tag{7.13}$$

It follows from Lemma 7.2 that the function

$$G_T(\rho) \;=\; \int_0^T \left(\int_\rho^\infty u^2 \right)^{\frac{n+p}{2}}$$

satisfies the inequality

$$G_T(\rho) \;\leq\; \frac{C\,T}{(\rho-r)^{\frac{(n+7p-6)\,(n+p)}{4}}}\; (G_T(r))^{\frac{n+p}{2}} \tag{7.14}$$

for all $0 < r < \rho$. Stampacchia's Lemma 7.3 (with $A = 0$) then implies $G_T(\rho_0(T)) = 0$ provided

$$\rho_0^\alpha \;\geq\; C\,T\,(G_T(0))^{\beta-1}, \tag{7.15}$$

where

$$\alpha \;=\; \tfrac{(n+7p-6)\,(n+p)}{4}, \quad \beta \;=\; \tfrac{n+p}{2}.$$

Since, in view of the boundedness of u, $G_T(0) \leq C\,T$, (7.15) is satisfied in particular if

$$\rho_0(T) \;\geq\; C\,T^{\frac{\beta}{\alpha}} \;=\; C\,T^{\frac{2}{n+7p-6}},$$

which proves (7.13). □

Proof of Theorem 2. In the sequel, C denotes a generic positive constant depending only on n, p. We will equivalently show that if $u_0(x) = 0$ in $(0,\infty)$, then

$$u(t,x) \;=\; 0 \;\text{ in }\; \left(C\, \|u_0\|_1^{\frac{n+p-2}{n+4(p-1)}}\, t^{\frac{1}{n+4(p-1)}}, \infty \right) \quad \forall\, t > 0. \tag{7.16}$$

The qualitative result in Proposition 7.1 implies that the diameter of the support of $u(t)$ is for all times finite. We let

$$\begin{aligned} r(t) &= \inf\{x > 0 : \ u(t,\xi) = 0 \ \forall \xi > x\}, \\ R(T) &= \sup_{t\in(0,T)} r(t). \end{aligned}$$

An integration by parts shows that for all $r > 0$

$$\begin{aligned} \int_r^\infty (x-r)^{\frac{3p-2}{2}} u(t) &= -\frac{2}{3p}\int_r^\infty (x-r)^{\frac{3p}{2}} u_x(t) \\ &\le C \left(\int_r^\infty (x-r)^{3p} u_x^2(t)\right)^{\frac{1}{2}} |\{u_x(t) \neq 0\} \cap (r,\infty)|^{\frac{1}{2}} \\ &\le C\sqrt{R(t)} \left(\int_r^\infty (x-r)^{3p} u_x^2(t)\right)^{\frac{1}{2}}. \end{aligned}$$

Combined with (7.2) and (7.8), this implies for arbitrary $0 < r < \rho$ that

$$\begin{aligned} \sup_{t\in(0,T)} \left(\int_\rho^\infty u(t)\right)^2 &+ R(T)\,(\rho-r)^2 \int_0^T \int_\rho^\infty \left|\left(u^{\frac{n+p}{p}}\right)_{xxx}\right|^p \\ &\le \frac{C\,R(T)}{(\rho-r)^{3p-2}} \int_0^T \int_r^\infty u^{n+p}. \qquad (7.17) \end{aligned}$$

The proof now closely follows the lines of the previous one: Gagliardo-Nirenberg, Hölder and Young inequalities lead to

$$\begin{aligned} \frac{1}{(\rho-r)^{3p-2}} \int_0^T \int_r^\infty u^{n+p} &\le \epsilon(\rho-r)^2 \int_0^T \int_r^\infty \left|\left(u^{\frac{n+p}{p}}\right)_{xxx}\right|^p \\ &\quad + C_\epsilon\,(\rho-r)^{-(n+4p-3)} \int_0^T \left(\int_r^\infty u\right)^{n+p}. \qquad (7.18) \end{aligned}$$

Combining (7.17) with (7.18) and using Lemma 7.4, we obtain

$$\begin{aligned} \sup_{t\in(0,T)} \left(\int_\rho^\infty u(t)\right)^2 &+ R(T)\,(\rho-r)^2 \int_0^T \int_\rho^\infty \left|\left(u^{\frac{n+p}{p}}\right)_{xxx}\right|^p \\ &\le \frac{C\,R(T)}{(\rho-r)^{(n+4p-3)}} \int_0^T \left(\int_r^\infty u\right)^{n+p}. \qquad (7.19) \end{aligned}$$

Introducing the function

$$G_T(r) = \int_0^T \left(\int_r^\infty u\right)^{n+p},$$

we see (raising (7.19) to the power $\frac{n+p}{2}$) that

$$G_T(\rho) \;\le\; \frac{C\,T\,(R(T))^\beta}{(\rho-r)^\alpha}\,(G_T(r))^\beta,$$

where

$$\alpha \;=\; \tfrac{(n+4p-3)(n+p)}{2}, \qquad \beta \;=\; \tfrac{n+p}{2}.$$

Noting that $G_T(0) \le \|u_0\|_1^{n+p}\,T$, Lemma 7.3 (with $A=0$) implies $G_T(\rho_0(T))=0$, where

$$\rho_0(T)^\alpha \;:=\; C\,T\,(R(T))^\beta\,\|u_0\|_1^{(n+p)(\beta-1)}\,T^{\beta-1}.$$

This means that

$$\operatorname{supp}(u(t)) \,\cap\, \mathbf{R}^+ \;\subset\; (0,\rho_0(T)) \quad \forall\, t<T,$$

and therefore, by the definition of $R(T)$,

$$R(T) \;\le\; \rho_0(T) \;=\; C\,(R(T))^{\frac{\beta}{\alpha}}\,\|u_0\|_1^{\frac{(n+p)\,(\beta-1)}{\alpha}}\,T^{\frac{\beta}{\alpha}}.$$

Solving the inequality with respect to $R(T)$ gives

$$R(T) \;\le\; C\,\|u_0\|_1^{\frac{n+p-2}{n+4(p-1)}}\,T^{\frac{1}{n+4(p-1)}},$$

and the proof of (7.16) is complete. □

Proof of Theorem 3. We assume without loss of generality $x_0=0$, and let $\rho_0>0$ to be chosen later. For any $0<\xi<\rho_0$ we let

$$G_T(\xi) \;=\; \int_0^T \left(\int_{\xi-\rho_0}^{\infty} u^2\right)^{\frac{n+p}{2}}.$$

Lemma 7.2, with $\rho=\xi-\rho_0$ and $r=\eta-\rho_0$, implies that

$$G_T(\xi) \;\le\; \frac{C\,T}{(\xi-\eta)^{\frac{(n+7p-6)\,(n+p)}{4}}}\left[(\xi-\eta)^{\frac{p+n-2}{2}}\int_{\eta-\rho_0}^{0}(x+\rho_0-\eta)^{3p}\,u_{0x}^2 + G_T(\eta)\right]^{\frac{n+p}{2}}.$$

Using the growth condition (1.20) we see that

$$(\xi-\eta)^{\frac{p+n-2}{2}}\int_{\eta-\rho_0}^{0}(x+\rho_0-\eta)^{3p}\,u_{0x}^2 \;\le\; (\rho_0-\eta)^{\frac{n+7p-2}{2}}\int_{\eta-\rho_0}^{0}u_{0x}^2 \;\le\; C\,(\rho_0-\eta)^{\frac{n+7p+4\gamma-4}{2}}$$

for a sufficiently small ρ_0, so that for all $0<\eta<\xi<\rho_0$

$$G_T(\xi) \;\le\; \frac{C\,T}{(\xi-\eta)^{\frac{(n+7p-6)\,(n+p)}{4}}}\left[(\rho_0-\eta)^{\frac{n+7p+4\gamma-4}{2}} + G_T(\eta)\right]^{\frac{n+p}{2}}.$$

A straightforward calculation shows that assumption (7.4) in Lemma 7.3 (with $A = 1$) is satisfied provided

$$\gamma \;\geq\; \tfrac{3p-2}{n+p-2}.$$

Since the validity of assumption (7.5) can be guaranteed by choosing $T = T^\star$ sufficiently small, we conclude that

$$G_{T^\star}(\rho_0) \;=\; \int_0^{T^\star}\left(\int_0^\infty u^2\right)^{\frac{n+p}{2}} \;=\; 0,$$

and the proof is complete. □

8. Appendix

We have systematically used Gagliardo-Nirenberg's inequality [21], [30]. In the formulation we provide below, some of the integrability exponents are allowed to be less than one; a proof of this extension may be found in [17].

Theorem 8.1 (Gagliardo-Nirenberg). *Let $0 < q < p$, $1 \leq r \leq \infty$, $m \in \mathbf{N}$, $m > 0$. Let $\Omega \subset \mathbf{R}^N$ be open bounded with $\partial\Omega$ piecewise smooth. Suppose u belongs to $L^q(\Omega)$ and its derivatives of order m belong to $L^r(\Omega)$. Then the following inequalities hold (with constants C_1, C_2 depending only on Ω, m, q, r):*

$$\|u\|_p \;\leq\; C_1\, \|D^m u\|_r^\Theta \cdot \|u\|_q^{1-\Theta} + C_2\, \|u\|_q$$

where

$$\tfrac{1}{p} \;=\; \Theta\left(\tfrac{1}{r} - \tfrac{m}{N}\right) + (1-\Theta)\,\tfrac{1}{q}$$

for all Θ in the interval $[0,1)$. The result continues to hold if Ω is unbounded and diffeomorphic to a cone, and in this case $C_2 = 0$.

The first step in the proof of Theorem 1 requires an existence result for non-degenerate problems of the form

$$\begin{cases} u_t \;=\; [-m(u)\,A(u_{xxx}) + L(u_x)]_x + g(u)\,A(u_{xx}) & \text{in } \mathbf{R}^+ \times \Omega \\ u_x \;=\; u_{xxx} \;=\; 0 & \text{on } \mathbf{R}^+ \times \partial\Omega \\ u(0,x) \;=\; u_0(x) & x \in \Omega, \end{cases} \tag{8.1}$$

where A and $L = L_\epsilon$ are defined respectively by (1.14) and (1.12), and

$(H0)$ *$m, g \in C(\mathbf{R}; [c_1, c_2])$ for some $0 < c_1 < c_2$;*

$(H1)$ *$u_0 \in H^1(\Omega)$.*

The next proposition provides such information.

Proposition 8.2. *Assume (H0) and (H1). A function* $u \in L^p_{loc}([0,\infty); W^{3,p}(\Omega)) \cap C_{loc}([0,\infty); H^1(\Omega))$ *with* $u_t \in L^{\frac{p}{p-1}}_{loc}([0,\infty); (W^{1,p}(\Omega))')$ *exists, which solves (8.1) in the following sense:*

(*i*) *for all* $T < \infty$ *and all* $\varphi \in L^p((0,T); W^{1,p}(\Omega))$

$$\int_0^T < u_t, \varphi > \, dt \; = \; \iint_{\Omega_T} \left[m(u)\, A(u_{xxx}) - L(u_x) \right] \varphi_x + \iint_{\Omega_T} g(u)\, A(u_{xx})\, \varphi; \tag{8.2}$$

(*ii*) $u(0) \; = \; u_0$ *in* $H^1(\Omega)$;

(*iii*) $u_x \; = \; 0$ *in* $L^p_{loc}([0,\infty) \times \partial\Omega)$.

Proof. We apply a Galerkin approximation. Let us denote by ψ_i, $i \in N$, the eigenfunctions of minus the Laplacian with Neumann boundary conditions:

$$\begin{cases} -(\psi_i)_{xx} & = \; \lambda_i\, \psi_i \quad x \in \Omega \\ (\psi_i)_x & = \; 0 \quad x \in \partial\Omega. \end{cases} \tag{8.3}$$

Without loss of generality, the eigenvalues are ordered so that $0 = \lambda_0 < \lambda_1 < ..$, and the eigenfunctions are taken to be orthonormal in $L^2(\Omega)$ and orthogonal in $H^1(\Omega)$. The Galerkin Ansatz for problem (8.1),

$$u^N(x,t) \; = \; \sum_{i=0}^N {a_i}^N(t)\, \psi_i(x), \tag{8.4}$$

yields the following initial value problem for $(a_0^N, \ldots, a_N^N)$:

$$(P_N) \begin{cases} \frac{da_j^N}{dt} & = \; \int_\Omega \left[m(u^N)\, A(u^N_{xxx}) - L(u^N_x) \right] \, {\psi_j}_x + \int_\Omega g(u^N)\, A(u^N_{xx})\, \psi_j \\ a_j^N(0) & = \; (u_0, \psi_j)_{L^2(\Omega)} \end{cases}$$

where $j \; = \; 0, .., N$. Here and in the rest of the proof, C denotes a generic positive constant independent of N. By standard ODE theory, (P_N) admits a unique local solution $(a_0^N, ..., a_N^N)$ for any $N \in \mathbf{N}$. From the choice of ψ_i we easily see that

$$\frac{d}{dt} \int_\Omega \tfrac{1}{2} (u^N_x(t))^2 \; \overset{(8.3)}{=} \; \sum_{i=0}^N \lambda_i\, a_i^N(t)\, \frac{da_i^N}{dt}(t). \tag{8.5}$$

Therefore, using the equations in (P_N) and integrating in time,

$$\begin{aligned} \tfrac{1}{2} \int_\Omega (u^N_x(t))^2 \; & = \; \tfrac{1}{2} \int_\Omega u^2_{0\,x} - \iint_{\Omega_t} m(u^N)\, |u^N_{xxx}|^p - \epsilon(p-1) \iint_{\Omega_t} |u^N_x|^{p-2}\, |u^N_{xx}|^2 \\ & - \; \epsilon \iint_{\Omega_t} |u^N_{xx}|^2 - \iint_{\Omega_t} g(u^N)\, |u^N_{xx}|^p. \end{aligned} \tag{8.6}$$

Together with $(H1)$, this yields in particular

$$\|u_x^N(t)\|_{L^2(\Omega)} \leq C, \tag{8.7}$$

which in turn implies that

$$\lambda_1 \sum_{i=1}^{N} (a_i^N(t))^2 \leq \sum_{i=1}^{N} \lambda_i \, (a_i^N(t))^2 = \int_\Omega (u_x^N(t))^2 \leq C. \tag{8.8}$$

Since in addition

$$\left|\frac{da_0^N}{dt}\right| = \left|\frac{1}{|\Omega|^{\frac{1}{2}}} \int_\Omega g(u^N)\, A(u_{xx}^N)\right| \overset{(H0)}{\leq} C \left(\int_\Omega g(u^N)\, |u_{xx}^N|^p\right)^{\frac{p-1}{p}},$$

by (8.6) we obtain

$$|a_0^N(t)| \leq C_T. \tag{8.9}$$

It follows from (8.8) and (8.9) that (P_N) admits a global solution for any N, and (together with (8.7)) that

$$\|u^N\|_{L^\infty_{loc}([0,\infty);H^1(\Omega))} \leq C. \tag{8.10}$$

Furthermore, from (H_0), (H_1) and (8.6) we deduce that $\|u_{xxx}^N\|_{L^p(\mathbf{R}^+\times\Omega)}$ is bounded uniformly with respect to N, which together with (8.10) implies

$$\|u^N\|_{L^p_{loc}([0,\infty);W^{3,p}(\Omega))} \leq C. \tag{8.11}$$

A uniform bound on the time derivative of u^N can be obtained by observing that

$$\begin{aligned} \iint_{\Omega_T} u_t^N\, \psi &= \iint_{\Omega_T} \left[m(u^N)\, A(u_{xxx}^N) - L(u_x^N)\right] \, (\Pi_N\, \psi)_x \\ &+ \iint_{\Omega_T} \left[g(u^N)\, A(u_{xx}^N)\right] \, (\Pi_N\, \psi) \end{aligned} \tag{8.12}$$

for any $\psi \in L^p((0,T);W^{1,p}(\Omega))$, where Π_N is the projection of $L^p(\Omega)$ on the space generated by $\psi_0, ..., \psi_N$ (Π_N is well defined since $p \geq 2$). Estimating the right hand side of (8.12) with the help of Hölder inequality and (8.6), we obtain

$$\|u_t^N\|_{L_{loc}^{\frac{p}{p-1}}([0,\infty);(W^{1,p}(\Omega))')} \leq C. \tag{8.13}$$

By the same argument, one easily sees that

$$\|\left(u_x^N\right)_t\|_{L_{loc}^{\frac{p}{p-1}}([0,\infty);(W_0^{1,p}(\Omega)\cap W^{2,p}(\Omega))')} \leq C. \tag{8.14}$$

Collecting (8.10), (8.11), (8.13), (8.14), and using also Simon compactness criterion [33, Corollary 8.4], it is possible to select a subsequence (still denoted by u^N) such that

$$\begin{aligned} u^N &\to u && \text{in} \quad L^p_{loc}([0,\infty);\, W^{2,p}(\Omega)), && (8.15)\\ u^N &\overset{*}{\rightharpoonup} u && \text{in} \quad L^\infty_{loc}([0,\infty);\, H^1(\Omega)), && (8.16)\\ u^N &\rightharpoonup u && \text{in} \quad L^p_{loc}([0,\infty);\, W^{3,p}(\Omega)), && (8.17)\\ u^N_t &\rightharpoonup u_t && \text{in} \quad L^{\frac{p}{p-1}}_{loc}([0,\infty);\, (W^{1,p}(\Omega))'), && (8.18)\\ \left(u^N_x\right)_t &\rightharpoonup u_{xt} && \text{in} \quad L^{\frac{p}{p-1}}_{loc}([0,\infty);\, (W^{1,p}_0(\Omega)\cap W^{2,p}(\Omega))'), && (8.19)\\ A(u^N_{xxx}) &\rightharpoonup \chi && \text{in} \quad L^{\frac{p}{p-1}}_{loc}([0,\infty); L^{\frac{p}{p-1}}(\Omega)). && (8.20) \end{aligned}$$

Of course, (8.15) implies (iii). In addition, as is well-known (see for instance [5, Prop. 3.1]), gathering the regularity of u given by (8.17) and (8.18), respectively (8.17) and (8.19), we get $u \in C_{loc}([0,\infty); L^2(\Omega))$, respectively $u_x \in C_{loc}([0,\infty); L^2(\Omega))$, so that

$$u \;\in\; C_{loc}([0,\infty); H^1(\Omega)), \tag{8.21}$$

which in particular implies (ii). Passing to the limit as $N \to \infty$ in (8.12), we obtain

$$\int_0^T < u_t, \varphi > \, dt \;=\; \iint_{\Omega_T} [m(u)\,\chi - L(u_x)]\,\varphi_x + \iint_{\Omega_T} g(u)\,A(u_{xx})\,\varphi \tag{8.22}$$

for all $T < \infty$ and all $\varphi \in L^p((0,T); W^{1,p}(\Omega))$, and our last task is to identify

$$\chi \;=\; A(u_{xxx}) \quad \text{in } L^{\frac{p}{p-1}}_{loc}([0,\infty); L^{\frac{p}{p-1}}(\Omega)). \tag{8.23}$$

To this aim, we introduce on $L^p((0,T); W^{3,p}(\Omega))$ the functional

$$X_N(v) \;:=\; \iint_{\Omega_T} m(u^N)\; \left[A(u^N_{xxx}) - A(v_{xxx})\right]\; \left[u^N_{xxx} - v_{xxx}\right].$$

In view of (3.31), we have

$$X_N(v) \;\geq\; 0 \quad \forall v \in L^p((0,T);\, W^{3,p}(\Omega)). \tag{8.24}$$

Using (8.6), we rewrite X_N as

$$\begin{aligned} X_N(v) \;=\; & -\iint_{\Omega_T} m(u^N)\,A(u^N_{xxx})\,v_{xxx} - \iint_{\Omega_T} m(u^N)\,A(v_{xxx})\,(u^N_{xxx} - v_{xxx})\\ & -\epsilon\,(p-1)\iint_{\Omega_T} |u^N_x|^{p-2}\,|u^N_{xx}|^2 - \epsilon\iint_{\Omega_T} |u^N_{xx}|^2 - \iint_{\Omega_T} g(u^N)\,|u^N_{xx}|^p\\ & + \tfrac{1}{2}\int_\Omega (u^N_{0x})^2 - \tfrac{1}{2}\int_\Omega (u^N_x(T))^2. \end{aligned} \tag{8.25}$$

As $N \to \infty$, the first two integrals on the right hand side of (8.25) converge in view of (8.17) and (8.15). For the remaining terms we use lower semicontinuity and (8.24) to conclude:

$$\begin{aligned} 0 \;\leq\; & -\iint_{\Omega_T} m(u)\,\chi\, v_{xxx} - \iint_{\Omega_T} m(u)\, A(v_{xxx})\,(u_{xxx} - v_{xxx}) \\ & -\; \epsilon\,(p-1) \iint_{\Omega_T} |u_x|^{p-2}\,|u_{xx}|^2 - \epsilon \iint_{\Omega_T} |u_{xx}|^2 - \iint_{\Omega_T} g(u)\,|u_{xx}|^p \\ & +\tfrac{1}{2}\int_\Omega (u_{0x})^2 - \tfrac{1}{2}\int_\Omega (u_x(T))^2. \end{aligned} \tag{8.26}$$

Since (8.21) implies in particular

$$\int_0^T < u_t, u_{xx} > \, dt \;=\; \tfrac{1}{2}\int_\Omega u_x^2(0) - \tfrac{1}{2}\int_\Omega u_x^2(T),$$

choosing $\varphi = -u_{xx}$ in (8.22) yields, after an integration by parts,

$$\begin{aligned} \iint_{\Omega_T} m(u)\,\chi\, u_{xxx} \;=\; & \tfrac{1}{2}\int_\Omega (u_{0x})^2 - \tfrac{1}{2}\int_\Omega (u_x(T))^2 - \epsilon\,(p-1) \iint_{\Omega_T} |u_x|^{p-2}\,|u_{xx}|^2 \\ & -\epsilon \iint_{\Omega_T} |u_{xx}|^2 - \iint_{\Omega_T} g(u)\,|u_{xx}|^p. \end{aligned} \tag{8.27}$$

Subtracting (8.27) from (8.26), we obtain

$$\iint_{\Omega_T} m(u)\,(\chi - A(v_{xxx}))\,(u_{xxx} - v_{xxx}) \;\geq\; 0 \tag{8.28}$$

for all $v \in L^p((0,T); W^{3,p}(\Omega))$. Fix now $w \in L^p((0,T); W^{3,p}(\Omega))$, and for $\lambda \in \mathbf{R}$ choose $v_\lambda = u - \lambda w$ in (8.28). Dividing (8.28) by λ and passing to the limit as $\lambda \to 0$, we conclude

$$\iint_{\Omega_T} m(u)\,(\chi - A(u_{xxx}))\, w_{xxx} \;\geq\; 0 \quad \forall\, w \in L^p((0,T); W^{3,p}(\Omega)).$$

Exchanging w with $-w$, we see that the equality holds. Since $m(u) \geq c_1 > 0$, this implies (8.23) and the proof is complete. □

We conclude with a simple, but essential (cf. (2.16)) calculus lemma:

Lemma 8.3. *Let $G_{\delta,\sigma}$ be defined by (2.11). For any $p > 2$, $\beta \in \left(1 - \frac{p-2}{4(p-1)}, 1\right)$, $n \in \left(\frac{p-1}{2}, 2p-1\right)$ and $M > 0$, a positive constant $C = C_{p,\beta,n,M}$ exists such that*

$$|G'_{\delta,\sigma}(s)|^p\, m^\beta_{\delta,\sigma}(s) \;\leq\; C + C\, \left[G_{\delta,\sigma}(s)\right]_+ \tag{8.29}$$

for any $s \in (0, M]$, $\delta \in (0,1)$ and $\sigma \in (0,1)$.

Proof. Using the fact that $s \in [0, M]$, it is easy to check that the inequality

$$|G'_{\delta,\sigma}(s)|^p\, m^{\beta}_{\delta,\sigma}(s) \;\le\; C\, \left(\sigma^{\frac{1}{p-1}}\, s^{-2} + s^{2-\frac{n}{p-1}} + \delta^{\frac{1}{p-1}}\, s^2\right) \tag{8.30}$$

implies (8.29). To prove (8.30) we distinguish two cases. In the case $n \neq p-1$, we have

$$G'_{\delta,\sigma}(s) \;=\; -\tfrac{1}{3}\sigma^{\frac{1}{p-1}}\, s^{-3} + \tfrac{(p-1)}{(p-1-n)}\, s^{1-\frac{n}{p-1}} + \delta^{\frac{1}{p-1}}\, s.$$

Simple computations show that

$$\begin{aligned}
|G'_{\delta,\sigma}(s)|^p\, m^{\beta}_{\delta,\sigma}(s) \;&\le\; C_{p,n}\, \frac{\left(\sigma\, s^{-3(p-1)} + s^{p-1-n} + \delta\, s^{p-1}\right)^{\frac{p}{p-1}}}{\left(\sigma\, s^n + s^{4(p-1)} + \delta\, s^{n+4(p-1)}\right)^{\beta}}\, s^{(n+4(p-1))\,\beta} \\
&\le\; C_{p,n,\beta}\, \left(\sigma^{\frac{p}{p-1}-\beta}\, s^{-3p+4\beta(p-1)} + s^{p-\frac{n\,p}{p-1}+\beta\, n} + \delta^{\frac{p}{p-1}-\beta}\, s^p\right) \\
&\le\; C_{p,n,\beta,M}\, \left(\sigma^{\frac{1}{p-1}}\, s^{-2} + s^{2-\frac{n}{p-1}} + \delta^{\frac{1}{p-1}}\, s^2\right).
\end{aligned}$$

In the last line we have used the fact that

$$\beta \;\in\; \left(1 - \tfrac{p-2}{4(p-1)}, 1\right) \;=\; \left(1 - (p-2)\, \min\left\{\tfrac{1}{4(p-1)}, \tfrac{1}{n}\right\}, 1\right),$$

so that $-3p+4\beta(p-1) > -2$ and $p - \frac{n\,p}{p-1} + \beta\, n > 2 - \frac{n}{p-1}$. In the case $n = p-1$, we have

$$G'_{\delta,\sigma}(s) \;=\; -\tfrac{1}{3}\sigma^{\frac{1}{p-1}}\, s^{-3} + \log s + \delta^{\frac{1}{p-1}}\, s.$$

If $\frac{1}{e} \le s \le M$ then $|\log s| \le C_M$ and (8.29) follows as before. Otherwise, for $0 < s < \frac{1}{e}$, since $|\log s| > 1$ we may write, arguing as before,

$$|G'_{\delta,\sigma}(s)|^p\, m^{\beta}_{\delta,\sigma}(s) \;\le\; C_{p,\beta}\, |\log s|^p\, \left(\sigma^{\frac{1}{p-1}}\, s^{-2+\epsilon_1} + s^{1+\epsilon_2} + \delta^{\frac{1}{p-1}}\, s^{2+\epsilon_3}\right),$$

where

$$\epsilon_1 = -3p + 2 + 4\beta(p-1),\; \epsilon_2 = \beta(p-1) - 1,\; \epsilon_3 = p-2$$

are positive exponents for $\beta > 1 - \frac{p-2}{4(p-1)}$ and $p > 2$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\} > 0$, we obtain

$$\begin{aligned}
|G'_{\delta,\sigma}(s)|^p\, m^{\beta}_{\delta,\sigma}(s) \;&\le\; C_{p,\beta,M}\, s^{\epsilon}\, |\log s|^p\, \left(\sigma^{\frac{1}{p-1}}\, s^{-2} + s + \delta^{\frac{1}{p-1}}\, s^2\right) \\
&\le\; C_{p,\beta,M}\, \left(\sigma^{\frac{1}{p-1}}\, s^{-2} + s + \delta^{\frac{1}{p-1}}\, s^2\right),
\end{aligned}$$

and the proof is complete. □

References

[1] R. Almgren (1996) Singularity formation in Hele-Shaw bubbles. *Phys. Fluids* **8**, 344-352.

[2] L. Ansini, L. Giacomelli (2002) Shear–thinning liquid films: Macroscopic and asymptotic behaviour by quasi self–similar solutions. *Nonlinearity* **15**, 2147-2164.

[3] G.I. Barenblatt, E. Beretta, M. Bertsch (1997) The problem of the spreading of a liquid film along a solid surface: A new mathematical formulation. *Proc. Acad. Sci. USA* **94**, 10024-10030.

[4] E. Beretta, M. Bertsch, R. Dal Passo (1995) Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Rational Mech. Anal.* **129**, 175-200.

[5] F. Bernis, (1988) Existence results for doubly nonlinear higher order parabolic equations on unbounded domains. *Math. Ann.* **279**, 373-394.

[6] F. Bernis, A. Friedman (1990) Higher order nonlinear degenerate parabolic equations. *J. Differential Equations* **83**, 179-206.

[7] F. Bernis (1996) Integral inequalities with applications to nonlinear degenerate parabolic equations. In Angell, T.S.; Cook, L.P.; Kleinman, R.E.; Olmstead, W.E. *Nonlinear Boundary Value Problems*. SIAM, Philadelphia.

[8] F. Bernis (1996) Finite speed of propagation and continuity of the interface for thin viscous flows. *Adv. Differential Equations* **1**, 337–368.

[9] F. Bernis (1996) Finite speed of propagation for thin viscous flows when $2 \leq n < 3$. *C.R. Acad. Sci. Paris; Sér.I Math.* **322**, 1169-1174.

[10] A.L. Bertozzi, M. Pugh (1994) The lubrication approximation for thin viscous films: regularity and long time behaviour of weak solutions. *Comm. Pure Appl. Math.* **49**, 85-123.

[11] M. Bertsch, R. Dal Passo, H. Garcke, G. Grün (1998) The Thin Viscous Flow Equation in Higher Space Dimension. *Adv. Differential Equations* **3**, 417-440.

[12] S.I. Betelú, M.A. Fontelos (2003) Capillarity driven spreading of power-law fluids. *Appl. Math. Lett.*, to appear.

[13] R. Dal Passo, H. Garcke (1999) Solutions of a fourth order degenerate parabolic equation with weak initial trace *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* **XXVIII**, 153-181.

[14] R. Dal Passo, H. Garcke, G. Grün (1998) On a fourth-order degenerate parabolic equation: Global entropy estimates, existence, and qualitative behavior of solutions. *SIAM J. Math. Anal.* **29**, 321-342.

[15] R. Dal Passo, L. Giacomelli, G. Grün (2001) A waiting time phenomenon for thin film equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30**, 437-463.

[16] R. Dal Passo, L. Giacomelli, G. Grün (2003) Waiting time phenomena for degenerate parabolic equations – a unifying approach. In *Geometric Analysis and Nonlinear Partial Differential Equations*, S. Hidebrandt and H. Karcher Eds., Springer-Verlag Berlin Heidelberg New York 2003, 637-648.

[17] R. Dal Passo, L. Giacomelli, A.E. Shishkov (2001) The thin film equation with nonlinear diffusion. *Comm. Partial Differential Equations* **26**, 1509-1557.

[18] P.G. de Gennes (1985) Wetting: Statics and dynamics. *Rev. Mod. Phys.* **57** 827–863.

[19] E. DiBenedetto (1993) *Degenerate parabolic equations*. Springer-Verlag, New York.

[20] E.B. Dussan V., S.H. Davis (1974) On the motion of fluid-fluid interface along a solid surface. *J. Fl. Mech.* **65** 71–95.

[21] E. Gagliardo (1959) Ulteriori properità di alcune classi di funzioni in piú variabili. *Ricerche di Mat. Napoli*, 24–51.

[22] G. Grün (2001) On free boundary problems arising in thin film flow. Habilitation thesis, University of Bonn, accepted.

[23] G. Grün (2001) On Bernis' interpolation inequalities in multiple space dimensions. *Z. Anal. Anwendungen* **20**, 987-998.

[24] G. Grün (2002) Droplet spreading under weak slippage: the optimal asymptotic propagation rate in the multi-dimensional case. *Interfaces Free Bound.* **4**, 309–323.

[25] G. Grün. Droplet spreading under weak slippage: the waiting time phenomenon. Submitted for publication.

[26] L. Giacomelli, F. Otto (2001) Variational formulation for the lubrication approximation of the Hele-Shaw flow. *Calc. Var. Partial Differential Equations* **13**, 377-403.

[27] L. Giacomelli, F. Otto (2002) Droplet spreading: Intermediate scaling law by PDE methods. *Communications in Pure and Applied Mathematics* **LV**, 217–254.

[28] C. Huh, L.E. Scriven (1971) Hydrodynamic model of a steady movement of a solid/liquid/fluid contact line. *J. Colloid Interface Sci.* **35** 85–101.

[29] J.R. King (2001) Two generalisations of the thin film equation. *Mathematical and Computer Modelling* **34** 737–756.

[30] L. Nirenberg (1959) On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa* **13**, 115–162.

[31] A. Oron, S.H. Davis, G. Bankoff (1997) Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69** 931–980.

[32] F. Otto (1998) Lubrication approximation with prescribed non-zero contact angle. *Comm. Partial Differential Equations* **23**, 2077-2164.

[33] J. Simon (1987) Compact Sets in the Space $L^p(0,T;B)$. *Annali Mat. Pura Appl.* **146**, 65-97.

[34] D.E. Weidner, L.W. Schwartz (1994) Contact-line motion of shear-thinning liquids. *Phys. Fluids* **6** 3535–3538.

Dipartimento Me.Mo.Mat. — Universitá di Roma "La Sapienza" — Via Scarpa, 16 — 00161 Roma — Italy