Thin-film equations with "partial wetting" energy: existence of weak solutions ⋆

Michiel Bertsch [a], Lorenzo Giacomelli [b,*], Georgia Karali [c]

[a] *Dipartimento di Matematica – Università di Roma "Tor Vergata" and Istituto per le Applicazioni del Calcolo "M. Picone" – C.N.R. – Rome*

[b] *Dipartimento Me.Mo.Mat. - Università di Roma "La Sapienza"*

[c] *Department of Mathematics - University of Toronto*

Abstract

The capillarity driven evolution with slip of a liquid thin film over a dry surface is considered in the regime of "partial wetting". The focus is on the simplest model case of a constant, non-zero dynamic contact angle in the lubrication approximation. For the analytical treatment of the corresponding free boundary problem, a new strategy is proposed, based on the introduction of an ad hoc class of disjoining pressures which tend to concentrate at triple junctions. A first investigation of this approach yields to the existence of weak solutions which satisfy the dissipation relation as an inequality and which are different from those with zero contact angle. A heuristic argument is also presented in order to clarify the connection between contact angle and dissipation relation: it shows that moving droplets which satisfy the dissipation relation as an equality are forced to have the prescribed contact angle.

Key words: Liquid thin films, fourth order degenerate parabolic equations, free boundary problems
PACS: 68.15.+e, 02.30.Jr

⋆ Supported by the RTN-Programme "Fronts-Singularities", HPRN-CT-2002-00274, and by GNAMPA Research Project "Interfacce e frontiere libere in problemi di evoluzione di ordine superiore". L.G. and G.K. are grateful to the *Institut für Angewandte Mathematik* in Bonn, respectively to the *Istituto per le Applicazioni del Calcolo "M. Picone"* in Rome, for their wonderful hospitality.
* corresponding author: Dipartimento Me. Mo.Mat. – via Scarpa, 16 – I-00161 Roma – Italy – giacomelli@dmmm.uniroma1.it

1 Introduction

The capillarity driven evolution of a Newtonian liquid film over a dry solid substrate is described, in lubrication approximation, by a class of fourth-order degenerate operators for the non-dimensionalized thickness u of the film, whose prototype is given by the so called thin-film equation:

$$u_t + (m(u)\, u_{xxx})_x \;=\; 0 \quad \text{on } \{u > 0\}. \tag{1.1}$$

Equation (1.1) is obtained when considering a two-dimensional setting $(x, y) \in \mathbf{R} \times [0, \infty)$, with the solid surface at $y = 0$, neglecting both external and intermolecular forces. The mobility m is of the form

$$m(u) = u^3 + u^n, \quad n \in (0, 3), \tag{1.2}$$

where n accounts for different forms of the slip condition at the liquid-solid interface ($n = 2$ for the Navier's one). We refer to [12] for a review. Specifying the positivity set of u in (1.1) emphasizes that the wetted region is also an unknown: the evolution is in fact a free boundary problem — the free boundaries being the triple junctions where liquid, solid and air meet, $\partial\{u > 0\}$ —, and the specification of free boundary conditions is required. Their number is determined by half the order of the equation, which formally implies well-posedness on a fixed domain, plus one — hence, in the present case, three. Two of them are obvious (at least in the supposed absence of a precursor layer): vanishing thickness, which defines the free boundary, and vanishing mass flux:

$$\lim_{\{u(t,\cdot)>0\} \ni x \to x_0 \in \partial\{u(t,\cdot)>0\}} m(u(t, x))\, u_{xxx}(t, x) \;=\; 0. \tag{1.3}$$

The third condition is less clear. The simplest approach, which we take as a model case, is to argue that

$$\text{dynamic contact angle} \;=\; \text{equilibrium contact angle} \;\; \theta_e. \tag{1.4}$$

In order to clarify (1.4), and to introduce our notion of solution, let us shortly review the statics. Consider a region $D \subset \mathbf{R} \times (0, \infty)$ filled with a liquid. The free energy of the system is given, under the aforementioned assumptions, by surface energy alone:

$$\mathcal{E}(D) \;=\; \gamma_{SL}\, \mathcal{H}^1(\partial D \cap \{y > 0\}) + (\gamma_{LV} - \gamma_{SV})\, \mathcal{H}^1(\partial D \cap \{y = 0\})$$

($\mathcal{H}^1(\gamma)$ denotes the one-dimensional Hausdorff measure of γ; i.e., its length). For an equilibrium configuration, taking first variations of $\mathcal{E}$ with respect to admissible displacements (which are mass preserving vector fields whose vertical component vanishes at triple junctions), one recovers Laplace's law in the bulk, $-p = \gamma_{SL}\kappa$, and Young's law at triple junctions (cf. [7]):

$$\gamma_{SL} \cos\theta_e = \gamma_{SV} - \gamma_{LV}. \tag{1.5}$$

Hence, condition (1.4) corresponds to the limiting case of a fast evolution at triple junctions, which instantaneously enforces local equilibrium, and a slow evolution in the bulk, which then reduces the free energy globally. Here we are interested in the "partial wetting" regime, $\theta_e \in (0, \pi/2)$. In view of Young's law, the free energy can then be rewritten as

$$\mathcal{E}(D) = \gamma_{SL}\left[\mathcal{H}^1(\partial D \cap \{y > 0\}) - \cos\theta_e\, \mathcal{H}^1(\partial D \cap \{y = 0\})\right].$$

In lubrication approximation (cf. [12], [10]), one assumes in particular that the ratio between the typical horizontal length-scale X and the typical vertical length-scale Y, is small: $\frac{Y}{X} \ll 1$. If we also assume that D is a subgraph,

$$D = \{(X\hat{x}, Y\hat{y}) \in \mathbf{R} \times (0, \infty) : 0 < \hat{y} < \hat{h}(\hat{x})\},$$

then we have

$$\begin{aligned}\mathcal{E}(D) &= X\gamma_{SL} \int_{\{\hat{h}>0\}} \left[\left(1 + \frac{Y^2}{X^2}\hat{h}_{\hat{x}}^2\right)^{\frac{1}{2}} - \cos\left(\frac{Y}{X}\hat{\theta}_e\right)\right] d\hat{x} \\ &\overset{Y \ll X}{\approx} \frac{Y^2}{X}\gamma_{SL} \int_{\{\hat{h}>0\}} \frac{1}{2}\left(\hat{h}_{\hat{x}}^2 + \hat{\theta}_e^2\right) d\hat{x},\end{aligned}$$

(in the model case of a Hele-Shaw flow in half space, these asymptotics have been rigorously justified in the dynamical context in [9], [10]). Therefore, at leading order one recovers the usual energy for the thin-film equation, plus a correction which depends on $\hat{\theta}_e$. This leads to define the "partial wetting" energy functional for the non-dimensionalized thickness u as

$$\mathcal{F}_{\hat{\theta}_e}(u) = \int_{\{u>0\}} \frac{1}{2}\left(u_x^2 + \hat{\theta}_e^2\right) dx,$$

with the contact angle condition (1.4) reading as

$$\lim_{\{u(t,\cdot)>0\} \ni x \to x_0 \in \partial\{u(t,\cdot)>0\}} |u_x(t,x)| = \hat{\theta}_e \tag{1.6}$$

(angles are identified with their tangents in lubrication approximation). The regime of "complete wetting" [8], corresponding to $\hat{\theta}_e = 0$, has been studied analytically by many authors in the last decade — we refer to the recent papers [1], [11], where further references may be found. On the other hand, in the regime of "partial wetting", the only available existence result is due to Otto [13]: He looks at the case $m(u) = u$, whose peculiarity is that the evolution can be understood as the gradient flow of $\mathcal{F}_{\hat{\theta}_e}$ with respect to the L^2-Wasserstein metric. This is probably the deepest insight behind his construction of a solution which satisfies condition (1.6) for almost every t.

Extending Otto's result to a generic mobility is a long-standing open question. Our scope is to propose and start to investigate a new strategy, which is based on the introduction of an *ad hoc* class of disjoining pressures which tend to *concentrate* at the triple junctions. It is inspired by the following remark of Schwartz. In [14] (see also the references therein) he considers the action of intermolecular forces, modelled by a disjoining pressure $\Pi_{u^*}(u)$ which maintains all over the substrate an ultrathin liquid layer of non-dimensional thickness $u = O(u^*)$; he points out that, at equilibrium, the force balance near the apparent contact line (where u attains its minimum) yields to a Young's type law of the form

$$\frac{1}{2}\left(\hat{\theta}_e^{app}\right)^2 \approx \int_{u^*}^{\infty} \Pi_{u^*}(u)\, du, \tag{1.7}$$

where $\hat{\theta}_e^{app}$, the apparent contact angle, is defined as the slope of u at the inflection point. In the limit of zero microscopic equilibrium thickness, $u^* \downarrow 0$, one would hope that microscopic and apparent contact angle coincide. But unfortunately, this is not the case if a standard two-term model of the form $\Pi_{u^*}(u) = B[(u^*/u)^{-m} - (u^*/u)^{-s}]$,$1 < m < s$, is employed, since then the right-hand side of (1.7) vanishes. Here we consider an ad hoc class of disjoining pressures which instead concentrate at $u = 0^+$ in the limit of zero microscopic thickness. It is given, according to (1.7), by

$$\Pi_\varepsilon(u) = \frac{1}{2}\hat{\theta}_e^2\, \delta_\varepsilon(u),$$

where

$$\delta_\varepsilon(u) = \left\{ \begin{array}{ll} \frac{1}{\varepsilon}\delta_1\left(\frac{u}{\varepsilon}\right) & \text{if } u > \varepsilon\overline{u} \\ -\eta_\varepsilon(u) & \text{if } 0 < u \le \varepsilon\overline{u} \end{array} \right\}, \tag{1.8}$$

with η_ε and δ_1 non-negative and such that

$$\int_{\overline{u}}^{\infty} \delta_1(v)\,dv \;=\; 1, \quad \lim_{u\to\infty} \delta_1(u) \;=\; 0, \tag{1.9}$$

$$\int_0^{\varepsilon\overline{u}} \eta_\varepsilon(u)\,du \;\stackrel{\varepsilon\downarrow 0}{\longrightarrow}\; 0. \tag{1.10}$$

We think of $\varepsilon\overline{u}$ as the microscopic equilibrium thickness. The cases $\overline{u} = 0$ (which we admit) or $\eta_\varepsilon \equiv 0$ correspond to purely attractive potentials, $\delta_\varepsilon \geq 0$. Given a non-negative u_0 such that $\mathcal{F}_{\hat\theta_e}(u_0)$ is finite, let us therefore consider solutions of

$$\begin{cases} u_t + \Big(m(u)\,\Big(u_{xx} - \frac{1}{2}\,\hat\theta_e^2\,\delta_\varepsilon(u)\Big)_x\Big)_x \;=\; 0 \\ u(0,x) \;=\; u_0(x) + \varepsilon\overline{u}. \end{cases} \tag{1.11}$$

Since $\delta_\varepsilon(u) \to 0$ as $\varepsilon \downarrow 0$ for any $u > 0$, the effect of the disjoining pressure vanishes in the bulk, and the limiting equation (1.1) is formally recovered. Furthermore, note that the free energy for (1.11) is given by

$$\int_{\{u>0\}} \frac{1}{2}\Big(u_x^2 + \hat\theta_e^2\,H_\varepsilon(u)\Big)\,dx,$$

where in view of (1.9)

$$H_\varepsilon(u) \;:=\; \int_{\varepsilon\overline{u}}^{u} \delta_\varepsilon(v)\,dv \;\stackrel{\varepsilon\downarrow 0}{\longrightarrow}\; \begin{cases} 1 \;\text{ if } u>0 & \text{in view of (1.9)} \\ 0 \;\text{ if } u=0 & \text{in view of (1.10).} \end{cases} \tag{1.12}$$

Therefore, in the limit $\varepsilon \downarrow 0$ one formally recovers $\mathcal{F}_{\hat\theta_e}$ for the free energy.

In Section 3, a first investigation of this strategy will be performed in the model case $\eta_\varepsilon \equiv 0$, showing that the limit procedure can be rigorously justified to the following extent: In the main result, Theorem 3.2, we obtain the existence of a *solution to (1.1)* which conserves mass,

$$\int_\Omega u(t)\,dx \;=\; \int_\Omega u_0\,dx, \tag{1.13}$$

and dissipates $\mathcal{F}_{\hat\theta_e}$ in the sense that

$$\mathcal{F}_{\hat\theta_e}(u(T))\,dx + \int_0^T \int_{\{u(t)>0\}} m(u)\,|u_{xxx}|^2\,dx\,dt \;\leq\; \mathcal{F}_{\hat\theta_e}(u_0)\,dx. \tag{1.14}$$

Here Ω is the spatial domain, and the space-time integral in (1.14) represents, at leading order in lubrication approximation, the dissipation of kinetic energy due to viscous friction. Hence, inequality (1.14) is fully consistent with the original energy landscape. In addition, in Proposition 3.3 we observe that such solutions indeed differ from those with zero contact angle.

We understand (1.13) and (1.14) as weak counterparts of (1.3), respectively (1.6). Note that (1.14) is, even as an equality, not sufficient to guarantee (1.6) (and, a fortiori, uniqueness of solutions, which anyway is still an open problem also in the zero contact angle case): Indeed, any overturned parabola with given mass is a steady state weak solution of (1.1) which satisfies (1.14) as an equality. In order that the evolution does not get stuck into these critical points, the free boundaries have to move; once they do it, we believe that the contact angle condition will be enforced by the dissipation relation at least when the latter holds true as an equality. In other words, for moving solutions the contact angle condition is of Neumann type — encoded in the dissipation relation instead of being imposed as a constraint on the ambient space. To clarify this issue, before going into the more analytical Section 3 (where precise statements of the results are given, furthermore discussed, and proved), we wish to present a simple heuristic argument showing that *moving* solutions with connected support which satisfy (1.14) as an *equality*, automatically satisfy (1.6).

2 The contact angle as a Neumann type condition

For $n \in (0,3)$, let us consider a mass-conserving and, for simplicity, even solution u of (1.1) with connected support, $\mathrm{supp}(u(t,\cdot)) = [-s(t), s(t)]$. Let us also assume that u satisfies (1.14) as an equality. Then we may write for almost every t

$$\frac{d}{dt}\mathcal{F}_{\hat\theta_e}(t) \;=\; -\int_{-s(t)}^{s(t)} m(u)\,|u_{xxx}|^2\,dx. \tag{2.1}$$

On the other hand, integrations by parts show that

$$\begin{aligned}&\frac{d}{dt}\int_{-s(t)}^{s(t)} \frac{1}{2}(u_x^2+\hat\theta_e^2)\,dx + \int_{-s(t)}^{s(t)} m(u)\,|u_{xxx}|^2\,dx\\ &= \frac{1}{2}\dot s(t)\,(u_x^2+\hat\theta_e^2)|_{x=-s(t)} + (u_x\,u_t + m(u)\,u_{xx}\,u_{xxx})|_{x=-s(t)}.\end{aligned} \tag{2.2}$$

Then (2.1) implies that for almost every t that the right-hand side of (2.2) has to be equal to zero. Since the speed of the contact line can be ascertained from (1.1) and (1.2) to be given by $\dot{s}(t) = u^{n-1} u_{xxx}$, this means that

$$B(u) := \Big[u^{n-1}\, u_{xxx}\, (u_x^2 + \hat{\theta}_e^2) + 2\, u_x\, u_t + 2\, u^n\, u_{xx}\, u_{xxx} \Big] \,|_{x=-s(t)} \;=\; 0. \quad (2.3)$$

Folklore suggests that the profile of $u(t,y) = u(t, x + s(t))$ near a moving contact line is selected among those of travelling wave solutions. This yields: A one parameter family of positive contact angle profiles given by

$$u(t,y) \;\sim\; \left\{ \begin{array}{ll} \theta\, y + A\, y^{4-n}, & 0 < n < 3,\; n \neq 2 \\ \theta\, y + A\, y^2 \log y, & n = 2 \end{array} \right\} \quad (2.4)$$

with $\theta > 0$ and $A = A(n, \dot{s}, \theta)$; a zero contact angle profile given by

$$u(t,y) \;\sim\; \left\{ \begin{array}{ll} A\, y^{\frac{3}{n}}, & \frac{3}{2} < n < 3,\; \dot{s} > 0, \\ A\, y^2\, (-\log y)^{\frac{2}{3}}, & n = \frac{3}{2},\; \dot{s} > 0, \\ A\, y^{\frac{3}{n}}, & 0 < n < \frac{3}{2},\; \dot{s} < 0 \end{array} \right\} \quad (2.5)$$

with $A = A(n, \dot{s}) > 0$; and, for $n \in (0, \frac{3}{2})$, a one parameter family of zero contact angle profiles given by

$$u(t,y) \;\sim\; \kappa\, y^2 + A\, y^{5-2n}, \quad 0 < n < \tfrac{3}{2} \quad (2.6)$$

with $\kappa > 0$ and $A = A(n, \dot{s}, \kappa)$. Substituting (2.4)-(2.6) into (2.3) gives, at leading order as $x \downarrow -s(t)$,

$$B(u(t,x)) \;\sim\; \left\{ \begin{array}{ll} A\, \theta^{n-1} (\theta^2 - \hat{\theta}_e^2)(n-4)(n-3)(n-2), & \text{for (2.4)},\; n \neq 2, \\ 2\, A\, \theta (\theta^2 - \hat{\theta}_e^2), & \text{for (2.4)},\; n = 2, \\ \frac{3}{n^3}\, A^n\, \hat{\theta}_e^2 (3-n)(3-2n), & \text{for (2.5)},\; n \neq \frac{3}{2}, \\ \frac{4}{3}\, A^{\frac{3}{2}}\, \hat{\theta}_e^2, & \text{for (2.5)},\; n = \frac{3}{2}, \\ 2\, A\, \kappa^{n-1} \hat{\theta}_e^2 (5-2n)(2n-3)(n-2), & \text{for (2.6)}. \end{array} \right.$$

Therefore, in view of (2.3), the only admissible profile is (2.4) with $\theta = \hat{\theta}_e$. This shows that, for moving solutions with connected support, the contact angle condition is of Neumann type, i.e. it is already contained in the dissipation relation (1.14) provided it holds true as an equality.

3 Existence of weak solutions

There will be no additional difficulty in slightly relaxing assumption (1.2) on the mobility:

$$m \in C([0,\infty)), \text{ increasing}, \ m(u) \sim u^n \text{ as } u \downarrow 0 \text{ for some } n \in (0,3). \quad (3.1)$$

Our notion of solution is the following:

Definition 3.1 *Let $\hat\theta_e > 0$, $\Omega = (-a,a)$, $a > 0$, and assume (3.1). We say that an $|\Omega|$-periodic (w.r.t. x) non-negative function $u \in C^{\frac{1}{8},\frac{1}{2}}([0,\infty) \times \overline{\Omega})$ is a weak solution of (1.1) with partial wetting energy if:*

(*i*) $u_{xxx} \in L^2_{loc}(\{u > 0\})$, $\sqrt{m(u)}\, u_{xxx} \in L^2(\{u > 0\})$;

(*ii*) *for all Ω-periodic $\varphi \in C_c^\infty((0,\infty) \times \bar\Omega)$,*

$$\int_0^\infty \int_\Omega u\, \varphi_t \, dx\, dt + \int_0^\infty \int_{\{u(t)>0\}} m(u)\, u_{xxx}\, \varphi_x \, dx\, dt \; = \; 0;$$

(*iii*) $\mathcal{F}_{\hat\theta_e}(u(t)) < \infty$ *for almost every $t > 0$, and inequality (1.14) holds true for almost every $T > 0$.*

The periodicity of u is a complementary boundary condition on $\partial\Omega$ which guarantees uniqueness of positive solutions. The main result of this paper is the following:

Theorem 3.2 *Let $\hat\theta_e > 0$, $\Omega = (-a,a)$, $a > 0$, and assume (3.1). For any non-negative u_0 such that $\mathcal{F}_{\hat\theta_e}(u_0) < \infty$, a weak solution of (1.1) with partial wetting energy exists in the sense of Definition 3.1, such that $u(0,x) = u_0(x)$.*

As a simple consequence, we also obtain the following:

Proposition 3.3 *Let u be a weak solution of (1.1) with partial wetting energy, and assume that $\mathcal{F}_{\hat\theta_e}(u_0) < a\,\hat\theta_e$. Then:*

(*i*) *$u(t)$ does not converge to its mean uniformly in Ω as $t \uparrow \infty$;*

(*ii*) *a sequence $t_j \overset{j\uparrow\infty}{\to} \infty$ exists such that $u(t_j,x) \overset{j\uparrow\infty}{\to} u_*(x)$ uniformly in $\overline{\Omega}$, where u_* is a non-degenerate parabola on each connected component of $\{u_* > 0\}$.*

Since zero contact angle solutions converge uniformly to their mean value (cf. [2], [3]), Proposition 3.3 implies that, starting from the same initial datum, solutions with partial wetting energy differ from those with zero contact angle. Hence, our result may also be seen as an example of non-uniqueness for (1.1) which is both not pathologic (in the sense that it does not rely on fine tuning of the parameters in the approximating scheme adopted for constructing a solution, as opposed to the non-uniqueness example in [2]), and generic (non-uniqueness of source-type solutions is well-known, cf. [5]). From the mathematical point of view, we should also notice that an approximating scheme analogous to the one we adopt, has been used by Caffarelli and Vazquez in [6] to prove the existence of solutions for a second order parabolic equation with prescribed jump in u_x.

Theorem 3.2 is far from being exhaustive. A more complete picture would require capturing the contact angle condition (1.6) for almost every t, as obtained by Otto for $m(u) = u$. The difficulty is related to the need of controlling the topology of the support, and is ultimately due to the highly nonlinear character of the intrinsic metric of the evolution, as visualized by the space-time integral in (1.14). All this is of course related to the need of developing a more robust regularity theory for thin-film equations. A less conclusive but yet interesting achievement, would be to infer the contact angle condition assuming some knowledge on the topology of the support. We hope that our method will serve as a base for further investigations in these directions.

Proof of Theorem 3.2. By the scale invariance of the equation, we may set, without loss of generality, $\hat{\theta}_e = 1$. We consider the following approximating problems:

$$\begin{cases} u_t + \left(m(u)\left(u_{xx} - \frac{1}{2}\,\delta_\varepsilon(u)\right)_x\right)_x = 0 \quad \text{on } \{u > 0\} \\ u(t,\cdot) \ \ |\Omega|\text{-periodic} \\ u(0,x) = u_0(x) + \varepsilon\overline{u} \qquad\qquad x \in \Omega. \end{cases} \tag{3.2}$$

Since our result is oblivious of the specific form of the short-range term η_ε in the definition (1.8) of δ_ε, we set it to be zero: $\eta_\varepsilon \equiv 0$. Concerning δ_1, we assume it to be a non-negative, Lipschitz continuous function in $([\overline{u}, \infty))$ such that $\delta_1(\overline{u}) = 0$ and (1.9) holds.

Remark 3.4 At least under these assumptions, one could avoid adding an initial microscopic layer of thickness $\varepsilon\overline{u}$; the proof would go through for $u(0,x) = u_0(x)$ with no modification.

Our starting point is the following existence result for (3.2):

Lemma 3.5 *For any $\varepsilon > 0$ and any non-negative u_0 such that $\mathcal{F}_{\hat{\theta}_e}(u_0) < \infty$, an $|\Omega|$-periodic, non-negative function $u \in C^{\frac{1}{8},\frac{1}{2}}([0,\infty) \times \overline{\Omega})$ exists such that:*

(i) $u \in C^{1,4}(\{u > 0\})$, $\sqrt{m(u)}\,u_{xxx} \in L^2(\{u > 0\})$;

(*ii*) *for all* $|\Omega|$*-periodic* $\varphi \in C_c^\infty((0,\infty)\times\bar\Omega)$

$$\int_0^\infty\int_\Omega u\,\varphi_t\,dx\,dt + \int_0^\infty \int_{\{u(t)>0\}} u^n \left(u_{xxx} - \tfrac{1}{2}\delta_\varepsilon'(u)\,u_x\right)\varphi_x\,dx\,dt \;=\; 0;$$

(*iii*) $u(0,x) = u_0(x) + \varepsilon\overline{u}$;

(*iv*) *for almost every* $T > 0$,

$$\int_\Omega \frac{1}{2}\left(u_x^2(T) + H_\varepsilon(u(T))\right)\,dx + \int_0^T \int_{\{u(t)>0\}} m(u)\left(u_{xxx} - \tfrac{1}{2}\delta_\varepsilon'(u)\,u_x\right)^2\,dx\,dt$$
$$\le \int_\Omega \frac{1}{2}\left(u_{0x}^2 + H_\varepsilon(u_0)\right)\,dx. \tag{3.3}$$

Lemma 3.5 is essentially a consequence of the work by Bertozzi and Pugh [4] on long-wave unstable thin-film equations; more precisely, of Theorem 3.4. As a matter of fact, there the function $g(s) = -m(s)\delta_\varepsilon'(s)$ is assumed to be positive, whereas here it changes sign. But since the positive sign of g is the "bad" sign, it is easy to see that their proof covers the case under consideration, too. Also, note that item (*iv*) is not contained in their statement; anyhow, it follows from the properties of their approximating solutions, by exactly the same argument we shall use below to infer the corresponding item for our limit solution.

Let u_ε be the solution of (3.2) as given by Lemma 2.1. We denote by C a generic positive constant independent of ε. Conservation of mass

$$\int_\Omega u_\varepsilon(t)\,dx \;=\; \int_\Omega u_0\,dx, \tag{3.4}$$

follows from choosing (after a straightforward density argument) $\varphi = \chi_{(0,t)}$ in (*ii*). It follows immediately from (3.3) that

$$\sup_{t\in(0,\infty)} \int_\Omega u_{\varepsilon x}^2(t)\,dx \;\le\; C, \tag{3.5}$$

$$\int_0^\infty \int_{\{u_\varepsilon(t)>0\}} m(u_\varepsilon)\left(u_{\varepsilon xxx} - \tfrac{1}{2}\delta_\varepsilon'(u_\varepsilon)\,u_{\varepsilon x}\right)^2\,dx\,dt \;\le\; C. \tag{3.6}$$

Bounds (3.4) and (3.5) combine into

$$\|u_\varepsilon\|_{L^\infty((0,\infty);H^1(\Omega))} \;\le\; C. \tag{3.7}$$

From (3.5) we also see that

$$|u_\varepsilon(t,x_1) - u_\varepsilon(t,x_2)| \;\le\; C\,|x_1 - x_2|^{\frac{1}{2}}. \tag{3.8}$$

Bounds (3.6)-(3.8) combine into Hölder continuity in time:

$$|u_\varepsilon(t_1,x_0) - u_\varepsilon(t_2,x_0)| \;\le\; C\,|t_1 - t_2|^{\frac{1}{8}}. \tag{3.9}$$

To see this, consider a non-negative cut-off function $\varphi_1 \in C_c^\infty(\mathbf{R})$ such that $\operatorname{supp}(\varphi) \subset (-2,2)$ and $\int_{\mathbf{R}} \varphi(s)\,ds = 1$, and let $\varphi_\delta(x) = \delta^{-1}\varphi(\delta^{-1}(x - x_0))$. Writing

$$\begin{aligned}
|u_\varepsilon(t_1,x_0) - u_\varepsilon(t_2,x_0)| &\le \int_\Omega \varphi_\delta(x - x_0)|u_\varepsilon(t_1,x_0) - u_\varepsilon(t_1,x)|\,dx \\
&\quad + \int_\Omega \varphi_\delta(x - x_0)|u_\varepsilon(t_2,x_0) - u_\varepsilon(t_2,x)|\,dx \\
&\quad + \left|\int_\Omega \varphi_\delta(x - x_0)(u_\varepsilon(t_1,x) - u_\varepsilon(t_2,x))\,dx\right| \\
&=: I_1 + I_2 + I_3,
\end{aligned} \tag{3.10}$$

we have for the first two integrals

$$I_1 + I_2 \overset{(3.8)}{\le} C\,\delta^{\frac{1}{2}}. \tag{3.11}$$

It follows from (ii) in Lemma 3.5 that

$$\begin{aligned}
I_3 &= \left|\int_{t_1}^{t_2} \int_{\{u_\varepsilon(t)>0\}} m(u_\varepsilon)\left(u_{\varepsilon xxx} - \tfrac{1}{2}\,\delta_\varepsilon'(u_\varepsilon)\,u_{\varepsilon x}\right)\varphi_{\delta x}\,dx\,dt\right| \\
&\le C\left(\int_{t_1}^{t_2} \int_{\{u_\varepsilon(t)>0\}} m(u_\varepsilon)\left(u_{\varepsilon xxx} - \tfrac{1}{2}\,\delta_\varepsilon'(u_\varepsilon)\,u_{\varepsilon x}\right)^2 dx\,dt\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{t_1}^{t_2} \int_\Omega m(u_\varepsilon)\varphi_{\delta x}^2\,dx\,dt\right)^{\frac{1}{2}} \\
&\overset{(3.6),(3.7)}{\le} C\,\delta^{-\frac{3}{2}}|t_1 - t_2|^{\frac{1}{2}}.
\end{aligned} \tag{3.12}$$

Collecting (3.11) and (3.12) into (3.10) and minimizing the right-hand side with respect to δ yields (3.8).

Inequalities (3.8) and (3.9) allow (by Ascoli-Arzelà Theorem) to select a subsequence (still indexed by ε) such that

$$u_\varepsilon \;\to\; u \text{ in } C^{\frac{1}{8},\frac{1}{2}}([0,\infty)\times\overline{\Omega}) \quad \text{as} \quad \varepsilon\downarrow 0 \tag{3.13}$$

with $u(t,\cdot)$ $|\Omega|$-periodic. By (3.7)

$$u_\varepsilon \;\overset{*}{\rightharpoonup}\; u \text{ in } L^\infty((0,\infty);H^1(\Omega)) \quad \text{as} \quad \varepsilon\downarrow 0. \tag{3.14}$$

Bound (3.6), combined with (3.13), gives in particular that

$$\iint_K |u_{\varepsilon xxx}|^2\,dx\,dt \;\le\; C(K) \quad \text{on} \quad \text{each } K \subset\subset \{u>0\}.$$

In turn, this implies (by a diagonal procedure) that

$$u_{\varepsilon xxx} \;\rightharpoonup\; u_{xxx} \text{ in } L^2_{loc}(\{u>0\}) \quad \text{as} \quad \varepsilon\downarrow 0. \tag{3.15}$$

Let

$$f_\varepsilon \;=\; \begin{Bmatrix} \sqrt{m(u_\varepsilon)}\left(u_{\varepsilon xx}-\frac{1}{2}\delta_\varepsilon(u_\varepsilon)\right)_x & \text{on } \{u_\varepsilon>0\} \\ 0 & \text{elsewhere.} \end{Bmatrix}$$

By (3.6)

$$f_\varepsilon \;\overset{\varepsilon\downarrow 0}{\rightharpoonup}\; f\in L^2((0,\infty)\times\Omega), \tag{3.16}$$

and by (3.13), (3.15) and (1.9)

$$f_\varepsilon \;\overset{\varepsilon\downarrow 0}{\rightharpoonup}\; \sqrt{m(u)}\,u_{xxx} \quad \text{in} \quad L^2_{loc}(\{u>0\}.$$

Hence

$$f=\sqrt{m(u)}u_{xxx} \quad \text{on} \quad \{u>0\}, \tag{3.17}$$

which together with (3.16) implies (i) in Definition 3.1. To prove (ii) in Definition 3.1, we pass to the limit as $\varepsilon\to 0$ in (ii) of Lemma 3.5, which we may rewrite as

$$\int_0^\infty\!\!\int_\Omega u_\varepsilon\varphi_t\,dx\,dt+\int_0^\infty\!\!\int_\Omega \sqrt{u_\varepsilon}\,f_\varepsilon\,\varphi_x\,dx\,dt \;=\; 0$$

for all $|\Omega|$-periodic $\varphi \in C_c^\infty((0,\infty)\times\overline{\Omega})$. The passage to the limit on the first integral is straightforward in view of (3.13). For the second integral, using (3.13) and (3.16) we obtain

$$\begin{aligned}\int_0^\infty\int_\Omega \sqrt{m(u_\varepsilon)}\, f_\varepsilon\, \varphi_x \,dx\,dt &\overset{\varepsilon\downarrow 0}{\to} \int_0^\infty\int_\Omega \sqrt{m(u)}\, f\, \varphi_x\,dx\,dt \\ &= \int_0^\infty\int_{\{u(t)>0\}} \sqrt{m(u)}\, f\, \varphi_x\,dx\,dt \\ &\overset{(3.17)}{=} \int_0^\infty\int_{\{u(t)>0\}} m(u)\, u_{xxx}\, \varphi_x\,dx\,dt.\end{aligned}$$

Our last goal is to pass to the limit as $\varepsilon\downarrow 0$ in the energy estimate (3.3). The passage to the limit on the left-hand side follows immediately from dominated convergence Theorem, since by (1.12)

$$H_\varepsilon(u_\varepsilon(0,x)) \;=\; H_\varepsilon(u_0(x)+\varepsilon\overline{u}) \;\overset{\varepsilon\downarrow 0}{\longrightarrow}\; \begin{cases} 1 & \text{if } u_0(x)>0 \\ 0 & \text{if } u_0(x)=0\end{cases} \quad \text{for all } x\in\overline{\Omega}.$$

For the right-hand side, by lower semi-continuity we obtain in view of (3.14) and (3.16)

$$\begin{aligned}&\liminf_{\varepsilon\to 0}\int_\Omega u_{\varepsilon x}^2(T)\,dx \ge \int_\Omega u_x^2(T)\,dx \quad \text{for a.e. } T,\\ &\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega f_\varepsilon^2\,dx\,dt \ge \int_0^T\int_\Omega f^2\,dx\,dt \overset{(3.17)}{\ge} \int_0^T\int_{\{u(t)>0\}} m(u)\,|u_{xxx}|^2\,dx\,dt.\end{aligned}$$

Finally, (3.13) implies that for all $\eta>0$

$$\int_\Omega H_\varepsilon(u_\varepsilon(T))\,dx \;\ge\; \int_{\{u(T)>\eta\}} H_\varepsilon(u_\varepsilon(T))\,dx \;\overset{\varepsilon\lll 1}{=}\; \int_{\{u(T)>\eta\}} 1\,dx,$$

and therefore

$$\liminf_{\varepsilon\to 0}\int H_\varepsilon(u_\varepsilon(T))\,dx \overset{(1.12)}{\ge} \int_{\{u(T)>0\}} 1\,dx \quad \text{for all } T.$$

This proves (iii) and completes the proof of Theorem 1.

Proof of Proposition 3.3. Once again, by scale invariance we consider $\hat{\theta}_e = 1$ without loss of generality. The first assertion is straightforward: if u converged to its mean uniformly in Ω as $t \uparrow \infty$, then we would have $\mathcal{F}_1(u(T)) \geq a$ for $T \gg 1$, in contradiction with (1.14) and $\mathcal{F}_1(u_0) < a$. To prove the second assertion, we observe that by (1.14) a sequence $t_j \stackrel{j\uparrow\infty}{\to} \infty$ exists such that

$$\int\limits_{\{u(t_j)>0\}} m(u(t_j))\,|u_{xxx}(t_j)|^2\,dx \;\stackrel{j\uparrow\infty}{\longrightarrow}\; 0. \tag{3.18}$$

On the other hand, we have

$$\int\limits_{\Omega} u(t_j)\,dx \;=\; \int\limits_{\Omega} u_0\,dx, \quad \int\limits_{\Omega} |u_x(t_j)|^2\,dx \;\leq\; C,$$

so that by Ascoli-Arzelá Theorem

$$u(t_j,x) \;\stackrel{j\uparrow\infty}{\longrightarrow}\; u_*(x) \quad \text{uniformly in } \overline{\Omega}, \quad \text{and} \quad \int\limits_{\Omega} u^*\,dx \;=\; \int\limits_{\Omega} u_0\,dx \tag{3.19}$$

for a subsequence (still indexed by j), with u_* non-negative and $|\Omega|$-periodic. The support of u^* is not empty (by (3.19)) and strictly contained in Ω since $\mathcal{F}_1(u_0) < a$. In view of (3.18) and (3.19), we have

$$\begin{aligned} u_{xxx}(t_j) &\stackrel{j\uparrow\infty}{\rightharpoonup} 0 \quad \text{in } L^2_{loc}(\{u_* > 0\}), \\ u &\stackrel{j\uparrow\infty}{\to} u_* \quad \text{in } C^2_{loc}(\{u_* > 0\}). \end{aligned}$$

This means that u_* has to be a (possibly degenerate) parabola on each connected component I of $\{u_* > 0\}$. Since $|I| < |\Omega|$, the degenerate case ($u_*'' = 0$) would violate the continuity of u_* in $\overline{\Omega}$ (taking of course periodicity into account). This completes the proof of Proposition 3.3.

References

[1] Ansini, L.; Giacomelli, L. Doubly nonlinear thin-film equations in one space dimension. *Arch. Rat. Mech. Anal.* (2004), in press.

[2] Beretta, E.; Bertsch, M.; Dal Passo, R. Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Rational Mech. Anal.* **129** (1995), 175-200.

[3] Bertozzi, A.; Pugh, M. The lubrication approximation for thin viscous films: regularity and long time behavior of weak solutions. *Comm. Pure Appl. Math.* **49** (1996), 85-123.

[4] Bertozzi, A.; Pugh, M. Long-wave instabilities and saturation in thin film equations. *Comm. Pure Appl. Math.* **51** (1998), 625-661.

[5] Bertsch, M; Dal Passo, R.; Davis, S.H.; Giacomelli, L. Effective and microscopic contact angles in thin film dynamics. *Euro. J. Appl. Math.* **11** (2000), 181-201.

[6] Caffarelli, L.A.; Vazquez, J.L. A free-boundary problem for the heat equation arising in flame propagation. *Trans. Am. Math. Soc.* **347** (1995), 411-441.

[7] Finn, R. Equilibrium capillary surfaces. Grundlehren der Mathematischen Wissenschaften, 284. Springer-Verlag, New York, 1986.

[8] de Gennes, P.G. Wetting: Statics and dynamics. *Rev. Mod. Phys.* **57** (1985), 827-863.

[9] Giacomelli, L.; Otto, F. Variational formulation for the lubrication approximation of the Hele-Shaw flow. *Calc. Var. Partial Differential Equations* **13** (2001), 377-403.

[10] Giacomelli, L.; Otto, F. Rigorous lubrication approximation. *Interfaces Free Bound.* **5** (2003), 483-529.

[11] Grün, G. Droplet spreading under weak slippage: existence for the Cauchy problem. Preprint SFB 611 n. 112, University of Bonn, submitted for publication.

[12] Oron, A.; Davis, S. H.; Bankoff, G. Long-scale evolution of thin liquid films. *Rev. Mod. Phys.* **69** (1997) 931-980.

[13] Otto, F. Lubrication approximation with prescribed non-zero contact angle. *Comm. Partial Differential Equations* **23** (1998), 2077-2164.

[14] Schwartz, L.W. Hysteretic effects in droplet motions on heterogeneous substrates: Direct numerical simulations. *Langmuir* **1988**, n. 14, 3440-3453.