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Flexible Gabor-wavelet atomic decompositions for L^2-Sobolev spaces

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Abstract. In this paper we present a general construction of frames, which allows to ensure that certain families of functions (atoms) obtained by a suitable combination of translation, modulation and dilation form Banach frames for the family of L^2-Sobolev spaces on $\mathbb{R}$ of any order. In this construction a parameter $\alpha \in [0, 1)$ governs the dependence of the dilation factor on the frequency parameter. The well-known Gabor and wavelet frames (also valid for the same scale of Hilbert spaces) using suitable Schwartz functions as building blocks arise as special cases ($\alpha = 0$) and limiting case ($\alpha \to 1$) respectively. In contrast to those limiting cases it is no longer possible to use group theoretical arguments. Nevertheless we will show how to ensure *constructively* that for Schwartz analyzing atoms and any sufficiently dense, but discrete and well structured family of parameters one can guarantee the frame property. As a consequence of this novel constructive technique, one can generate quasi-coherent dual frames by an iterative algorithm. As will be shown in a subsequent paper the new frames introduced here generate Banach frames for corresponding families of α-modulation spaces.

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1. Introduction and motivations

The theory of *frames* or stable redundant non-orthogonal expansions, introduced by Duffin and Schaeffer [12] plays an important role in wavelet theory [9, 10] as well as in Gabor analysis [37, 25]. Many relevant contributions, just to mention [1, 5, 6, 10, 9, 19–21, 42–45], describe Gabor and wavelet analysis as two parallel theories with similar, but different structures and

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typically different applications. We shall propose a general construction of suitably structured frames and its application to present a unified theory for L^2-Sobolev spaces of Gabor and wavelet frames, usually treated separately. Our results want to be a further answer to the theoretical need of a common interpretation and framework and to define more general intermediate time-frequency tools for applications. To fix notations, consider the following unitary operators on $L^2(\mathbb{R})$ of modulation, translation and dilation respectively:

$$M_\omega(f)(t) = e^{2\pi i\omega\cdot t}f(t), \tag{1}$$

$$T_x(f)(t) = f(t-x), \tag{2}$$

$$D_a(f)(t) = |a|^{-1/2}f(t/a). \tag{3}$$

In particular, we shall study families of functions given by:

$$[\pi(x,\omega,\eta_\alpha(\omega))g] = M_\omega T_x D_{\eta_\alpha(\omega)^{-1}}g, \quad \text{for} \quad (x,\omega)\in\mathbb{R}^2, \tag{4}$$

where $\eta_\alpha(\omega) = (1+|\omega|)^\alpha$, for $\alpha\in[0,1]$ and the corresponding *flexible Gabor-wavelet transform*:

$$V_g^\alpha(f)(x,\omega) = \langle f, M_\omega T_x D_{\eta_\alpha(\omega)^{-1}}g\rangle, \quad \forall(x,\omega)\in\mathbb{R}^2. \tag{5}$$

We shall show that for suitable discrete sampling of (4), one can generate Banach frames for L^2-Sobolev spaces $H^s(\mathbb{R})$, if only g is sufficiently well localized in the time-frequency plane. One can easily verify that for $\alpha = 0$ the family (4) is just a Gabor family, while for $\alpha\to 1$ the family tends to the situation encountered in the wavelet context, in a very natural (geometric) sense.

Cordoba and Fefferman [7], Folland [28] and Holschneider and Nazaret [42] introduced such families and transforms as new time-frequency tools to study certain pseudo-differential operators. Moreover, L^2-Sobolev spaces are both modulation spaces [13,16,37], whose natural atomic decompositions provide (Banach) Gabor frames, and Besov spaces [34,47,46], for which wavelet expansions are atomic decompositions. Hence, L^2-Sobolev spaces appear as natural common spaces for which one can develop an intermediate theory. Furthermore, they are special instances of the more general family of α-modulation spaces [14,35] for each $\alpha\in[0,1]$. Although interesting group theoretical approaches in generalizing Gabor and wavelet frames are presented in [1,2,41,43,44], as matter of fact, classical unified (coorbit-space) theories [19–21,36] cannot be applied to generate discrete flexible (parametric) Gabor-wavelet frames. Our approach overcomes these difficulties. We mention however that a recent generalization of the coorbit space theory [8] may open up an alternative approach.
In approximation theory, one might use the parameter α as a *tuning* parameter in analyzing signals. In audio signal encoding, for example, one can use different bases to model different components of signals. The non-linear approximation of the *transient* component (shock waves) is very effective using

wavelet decompositions, but the *tonal* component (harmonic or stationary signal) is much better represented by Gabor type frames (or by local Fourier bases) [11]. Presently it is claimed that possible hybrid versions of Gabor and wavelet expansions can achieve better and more efficient representations of signals. In particular, the almost complementary quality and capability of approximation of Gabor and wavelet expansions has been shown in [11]. This is a further motivation to investigate possible intermediate frames, to introduce more flexible tools of approximation.

The paper is organized as follows. Section 2 lists the elementary properties of frames and few notational conventions. In Section 3 we briefly recall the coorbit space theory and the unified approach to Gabor and wavelet Banach frames developed by Feichtinger and Gröchenig [19–21,36]. We explain why it cannot be applied directly to derive any intermediate theory and time-frequency tool. In Section 4 we collect the relevant tools and technical results we need. The concept of admissible coverings of $\mathbb{R}^d$ and BAPUs (bounded admissible partitions of unity), decomposition spaces, in particular Wiener amalgam spaces, and results on series expansions for band-limited functions are presented. In Section 5 we introduce a new constructive technique to generate admissible (discrete) coverings from certain continuous coverings of the real line. In particular, each interval of an admissible covering is identified by a function P, parametrizing the *position* of the interval, and a function S, governing the *size* of the interval. Next we study how to check the Bessel condition for a family of functions of the type (4) associated to such admissible coverings. Finally, we state the definition of α-admissibility for an analyzing function g. In Section 6 we combine the results and tools established in Sections 4 and 5 to give unified sufficient conditions for the existence of *flexible Gabor-wavelet (discrete) frames* for L^2-Sobolev spaces.

The main result can be formulated as follows.

Theorem 1. *Let $\alpha \in [0,1)$, $s \in \mathbb{R}^+$ and $g \in H^s(\mathbb{R})$ α-admissible. Denote*

$$P_\alpha(j) = sgn(j)\left((1+(1-\alpha)\cdot b\cdot |j|)^{\frac{1}{1-\alpha}} - 1\right)$$

and

$$S_\alpha(j) = b\cdot(1+(1-\alpha)\cdot b\cdot(|j|+1))^{\frac{\alpha}{1-\alpha}},$$

position and size functions respectively, where $sgn(\cdot)$ is the sign function. Then, for all $b > 0$ and all sufficiently small values $a > 0$, the set of functions

$$\{g_{\alpha,a,b}^{j,k} = M_{P_\alpha(j)} T_{a\cdot S_\alpha(j)^{-1}\cdot k} D_{S_\alpha(j)^{-1}} g\}_{j,k\in\mathbb{Z}} \tag{6}$$

has the property that any $f \in H^s$ can be written as the unconditionally convergent series in $H^s(\mathbb{R})$

$$f = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_{\alpha,a,b}^{j,k}(f) g_{\alpha,a,b}^{j,k}, \tag{7}$$

$$\|f\|_{H^s}^2 \asymp \sum_{k,j} |c_{\alpha,a,b}^{j,k}(f)|^2 \left(1 + (1-\alpha) \cdot b \cdot |j|\right)^{\frac{2s}{1-\alpha}} , \tag{8}$$

for a suitable set of linear functionals $c_{\alpha,a,b}^{j,k}$. *In particular, the renormed sequence* $\{(1 + (1-\alpha) \cdot b \cdot |j|)^{\frac{s}{\alpha-1}} \cdot g_{\alpha,a,b}^{j,k}\}_{j,k \in \mathbb{Z}}$ *is a frame, while* $\{g_{\alpha,a,b}^{j,k}\}_{j,k \in \mathbb{Z}}$ *is a Banach frame for* $H^s(\mathbb{R})$. *Thus, the space* $H^s(\mathbb{R})$ *can be interpreted as a generalized coorbit space with respect to (4).*

The strategy of the proof has already been introduced in [30] and applied for the Gabor frames case only in [31]. It is essentially inspired by the fundamental works of Frazier and Jawerth [34] and Feichtinger [16], and it works as follows

- decompose the space into suitable subspaces of band-limited functions;
- construct a local family of atoms of translates for each subspace;
- show that the union of these local families is a Bessel sequence;
- the global system so built is a frame.

Then, one can apply a perturbation result to extend the frame property to larger classes of generating atoms. In fact, under suitable decay-smoothness conditions on g (α-admissibility), one starts to construct a local family of atoms by translates of a band-limited approximation g_ρ of g, which generates global frames. Then, by means of the Neumann series inversion of certain synthesis operators, one proves that the frame property applies also for g.

The α-admissibility condition essentially involves only the upper frame bound condition (Bessel condition) of the system, while the lower bound condition, typically harder to be checked, is automatically fulfilled by a quite large class of analyzing functions, as a consequence of the general and unified construction. This result is an improvement of earlier contributions, for example [5,9,41], where it was necessary to check the lower bound condition, separately, for Gabor or wavelet frames and it is a generalization of the results of [3, Section 3.6]. Moreover, since the developed technique is constructive, we provide a method to calculate suitable coefficients $\{c_{\alpha,a,b}^{j,k}(f)\}_{j,k}$, by means of the choice of (approximated) dual families. A constructive technique to derive explicit formulas of suitable time-frequency-scale sampling points $(x_{j,k}^\alpha, \omega_j^\alpha)$ for the family (4) to define the corresponding discrete frames is also presented. Hence, our approach should be distinguished from those results, see, for example, [41] Theorem 7.1 and [19–21], where the points were required to be just *sufficiently dense*. The construction is also generalizable to describe *irregular* Gabor and wavelet frames [32].

The present paper is the third of a series [30,31] on constructive methods to generate structured Banach frames for function spaces. We have also developed ([29, Chapter 5]) the extension of these expansions as Banach frames for all the class of the α-modulation spaces as generalization of the known coorbit-space theory. These results will be detailed in a subsequent paper [32] and they will make use of the present L^2-Sobolev theory, combined with a novel generalization [33] of the theory of localization of frames recently developed by Gröchenig [38].

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2. Preliminaries

Definition 1. *A sequence* $\{f_n\}_{n\in\mathbb{N}}$ *in* $\mathbb{H}$ *is a frame for the Hilbert space* $\mathbb{H}$ *if there exist two positive constants* $A, B > 0$ *such that*

$$A \cdot \|f\|^2 \le \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \le B \cdot \|f\|^2, \quad \forall f \in \mathbb{H}. \tag{9}$$

The upper bound in condition (9) is also known as the Bessel condition for the sequence $\{f_n\}_{n\in\mathbb{N}}$ *and whenever it holds the sequence* $\{f_n\}_{n\in\mathbb{N}}$ *is called a Bessel sequence.*

Condition (9) ensures that the *frame operator* $S : \mathbb{H} \to \mathbb{H}$ given by

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n, \tag{10}$$

is an invertible operator. This implies that

$$f = SS^{-1}f = \sum_{n\in\mathbb{N}} \langle f, S^{-1}f_n\rangle f_n. \tag{11}$$

The sequence $\{S^{-1}f_n\}_{n\in\mathbb{N}}$ is again a frame and it is called the *canonical dual* frame with frame operator S^{-1}. For a frame the coefficient functionals $\{c_n\}_{n\in\mathbb{N}}$ such that

$$f = \sum_{n\in\mathbb{N}} c_n(f) f_n \tag{12}$$

are in general not unique. Typically many dual Bessel sequences $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ in $\mathbb{H}$ exist such that for all $f \in \mathbb{H}$

$$f = \sum_{n\in\mathbb{N}} \langle f, \tilde{f}_n\rangle f_n. \tag{13}$$

Such redundancy of a frame can play an important role in practical problems where robustness and error tolerance are fundamental as, for example, denoising, irregular sampling problems [22–24], pattern matching, data transmission, and communication. We refer to [6] for a recent book on frames and non-orthogonal expansions and we refer to [3,8,36,38] for the more general notion of *Banach frame.*

We write $\mathbb{R}^+$ for the non-negative reals. For a given function or tempered distribution f, we denote the Fourier transform of f by $\hat{f}$ or $\mathcal{F}f$. Moreover,

f is called *band-limited* if $\mathrm{spec}(f) := \mathrm{supp}(\hat{f}) \subset \Omega$, for some Ω compact subset of $\mathbb{R}^d$. We write $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ for the Schwartz space. If $B(\mathbb{R}^d)$ is a Banach space of functions on $\mathbb{R}^d$, sometimes we write simply B instead of $B(\mathbb{R}^d)$. For positive quantities F and G, we will write $F \lesssim G$ whenever $F(x) \leq C \cdot G(x)$ for some universal constant $C > 0$ and for all arguments x. If $F \lesssim G$ and $G \lesssim F$ then we will write $F \asymp G$. By convention we write w_s for the weight function $w_s(x) = (1 + |x|^2)^{s/2}$ and $L^p_s := L^p_{w_s}$, the space of functions f such that $f \cdot w_s \in L^p$, and in the following we assume $s \geq 0$. Let us also denote $\mathcal{A}$ the regular Banach algebra $\mathcal{A} := \mathcal{F}L^1$ and, finally, M the space of bounded measures, topological dual space of the continuous function space C^0.

3. The coorbit-space theory: Gabor and wavelet frames

A number of interesting papers in the field suggest a parallel description of Gabor and wavelet decompositions. Let us just mention few names here, such as Christensen [6,5], Daubechies [9,10], Triebel [47]. Inspired by the work of Grossmann et al. [39] Feichtinger and Gröchenig have presented a unified group theoretical approach in [19–21,36]. In that work one can switch between the Gabor and the wavelet context by exchanging the reduced Heisenberg group for the affine $ax + b$-group.

Let us recall their approach and then introduce a common presentation for these two types of coherent frames. In the sequel G is some (Hausdorff) locally compact group.

Definition 2. *A strongly continuous representation of* G *on* $\mathbb{H}$ *is a mapping* π *from* G *into the bounded linear operators on* $\mathbb{H}$ *for which*

i) $\pi(xy) = \pi(x)\pi(y)$, *for all* $x, y \in G$.

ii) for all $f \in \mathbb{H}$ *the mapping* $x \mapsto \pi(x)f$ *is continuous from* G *to* $\mathbb{H}$.

We further say that

iii) π *is unitary if all the operators* $\{\pi(x)\}_{x\in G}$ *are unitary.*

iv) π *is irreducible if the only closed subspaces of* $\mathbb{H}$ *which are invariant under all the operators* $\{\pi(x)\}_{x\in G}$ *are* $\{0\}$ *and* $\mathbb{H}$.

v) π *is integrable if there exists* $g \in \mathbb{H}\backslash\{0\}$ *such that*

$$\int_G |\langle \pi(x)g, g\rangle| d\mu(x) < \infty,$$

where μ *is the Haar measure on* G. *The set of all* $g \in \mathbb{H}$ *for which the above integral is finite will be denoted by* $\mathbb{A}$ *or* $\mathbb{A}_\pi$.

Definition 3. *Let* X *be a locally compact space. A sequence of open, relatively compact subsets* $\mathcal{O} = \{\Omega_i\}_{i\in I} \subset X$ *is called an admissible covering of* X *if*

C1) (covering) $X = \bigcup_{\in I} \Omega_i$;

C2) (uniformly locally finite) $\sup_{i\in I} \#\{j \in I : \Omega_j \cap \Omega_i \neq \emptyset\} \leq N < \infty$.

Theorem 2. *Let π be an irreducible, unitary and integrable representation of G on $\mathbb{H}$. Then, there exists subspace $\mathbb{B} \subset \mathbb{A}$ which is dense in $\mathbb{H}$ such that for all $g \in \mathbb{B}\backslash\{0\}$ there exists a relatively compact subset Ω of G with non-void interior with the following property: for any admissible countable covering $\{x_j\Omega\}_{j\in I}$ of G there exists a bounded operator*

$$\Lambda : \mathbb{H} \to l^2(I), \quad \Lambda f = \{\lambda_j(f)\}_{j\in I}$$

for which

$$f = \sum_{j\in I} \lambda_j(f)\pi(x_j)g, \quad \forall f \in \mathbb{H} \tag{14}$$

Moreover $\{\pi(x_j)g\}_{j\in I}$ is a frame for $\mathbb{H}$.

This theorem is just a special case of the general Feichtinger-Gröchenig theory, based on a discretization of convolutions and iterative approximations of a reproducing formula. To a large class of Banach function spaces Y (including weighted L^p-spaces) one can associate a sequence space Y_d and a Banach space $Co_\pi(Y)$ (coorbit space) such that the result holds with $\mathbb{H}$ replaced by $Co_\pi(Y)$ and $l^2(I)$ by Y_d. In particular, Theorem 2 can be applied to obtain Gabor and wavelet frames:

Examples 1. We collect a few relevant examples of atomic decompositions for coorbit-spaces:

- (Gabor expansions and modulation spaces) For $\mathbb{H}_{red} = \mathbb{R}\times\mathbb{R}\times\mathbb{T}$, the *(reduced) Heisenberg group* with a suitable group law, and the *Schrödinger representation* on $L^2(\mathbb{R})$ given for $(z, y, x) \in \mathbb{H}_{red}$ by
$$[\pi(z, y, x)g](t) = x \cdot T_z M_y g(t) = x \cdot e^{2\pi i y(t-z)} g(t - z), \quad g \in L^2, \tag{15}$$
the resulting frames are called *Gabor frame*s and the corresponding coorbit spaces are $Co_\pi(L^p_{w_s}) = M^s_{p,p}$, belonging to the family of *modulation spaces* [16,37].
- (wavelet expansions and Besov-Triebel spaces) For $\mathbb{A}_{ff} = \mathbb{R} \ltimes \mathbb{R}^+$, the *affine* group, one can define the representation on $L^2(\mathbb{R})$ given for $(b, a) \in \mathbb{A}_{ff}$ by
$$[\pi(b, a)f](t) = T_b D_a f(t) = \frac{1}{\sqrt{a}} f(\frac{t-b}{a}), \quad f \in L^2. \tag{16}$$
Hence wavelet expansions appear by appropriate sampling of the representation. For the weight function $w(b, a) = |a|^s$, $s \in \mathbb{R}$, the corresponding coorbit spaces are $Co_\pi(L^p_w) = \dot{B}^{s-1/2+1/p}_{p,p}$, the *homogeneous Besov spaces* [34,46,45].
- (L^2-Sobolev spaces) Fractional L^2-Sobolev spaces H^s defined by
$$H^s(\mathbb{R}) = \{f \in \mathcal{S}' : \hat{f}(\omega)(1 + |\omega|^2)^{\frac{s}{2}} \in L^2(\mathbb{R})\}, \tag{17}$$
endowed with the scalar product $\langle f, g\rangle_{H^s} := \langle \hat{f}, \hat{g}\rangle_{L^2_{w_s}}$ are Hilbert spaces and $H^s = M^s_{2,2} = \dot{B}^s_{2,2} = Co_\pi(L^2_{w_s})$ for all s since belong to both families.

The Feichtinger-Gröchenig approach does *not* suggest any way to move from the Gabor to the wavelet decomposition theory and any possible intermediate time-frequency tools. In fact, Torresani [43,44] investigated the existence of subgroups G of the *affine-Heisenberg group* $G_{aWH} = \mathbb{R} \times \mathbb{R} \ltimes \mathbb{R}^+ \times \mathbb{T}$, such that they admit mixed square integrable, unitary representations

$$[\pi(b, \omega, \eta_G(\omega))g] = M_\omega T_b D_{\eta_G(\omega)^{-1}} g, \quad \text{for all } (b, \omega) \in \mathbb{R}^2 \qquad (18)$$

giving as a unique solution $\eta_G(\omega) = \eta_\lambda(\omega) = (1 + \lambda\omega)$. For $\lambda \neq 0$ it gives a trivial modification of the standard wavelet theory, and for $\lambda = 0$ it gives the standard Gabor theory. Hence, *the Feichtinger-Gröchenig theory cannot be used to provide an intermediate theory between Gabor and wavelet decompositions.* However, as we shall show below, an *intermediate theory* is still possible (see also [41]) with a respective coorbit space theory [29, 32] by using the *decomposition of Hilbert space* method illustrated in [30] and already used in [31] for generating Gabor frames only. Such an *intermediate* theory provides a new flexible tool which potentially combines the advantages of the two time-frequency techniques to represents signals and operators in a parametric way.

In the construction of intermediate frames we follow the approach of Cordoba-Fefferman [7] and Folland [28] for the representation of functions as continuous superpositions of modulated, translated and dilated *wave packets.* They introduced such family of functions to study classes of *pseudo-differential operators*, and they are generated by the action of a mixed family of operators given by

$$[\pi(x, \omega, \eta_\alpha(\omega))g] = M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g, \quad \text{for all } (x, \omega) \in \mathbb{R}^2, \qquad (19)$$

where $\eta_\alpha(\omega) = (1 + |\omega|)^\alpha$, for $\alpha \in [0, 1]$, producing a new transform which we want to call *flexible Gabor-wavelet transform* or *α-transform*:

$$V_g^\alpha(f)(x, \omega) = \langle f, M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g \rangle, \quad \text{for all } (x, \omega) \in \mathbb{R}^2, \qquad (20)$$

Note that, for $\alpha = 0$, V_g^α is just the short time Fourier transform (STFT) or the windowed Fourier transform and, for $\alpha = 1$, V_g^α is just a slight modification of the continuous wavelet transform (CWT). Further interesting results in this direction were also proposed by Holschneider and Nazaret in [42], and we briefly collect some of them in the following

Theorem 3. *For $g \in \mathcal{S}(\mathbb{R})$ define*

$$\tilde{\sigma}_g^\alpha(\omega) = \int_{\mathbb{R}} |\hat{g}((1 + |\xi|)^{-\alpha}(\omega - \xi))|^2 (1 + |\xi|)^{-\alpha} d\xi. \qquad (21)$$

Assume that there exists $A > 0$ such that

$$0 < A^{-1} \leq \tilde{\sigma}_g^\alpha(\omega) \leq A < \infty, \quad \textit{for a.e} \quad \omega \in \mathbb{R}. \qquad (22)$$

Then

$$f \in H^s(\mathbb{R}) \quad \textit{if and only if} \quad V_g^\alpha f \in L^2_{(1+|\omega|^2)^{\frac{s}{2}}}(\mathbb{R}^2). \qquad (23)$$

In particular, for $s = 0$ *the system* $\{M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g\}_{\omega, x \in \mathbb{R}}$ *is a continuous frame [2] and for all* $f \in L^2$

$$f = \int_{\mathbb{R}^2} c_{\omega,x}(f) M_\omega T_x D_{\eta_\alpha(\omega)^{-1}} g \quad dx d\omega \quad \textit{and} \quad \|c_{\omega,b}(f)\|_{L^2(\mathbb{R}^2)} \asymp \|f\|_{L^2}. \tag{24}$$

A representative function satisfying (22) is the Gaussian.

It can be interesting to the reader to compare the previous theorem with Theorem 1, which can be considered its discrete version, similar results in [41] and [1, Proposition 15.2.1].

4. Wiener amalgams, series expansions of band-limited functions

We want to collect in this section those relevant tools we need, for a better and self-contained presentation. For more details we refer the reader to [14, 35,15,17,40,23]

Definition 4. *Given an admissible covering* $\mathcal{O} = \{\Omega_i\}_{i \in I}$ *of* $\mathbb{R}^d$*, we call a family* $\Phi = \{\varphi_i\}_{i \in I}$ *in* $\mathcal{A} := \mathcal{F}L^1$ *(see [14, Theorem 4.2]) an* $\mathcal{O}$*-BAPU (Bounded Admissible Partition of Unity) subordinate to* $\mathcal{O}$*, if* Φ *fulfills the following properties*

i) $\sup_{i \in I} \|\varphi_i\|_{\mathcal{A}} = C_0 < \infty$*;*
ii) $\mathrm{supp}(\varphi_i) \subseteq \Omega_i$*,* $\forall i \in I$*;*
iii) $\sum_{i \in I} \varphi_i(x) = 1$*.*

Definition 5. *For a Banach space* $(B, \|\cdot\|_B)$ *continuously embedded in* $\mathcal{S}'$*, an* $\mathcal{O}$*-BAPU* $\Phi = \{\varphi_i\}_{i \in I}$ *in* $\mathcal{A}$ *acts boundedly on* $(B, \|\cdot\|_B)$ *if*

$$\|\varphi_i f\|_B \leq \|\varphi_i\|_{\mathcal{A}} \|f\|_B, \forall i \in I, f \in B.$$

Definition 6. *Given a Banach space* $(B, \|\cdot\|_B)$ *continuously embedded in* $\mathcal{S}'$*, let* $\Phi = \{\varphi_i\}_{i \in I} \subset \mathcal{A}$ *an* $\mathcal{O}$*-BAPU acting boundedly on* $(B, \|\cdot\|_B)$ *and* w *a discrete (weight) strictly positive sequence on* I*. We define the corresponding decomposition space as:*

$$D(\mathcal{O}, B, l^q_w)(\mathbb{R}^d) = \{f \in \mathcal{S}' : f\varphi_i \in B \quad \forall i \in I, (\|f\varphi_i\|_B)_{i \in I} \in l^q_w(I)\}.$$

Moreover, one can define the natural norm:

$$\|f|D\| := \left(\sum_{i \in I} \|f\varphi_i\|^q_B w(i)^q \right)^{1/q},$$

for $1 \leq q < \infty$*. The usual modification applies for* $q = \infty$*.*

With such a norm the space $(D(\mathcal{O}, B, l^q_w), \| \cdot |D\|)$ is a Banach space. If, in addition to the already mentioned assumptions, the $\mathcal{O}$-BAPU is a Q-BUPU (Bounded Uniform Partition of Unity), i.e. if there exists $(x_i)_{i \in I} \subset \mathbb{R}^d$ such that $\Omega_i = x_i + Q$, with $Q \subset \mathbb{R}^d$ relatively compact, we call $D(\mathcal{O}, B, l^q_w)$ a Wiener amalgam space and we will write $W(B, l^q_w) := D(\mathcal{O}, B, l^q_w)$. Then the definition does not depend on the particular BUPU Φ taken, whenever the weight function w is assumed *submultiplicative*, that is $w(x+y) \leq w(x)w(y)$ for all $x, y \in \mathbb{R}^d$, e.g. w_s for $s \geq 0$. Moreover, there exist characterizations of equivalent general decomposition spaces, by means of equivalent coverings, see [14,35]. The properties of Wiener amalgams with respect to inclusions, Fourier transform, convolutions, were described and studied by Feichtinger [15,17]. We refer to [40] for further information.

We also have to recall and to develop some technical tools on expansions of band-limited functions by translates of a single band-limited function g.

Proposition 1 (Regular case). *Given a band-limited function $g \in L^1(\mathbb{R}^d)$ such that $\hat{g} \neq 0$ on a relatively compact set $\Omega \subset \mathbb{R}^d$ then there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ $\{g(\cdot - \delta k)\}_{k \in \mathbb{Z}^d}$ is a local family of atoms [27] (see also [30, Definition 3]) for $L^2_\Omega(\mathbb{R}^d) := \{f \in L^2 : \mathrm{supp}(\hat{f}) \subset \Omega\}$. This means that for all $f \in L^2_\Omega(\mathbb{R}^d)$*

$$f = \sum_{k \in \mathbb{Z}^d} c_k(f) T_{\delta k} g \quad \text{and} \quad \sum_{k \in \mathbb{Z}^d} |\langle f, T_{\delta k} g \rangle|^2 \leq B \cdot \|f\|_2^2,$$

for suitable bounded functionals c_k such that $\sum_{k \in \mathbb{Z}^d} |c_k(f)|^2 \leq \tilde{B} \cdot \|f\|_2^2$.

Proof. It is well known that, $\{\delta^{d/2} e^{2\pi i \delta k x}\}_{k \in \mathbb{Z}^d}$ is an orthonormal basis for $L^2([-1/(2\delta), 1/(2\delta)]^d)$. For $\delta > 0$ small enough $\Omega \subset [-1/(2\delta), 1/(2\delta)]^d$. By Wiener's lemma there exists $g_1 \in L^1$ band-limited function such that $\hat{g}_1 \cdot \hat{g} \equiv 1$ on Ω. Hence, for all $f \in L^2_\Omega(\mathbb{R}^d)$.

$$\hat{f}(\omega) = [\left(\hat{f}\hat{g}_1\right)\hat{g}](\omega) = \sum_{k \in \mathbb{Z}^d} \delta^d \langle \hat{f}\hat{g}_1, e^{2\pi i \delta k x} \rangle e^{2\pi i \delta k \omega} \hat{g}(\omega). \tag{25}$$

By applying the inverse Fourier transform

$$f = \sum_{k \in \mathbb{Z}^d} \delta^d (f * g_1)(\delta k) T_{\delta k} g, \tag{26}$$

where $*$ is the convolution operator. Observe now that, by Young's inequality

$$\sum_{k \in \mathbb{Z}^d} \delta^d |(f * g_1)(\delta k)|^2 = \|f * g_1\|_2^2 \leq \|f\|_2^2 \|g_1\|_1^2. \tag{27}$$

Let us denote $c_k(f) = \delta^d (f * g_1)(\delta k)$ and then one has:

$$\sum_{k \in \mathbb{Z}} |c_k(f)|^2 \leq \left(\delta^d \|g_1\|_1^2\right) \|f\|_2^2. \tag{28}$$

In particular, $\sum_{k\in\mathbb{Z}} |\langle f, T_{\delta k} g\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle \hat{f}, e^{2\pi i\delta k\omega}\hat{g}\rangle|^2 \leq \delta^{-d}\|f\|_2^2\|\hat{g}\|_\infty^2$. By formula (25) one has that $\{T_{\delta k}\delta^d\overline{g}_1\}_{k\in\mathbb{Z}^d}$ is a dual for the local family of atoms $\{T_{\delta k}g\}_{k\in\mathbb{Z}^d}$ (see [30, Remark]). Moreover, by (28) there exists a universal constant $\tilde{B} > 0$ depending only on the behavior of $\hat{g}$ on Ω such that for all $0 < \delta \leq \delta_0$, $\sum_{k\in\mathbb{Z}} |c_k(f)|^2 \leq \tilde{B}\|f\|^2$.

Similarly, the more general translation irregular sampling case is given by the following

Theorem 4 (Irregular case). *Let $\Omega \subset \mathbb{R}^d$ be a relatively compact set, and $g \in L^1(\mathbb{R}^d)$ a band-limited function such that $\hat{g}(\omega) \neq 0$ on Ω. Then there exist a relatively compact set $U_0 \subset \mathbb{R}^d$ and a constant $C = C(g, \Omega, U_0) > 0$, such that for any subset $\{y_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^d$ for which $\{y_k + U_0\}_{k\in\mathbb{Z}}$ is an admissible covering for $\mathbb{R}^d$ there exist linear mappings $f \mapsto c_k(f)$ such that*

$$f = \sum_{k\in\mathbb{Z}} c_k(f) T_{y_k} g, \quad \textit{and} \quad (\sum_{k\in\mathbb{Z}} |c_k(f)|^p)^{1/p} \leq C \cdot \|f\|_p, \tag{29}$$

for every $f \in L^p_\Omega(\mathbb{R}^d) = \{f \in L^p(\mathbb{R}^d) : spec(f) \subset \Omega\}$ and $p \in [1, \infty)$.

Proof. [23].

For $p = 2$ the set $\{\psi_k^0 = T_{y_k} g\}_{k\in\mathbb{Z}}$ is a local family of atoms for $\mathbb{W}_0 = L^2_\Omega(\mathbb{R}^d)$ in the sense of [30, Definition 3.]. In particular, the projection of this family onto $L^2_\Omega(\mathbb{R}^d)$ is a frame for that subspace. The constant C in (29) depends only on the behavior of $\hat{g}$ on Ω and on the *density* (see Definition 2.2 [23]) of the nodes $\{y_k\}_{k\in\mathbb{Z}}$. In fact, there exists $\delta = \delta(U_0) > 0$ such that

$$C = \frac{1}{1 - c(\delta, \Omega_1)\|h\|_1} \cdot \|g_1\|_1, \quad c(\delta, \Omega_1)\|h\|_1 < 1 \tag{30}$$

where

1) g_1 is a band-limited function $g_1 \in L^1$ with $\mathrm{supp}(\hat{g}_1) \subset \mathrm{supp}(\hat{g})$ and $\hat{g}_1 \cdot \hat{g} \equiv 1$ on Ω.
2) $h \in L^1$ is a band-limited function such that $\hat{h} \equiv 1$ on $\mathrm{supp}(\hat{g})$ and $\Omega_1 := \mathrm{supp}(\hat{h})$,
3) $c(\delta, \Omega_1) := \inf \|\mathrm{osc}_\delta p\|_1$, $\mathrm{osc}_\delta p(x) = \left(\sup_{z\in B_\delta(x)} |p(z) - p(x)|\right)$, where the infimum ranges over all functions $p \in L^1$ with $\hat{p} \equiv 1$ on Ω_1.

Remark 1. Observe that $\|h\|_1$ in (30) is independent of the size $|\mathrm{supp}(\hat{g})|$. In fact, for any band-limited function $h \in L^1$ such that $\hat{h} \equiv 1$ on Ω and for all band-limited $g \in L^1$ there exists $\rho > 0$ such that $h_\rho(t) = \rho^d \cdot h(\rho \cdot t) \in L^1$ is a band-limited function, $\hat{h}_\rho \equiv 1$ on $\mathrm{supp}(\hat{g})$ and $\|h_\rho\|_1 = \|h\|_1$.

Lemma 1. *Let U be a relatively compact subset of $\mathbb{R}^d$. Then, there exists an increasing function $C_U(\xi)$ on $\mathbb{R}^+$, $C_U(\xi) \to 0$ for $\xi \to 0$, such that for all constants $\delta > 0$ and $\rho > 0$*

$$c(\delta, \rho \cdot U) \leq C_U(\delta \cdot \rho). \tag{31}$$

Proof. Consider some band-limited function $p \in L^1(\mathbb{R}^d)$ with $\hat{p} \equiv 1$ on U. Hence $p_\rho(x) = \rho^d \cdot p(\rho \cdot x)$ is a band limited function with $\hat{p}_\rho \equiv 1$ on $\rho \cdot U$ and $c(\delta, \rho \cdot U) \leq \|\mathrm{osc}_\delta p_\rho\|_1$. By the mean-value theorem, one has

$$\mathrm{osc}_\delta p_\rho(x) \leq \delta \cdot |\nabla p_\rho|^\#_\delta(x), \tag{32}$$

where $f^\#_\delta(x) := \sup_{z \in B_\delta(x)} |f(z)|$ is the *local maximal function* of f. Observe now that $\nabla p_\rho(x) = \rho^{d+1} \cdot \nabla p(\rho \cdot x)$ and

$$|\nabla p_\rho|^\#_\delta(x) = \rho^{d+1} \cdot \sup_{z \in B_\delta(x)} |\nabla p(\rho \cdot z)| = \rho^{d+1} \cdot \sup_{\rho z \in B_{\rho \cdot \delta}(\rho x)} |\nabla p(\rho \cdot z)|.$$

Thus, one has

$$\|\mathrm{osc}_\delta p_\rho\|_1 \leq \delta \cdot \rho \cdot \| |\nabla p|^\#_{\delta\rho} \|_1. \tag{33}$$

Let us choose $C_U(\xi) = \xi \cdot \| |\nabla p|^\#_\xi \|_1$.

Remark 2. By the previous lemma, for any $\varepsilon > 0$ and $\rho > 0$, for the dilation of a fundamental domain Ω by ρ there exists $\delta > 0$, $\delta \asymp \rho^{-1}$, such that $c(\delta, \rho \cdot \Omega) < \varepsilon$. Hence, a fixed constant C in (30) can be used for all pairs of functions g and sampling sets $\{y_k\}_k$, as long as the product of maximal gap size of $\{y_k\}_k$ and the size of $\mathrm{supp}(\hat{g})$ is uniformly bounded.

In order to develop a constructive approach, we present a method to calculate a possible dual $\{\tilde{\psi}^0_k\}_k$ for $\{\psi^0_k\}_k$ (also valid for the irregular case):

Remark 3. ([LDA], **L**ocal **D**ual **A**lgorithm). Under the assumptions and notations of Theorem 4, let $\Phi = \{\varphi_k\}_{k \in \mathbb{Z}}$ be an U_0-BUPU associated to $\{y_k + U_0\}_{k \in \mathbb{Z}}$. One can define the discrete measure $D_\Phi f$ in $W(M, l^2)$ by the operator

$$D_\Phi : L^2 \to W(M, l^2), \quad D_\Phi f = \sum_{k \in \mathbb{Z}} \langle f, \varphi_k \rangle_{L^2} \delta_{y_k}. \tag{34}$$

Observe that for all $g \in W(C^0, l^2)$,

$$\langle g, D_\Phi f \rangle = \langle g, \sum_{k \in \mathbb{Z}} \langle f, \varphi_k \rangle_{L^2} \delta_{y_k} \rangle = \sum_{k \in \mathbb{Z}} \langle f, \varphi_k \rangle_{L^2} g(y_k) = \langle f, \overline{\sum_{k \in \mathbb{Z}} g(y_k) \varphi_k} \rangle_{L^2}. \tag{35}$$

Hence, one can also define

$$S_\Phi : W(C^0, l^2) \to L^2, \quad S_\Phi g = \sum_{k \in \mathbb{Z}} g(y_k) \varphi_k, \tag{36}$$

and $\langle g, D_\Phi f \rangle$[1]$= \langle f, \overline{S_\Phi g} \rangle_{L^2}$. Let us consider the convolution operator:

$$C_h f = f * h. \tag{37}$$

---

[1] Here the symbol $\sigma(g) = \langle g, \sigma \rangle$ is the action of the distribution σ on the function g as extension of the bilinear form $\langle g, \overline{\sigma} \rangle_{L^2} = \int_{\mathbb{R}^d} g(x)\sigma(x)dx$ when g and σ are both functions. This notation is in fact that used in [22, 23].

Denote $C_h^* g = h^\nabla * g = C_g(h^\nabla)$, where $h^\nabla = h(-x)$. By the proof of [23, Theorem 2, formula (64)], one has that the coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ can be calculated by means of the Neumann series:

$$c_k(f) = \langle \varphi_k, \sum_{n=0}^{\infty} \left(C_h(\mathcal{I} - D_\Phi)\right)^n f_0 \rangle, \tag{38}$$

where $f_0 = C_{g_1} f$. Hence, for $p = 2$, a dual sequence $\{\tilde{\psi}_k^0\}_{k\in\mathbb{Z}}$ for the local family of atoms $\{T_{y_k} g\}_{k\in\mathbb{Z}}$ satisfying $c_k(f) = \langle f, \tilde{\psi}_k^0 \rangle_{L^2}$ can be constructed by:

$$\tilde{\psi}_k^0 = \overline{C_{g_1}^* \left(\sum_{n=0}^{\infty} \left((\mathcal{I} - S_\Phi) C_h^*\right)^n \right) \varphi_k}. \tag{39}$$

This construction can be implemented by the algorithm:

$$\Psi_k^0 = \varphi_k, \quad \Psi_k^{n+1} = (h^\nabla * \Psi_k^n) - \sum_{j\in\mathbb{Z}} h^\nabla * \Psi_k^n)(y_j)\varphi_j, \quad \forall n \in \mathbb{N}, \quad \forall k \in \mathbb{Z}, \tag{40}$$

and

$$\tilde{\psi}_k^0 = \overline{g_1^\nabla * \left(\sum_{n=0}^{\infty} \Psi_k^n \right)}. \tag{41}$$

For the regular case, $y_k = \delta \cdot k$ and $\varphi_k = T_{\delta\cdot k}\varphi_0$ imply

$$[S_\Phi, T_{\delta\cdot k}] = [C_{g_1}^*, T_{\delta\cdot k}] = [C_h^*, T_{\delta\cdot k}] \equiv 0, \tag{42}$$

where $[A, B] = A \circ B - B \circ A$ is the commutator operator of A and B. these commutator equations implies together with formula (39) that

$$\tilde{\psi}_k^0 = T_{\delta\cdot k} \tilde{\psi}_0^0. \tag{43}$$

where

$$\tilde{\psi}_0^0 = \overline{C_{g_1}^* \left(\sum_{n=0}^{\infty} \left((\mathcal{I} - S_\Phi) C_h^*\right)^n \right) \varphi_0}. \tag{44}$$

In this case we will say that the dual is (locally) *coherent* because it preserves the structure of the original local family of atoms.

5. Admissible coverings and α-admissibility

Let us assume in the following $d = 1$. The multidimensional theory will be the subject of subsequent contributions. We want to show how to define suitable decompositions ([30, Definition 2], [31]) of H^s, as a Hilbert space, in order to construct frames derived for sampling of (19) in the time-frequency-scale domain. This will be done by suitable partitioning of the frequency line and corresponding decomposition of H^s into suitable subspaces of band-limited functions. For that, the following technical results are useful:

Lemma 2. *Let* $P, S : \mathbb{R}^+ \to \mathbb{R}^+$ *two positive non decreasing (unbounded) functions,* $P \in C^1(\mathbb{R}^+)$*, such that*

$$\frac{dP}{dw}(w) = S(w), \quad P(0) = 0. \tag{45}$$

For all $b > 0$ *and* $j \in \mathbb{N}$ *denote by* $\Omega_j^{P,S}$ *the interval* $P(b{\cdot}j) + [0, b{\cdot}S(b(j+1))]$. *Then* $\mathcal{O} = \{\Omega_j^{P,S}\}_{j \in \mathbb{Z}}$ *is an admissible covering for* $\mathbb{R}^+$. *We call* P *the position function and* S *the size function of the covering.*

Proof. By the mean-value theorem, one has

$$P(b(j+1)) - P(b \cdot j) = b \cdot S(b \cdot \xi_j),$$

for some $\xi_j \in (j, j+1)$. Since S is non-decreasing, one has also:

$$P(b(j+1)) \leq P(b \cdot j) + b \cdot S(b(j+1)).$$

Moreover, P is also non-decreasing and then

$$P(b(j{+}1)) \leq P(b{\cdot}j) + b{\cdot}S(b(j{+}1)) \leq P(b(j{+}1)) + b{\cdot}S(b(j{+}1)) \leq P(b(j{+}2)).$$

Hence, for all j, $\Omega_j^{P,S}$ intersects $\Omega_{j+1}^{P,S}$, but no further successive elements of the covering and

$$\sup_{j \in \mathbb{N}} \#\{i \in \mathbb{N} : \Omega_i^{P,S} \bigcap \Omega_j^{P,S} \neq \emptyset\} \leq 3. \tag{46}$$

Lemma 3. *For all* $\alpha \in [0, 1)$*, the functions*

$$P_\alpha(\omega) = (1 + (1-\alpha)\omega)^{\frac{1}{1-\alpha}} - 1, \quad S_\alpha(\omega) = (1 + (1-\alpha)\omega)^{\frac{\alpha}{1-\alpha}} \tag{47}$$

are position and size functions. Then, for each fixed constant $b > 0$*, denote*

$$\Omega_j^\alpha = \left((1 + (1-\alpha) \cdot b \cdot j)^{\frac{1}{1-\alpha}} - 1\right) + [0, b \cdot ((1 + (1-\alpha) \cdot b \cdot (j+1))^{\frac{\alpha}{1-\alpha}}]. \tag{48}$$

Hence $\mathcal{O}^\alpha = \{\Omega_j^\alpha\}_{j \in \mathbb{N}}$ *is an admissible covering of* $\mathbb{R}^+$.

Proof. Consider the functions

$$\tilde{P}(\omega) = \omega, \quad \tilde{S}(\omega) = \eta_\alpha(\omega) = (1 + \omega)^\alpha. \tag{49}$$

They are position and size functions only for $\alpha = 0$. In order to generate some position and size functions, it suffices to show that there exists a positive smooth parameterizing function $h(\omega)$ such that

$$P_\alpha(\omega) = \tilde{P}(h(\omega)), \quad S_\alpha(\omega) = \tilde{S}(h(\omega)) \tag{50}$$

satisfy (45). This means that one has to solve the following differential equation:

$$\tilde{P}'(h(\omega))h'(\omega) = \tilde{S}(h(\omega)). \tag{51}$$

In our particular case one has to find h such that:

$$h'(\omega) = (1 + h(\omega))^{\alpha}. \tag{52}$$

The general solution of (52) is given by:

$$h_C(\omega) = (C + (1-\alpha)\omega)^{\frac{1}{1-\alpha}} - 1. \tag{53}$$

We should choose C such that $P(0) = 0$ and hence $C = 1$. One concludes by using Lemma 2.

Remark 4. Observe that for $\alpha = 0$, $\mathcal{O}^0$ is a regular covering of positions $b \cdot j$ and fixed size b. But, for $\alpha \to 1$, $\mathcal{O}^\alpha$ tends to an *exponential* covering $\mathcal{O}^1$, where position and size are of the type e^{bj}. In particular, for $b = \ln(2)$, $\mathcal{O}^1$ is a dyadic covering of $\mathbb{R}^+$. Moreover, up to mirroring, $\mathcal{O}^\alpha$ defines an admissible covering for all $\mathbb{R}$.

Remark 5. We will say that a couple of functions ($\tilde{P}$,$\tilde{S}$) is an *admissible parameterization* of the frequency axis if there exists a smooth, increasing and non-negative function h such that $\lim_{x\to\infty} h(x) = \infty$ and

$$P(\omega) = \tilde{P}(h(\omega)), \quad S(\omega) = \tilde{S}(h(\omega)),$$

are position and size functions. In particular, let us observe that for $\alpha > 1$ there does *not* exist any h satisfying the requirements.
The following results, up to suitable modifications, will hold also for more general admissible parameterizations of the frequency axis and equivalent admissible covering are defined as in [18, Definition 3.3].

In [6,5,3] sufficient conditions to have Gabor or wavelet L^2-frames are presented separately. In the following we generalize those sufficient conditions (see in particular [5, Theorem 5.1]) to check the frame upper bound (Bessel) condition for a more general class of sequences in L^2-Sobolev spaces [3], related to suitable positions and size functions.

Lemma 4. *For $\omega \in \mathbb{R}$ and $j \in \mathbb{Z}$ write $\xi_{\omega,j} = \frac{(\omega - P(j))}{S(j)}$. If for some $c > 0$ and $s \in \mathbb{R}^+$ a function $g \in H^s(\mathbb{R})$ has the property that for almost all $\omega \in \mathbb{R}$*

$$(B) \quad \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left| \hat{g}\left(\xi_{\omega,j}\right) \hat{g}\left(\xi_{\omega,j} - \frac{k}{c}\right) \right| \left(\frac{1+|\omega|^2}{1+|P(j)|^2} \right)^s \leq B_c < \infty, \tag{54}$$

then the system

$$\{g^{j,k}_{P,S,c} = (1+|P(j)|^2)^{-\frac{s}{2}} \cdot M_{P(j)} D_{S(j)^{-1}} T_{c\cdot k}\, g\}_{j,k\in\mathbb{Z}} \tag{55}$$

is a Bessel family for H^s with Bessel bound B_c/c.

Proof. We first assume $f \in \mathcal{S}$ band-limited. The general case follows later by a standard density argument.

[Step 1.] For a fixed $j \in \mathbb{Z}$ one has

$$\begin{aligned}
\int_0^{S(j)/c} \sum_{k\in\mathbb{Z}} & \left|\hat f(\omega - \frac{S(j)k}{c})\hat g\left(\xi_{\omega,j} - \frac{k}{c}\right)\right| (1+|\omega - \frac{S(j)k}{c}|^2)^s d\omega \\
&= \int_{\mathbb{R}} \left|\hat f(\omega)\hat g\left(\xi_{\omega,j}\right)\right| (1+|\omega|^2)^s d\omega \\
&\leq \|f\|_{H^s} \cdot \left(\int_{\mathbb{R}} \left|\hat g\left(\xi_{\omega,j}\right)\right|^2 (1+|\omega|^2)^s d\omega\right)^{1/2} \\
&\lesssim S(j)(1+S(j))^s(1+P(j))^s \cdot \|f\|_{H^s} \cdot \|g\|_{H^s}.
\end{aligned}$$

Therefore the periodic function

$$F_j(\omega) = \sum_{k\in\mathbb{Z}} \hat f(\omega - \frac{S(j)k}{c})\overline{\hat g\left(\xi_{\omega,j} - \frac{k}{c}\right)}(1+|\omega - \frac{S(j)k}{c}|^2)^s$$

is in $L^1[0, S(j)/c]$ and one can show that it is also in $L^2[0, S(j)/c]$. Moreover:

$$\int_{\mathbb{R}} \hat f(\omega)\overline{\hat g\left(\xi_{\omega,j}\right)}e^{2\pi i m S(j)^{-1}c\omega} \cdot (1+|\omega|^2)^s d\omega = \int_0^{\frac{S(j)}{c}} F_j(\omega)e^{2\pi i m S(j)^{-1}c\omega}d\omega. \tag{56}$$

By Fourier series, one has also

$$\sum_{m\in\mathbb{Z}} \left|\int_0^{\frac{S(j)}{c}} F_j(\omega)e^{2\pi i m S(j)^{-1}c\omega}d\omega\right|^2 = \frac{S(j)}{c}\int_0^{\frac{S(j)}{c}} |F_j(\omega)|^2 d\omega. \tag{57}$$

[Step 2.]

$$\sum_{j,k\in\mathbb{Z}} |\langle f, g^{j,k}_{P,S,c}\rangle_{H^s}|^2 = \sum_{j,m\in\mathbb{Z}} (1+|P(j)|^2)^{-s} \cdot |\langle \hat f, M_{cmS(j)^{-1}}T_{P(j)}D_{S(j)}\hat g\rangle_{L^2_{w_s}}|^2$$

$$= \sum_{j,m\in\mathbb{Z}} S(j)^{-1} \cdot (1+|P(j)|^2)^{-s} \cdot \left|\int_{\mathbb{R}} \hat f(\omega)\overline{\hat g\left(\xi_{\omega,j}\right) e^{2\pi i m S(j)^{-1}c\omega}} \cdot (1+|\omega|^2)^s d\omega\right|^2 .$$

By (56) and (57) one has

$$
\begin{aligned}
&\sum_{j,k\in\mathbb{Z}} |\langle f, g^{j,k}_{P,S,c}\rangle_{H^s}|^2\\
&= \sum_{j\in\mathbb{Z}} c^{-1}\cdot(1+|P(j)|^2)^{-s}\cdot \int_0^{\frac{S(j)}{c}} \left|\sum_{k\in\mathbb{Z}} \hat{f}(\omega-\frac{S(j)k}{c})\overline{\hat{g}\left(\xi_{\omega,j}-\frac{k}{c}\right)}(1+|\omega-\frac{S(j)k}{c}|^2)^s\right|^2 d\omega\\
&\leq c^{-1}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\int_0^{\frac{S(j)}{c}}\sum_{l\in\mathbb{Z}}\left|\hat{f}(\omega-\frac{S(j)l}{c})\hat{g}\left(\xi_{\omega,j}-\frac{l}{c}\right)\right|\left(1+|\omega-\frac{S(j)l}{c}|^2\right)^s\times\\
&\times\sum_{k\in\mathbb{Z}}\left|\hat{f}(\omega-\frac{S(j)k}{c})\hat{g}\left(\xi_{\omega,j}-\frac{k}{c}\right)\right|\left(1+|\omega-\frac{S(j)k}{c}|^2\right)^s d\omega\\
&= c^{-1}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\sum_{l\in\mathbb{Z}}\int_0^{\frac{S(j)}{c}}\left|\hat{f}(\omega-\frac{S(j)l}{c})\hat{g}\left(\xi_{\omega,j}-\frac{l}{c}\right)\right|\left(1+|\omega-\frac{S(j)l}{c}|^2\right)^s\times\\
&\times\sum_{k\in\mathbb{Z}}\left|\hat{f}(\omega-\frac{S(j)k}{c})\hat{g}\left(\xi_{\omega,j}-\frac{k}{c}\right)\right|\left(1+|\omega-\frac{S(j)k}{c}|^2\right)^s d\omega\\
&= c^{-1}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\int_{\mathbb{R}}\left|\hat{f}(\omega)\hat{g}\left(\xi_{\omega,j}\right)\right|(1+|\omega|^2)^s\times\\
&\times\sum_{k\in\mathbb{Z}}\left|\hat{f}(\omega-\frac{S(j)k}{c})\hat{g}\left(\xi_{\omega,j}-\frac{k}{c}\right)\right|\left(1+|\omega-\frac{S(j)k}{c}|^2\right)^s d\omega\\
&= c^{-1}\int_{\mathbb{R}}|\hat{f}(\omega)|^2(1+|\omega|^2)^s\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\left|\hat{g}\left(\xi_{\omega,j}\right)\right|^2(1+|\omega|^2)^s d\omega\\
&+c^{-1}\sum_{k\neq 0}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\int_{\mathbb{R}}|\hat{f}(\omega)\hat{f}(\omega-\frac{S(j)k}{c})|(1+|\omega|^2)^s\times\\
&\times\left|\hat{g}\left(\xi_{\omega,j}\right)\hat{g}\left(\xi_{\omega,j}-\frac{k}{c}\right)\right|\left(1+|\omega-\frac{S(j)k}{c}|^2\right)^s d\omega\\
&= c^{-1}\int_{\mathbb{R}}|\hat{f}(\omega)|^2(1+|\omega|^2)^s\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\left|\hat{g}\left(\xi_{\omega,j}\right)\right|^2(1+|\omega|^2)^s d\omega+c^{-1}R.
\end{aligned}
$$

We now estimate the second term R. Using Cauchy-Schwarz first on the integral and then on the sum over k as in the proof of [5, Theorem 5.1], we obtain:

$$R\leq\int_{\mathbb{R}}|\hat{f}(\omega)|^2(1+|\omega|^2)^s\sum_{k\neq 0}\sum_{j\in\mathbb{Z}}(1+|P(j)|^2)^{-s}\cdot\left|\hat{g}\left(\xi_{\omega,j}\right)\hat{g}\left(\xi_{\omega,j}-\frac{k}{c}\right)\right|(1+|\omega|^2)^s d\omega.$$

Hence, by using property (B), one finally has:

$$\sum_{j,k\mathbb{Z}}|\langle f, g^{j,k}_{P,S,c}\rangle_{H^s}|^2\leq\frac{B_c}{c}\cdot\|f\|_{H^s}. \tag{58}$$

This concludes the proof.

On the basis of the previous lemma we give the following

Definition 7. *Let* $P_\alpha(j) = sgn(j)\left((1+(1-\alpha)\cdot b\cdot|j|)^{\frac{1}{1-\alpha}}-1\right)$ *and* $S_\alpha(j) = b\cdot(1+(1-\alpha)\cdot b\cdot(|j|+1))^{\frac{\alpha}{1-\alpha}}$*, where* $\alpha \in [0,1)$*, and* $s \in \mathbb{R}^+$*. A function* $g \in H^s(\mathbb{R})$ *is called* α*-admissible with respect to* (P_α, S_α) *if*

AD1) $g \in L^1$ *and* $\hat{g} \neq 0$ *on* $\Omega = [-1,1]$ *or* g *is band-limited,* $\hat{g} \in L^\infty$ *and* $\hat{g} \neq 0$ *on* $spec(g)$*,* $\Omega \subset spec(g)$*.*

AD2) g *satisfies the condition (B) of Lemma 4 for all* $a > 0$*:*

$$\sigma^\alpha_{g,a}(\omega) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left|\hat{g}\left(\frac{\omega-P_\alpha(j)}{S_\alpha(j)}\right)\hat{g}\left(\frac{\omega-P_\alpha(j)}{S_\alpha(j)}-\frac{k}{a}\right)\right|\left(\frac{1+|\omega|^2}{1+|P_\alpha(j)|^2}\right)^s \leq B_{g,a} < \infty, \tag{59}$$

for a.e. $\omega \in \mathbb{R}$*;*

AD3) for all $\rho, a > 0$ *and all* g_ρ *band-limited approximation of* g*, i.e.* $g_\rho = g * \gamma_\rho$ *where* $\gamma \in \mathcal{S}$ *is a band-limited function, with* $\hat{\gamma} \equiv 1$ *on* Ω *and* $\gamma_\rho(t) = \rho^d\gamma(\rho\cdot t)$*, there exist* $B_{g_\rho,a} > 0$ *and* $B_{g-g_\rho,a} > 0$ *such that* $\sigma^\alpha_{g_\rho,a}(\omega) \leq B_{g_\rho,a} < \infty$ *and* $\sigma^\alpha_{g-g_\rho,a}(\omega) \leq B_{g-g_\rho,a} < \infty$ *a.e.* $\omega \in \mathbb{R}$*;*

AD4) If $a = a(\rho) = c_0^{-1}\rho^{-1}$ *then* $a^{-1}\cdot B_{g-g_\rho,a} \to 0$*, for* $\rho \to +\infty$*.*

Lemma 5. *For* $s \geq 0$ *and* $\alpha \in [0,1)$ *write* $(s;\alpha) = \frac{2s}{1-\alpha}$*. Then every function* $g \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ *such that* $|\widehat{g}| \lesssim h_t$*, with* $h_t(\omega) = (1+|\omega|^2)^{-\frac{t}{2}}$*,* $t \geq 2 + (s;\alpha)$*, and* $\hat{g}(\omega) \neq 0$ *on* Ω*, is* α*-admissible with respect to* (P_α, S_α)*.*

Proof. Assume for simplicity $b = 1$. By assumption $\widehat{g}(\omega) \neq 0$ for $\omega \in \Omega$ (AD1). Let $\xi_{\omega,j} = \frac{\omega-P_\alpha(j)}{S_\alpha(j)}$ and since $|\widehat{g}| \lesssim h_t$ and h_t is a symmetric decreasing function one has:

$$\sum_{k\in\mathbb{Z}}|\widehat{g}(\xi_{\omega,j}-\frac{k}{a})| \leq K\|h_t\|_1 := b_{g,a} < \infty, \tag{60}$$

where K is uniform with respect to $a \in (0,1]$.

Hence, one also has:

$$\sigma^\alpha_{g,a}(\omega) \leq b_{g,a}\cdot\sum_{j\in\mathbb{Z}}|\widehat{g}(\xi_{\omega,j})|\left(\frac{1+|\omega|^2}{1+|P_\alpha(j)|^2}\right)^s. \tag{61}$$

By standard analysis and calculations, one can check that $|\widehat{g}(\xi_{\omega,j})| \lesssim h_t\left((|\omega|+1)^{1-\alpha}-1-\mathrm{sgn}(j)(1-\alpha)j\right)$. This implies that

$$\begin{aligned}
\sup_{\omega\in\mathbb{R}}\sigma^\alpha_{g,a}(\omega) &\leq K_1 b_{g,a}\cdot\sup_{\omega\in\mathbb{R}}\sum_{j\in\mathbb{Z}}h_t\left((|\omega|+1)^{1-\alpha}-1-\mathrm{sgn}(j)(1-\alpha)j\right)\left(\frac{1+|\omega|^2}{1+|P_\alpha(j)|^2}\right)^s \\
&= K_1\cdot b_{g,a}\cdot\sup_{\xi\in\mathbb{R}}\sum_{j\in\mathbb{Z}}h_t\left(|\xi|-\mathrm{sgn}(j)(1-\alpha)j\right)\left(\frac{1+|P_\alpha(\frac{\xi}{1-\alpha})|^2}{1+|P_\alpha(j)|^2}\right)^s \\
&\leq K_2\cdot b_{g,a}\cdot\sup_{\xi\in\mathbb{R}}\sum_{j\in\mathbb{Z}}h_t\left(\xi-(1-\alpha)j\right)\left(\frac{1+|P_\alpha(\frac{\xi}{1-\alpha})|^2}{1+|P_\alpha(j)|^2}\right)^s \ (*)
\end{aligned}$$

Observe that $h_t \in W(C^0, l^1_{(s;\alpha)})$, hence $\sum_{j\in\mathbb{Z}}(1+|P_\alpha(j)|^2)^{-s}\delta_{(1-\alpha)j} \in W(M, l^\infty_{(s;\alpha)})$, and

$$\sum_{j\in\mathbb{Z}} \frac{h_t\left(\xi-(1-\alpha)j\right)}{(1+|P_\alpha(j)|^2)^s} = \left(\sum_{j\in\mathbb{Z}}(1+|P_\alpha(j)|^2)^{-s}\delta_{(1-\alpha)j}\right) * h_t.$$

Applying standard results concerning the behavior of Wiener amalgams under convolution and pointwise multiplication [15,17,40] may recall:

$$W(M, l^\infty_{(s;\alpha)}) * W(C^0, l^1_{(s;\alpha)}) \subset W(C^0, l^\infty_{(s;\alpha)}).$$

and

$$W(C^0, l^\infty_{-(s;\alpha)}) \cdot W(C^0, l^\infty_{(s;\alpha)}) \subset W(C^0, l^\infty).$$

In particular, $(1+|P_\alpha(\frac{\xi}{1-\alpha})|^2)^s \in W(C^0, l^\infty_{-(s;\alpha)})$, which implies (AD2):

$$(*) \leq K_2 b_{g,a} \cdot \|h_t\|_{W(C^0, l^1_{(s;\alpha)})} = K_3 \cdot \|h_t\|_1 \cdot \|h_t\|_{W(C^0, l^1_{(s;\alpha)})} := B_{g,a}. \quad (62)$$

In the same way one shows that for g_ρ, and denoting $g^\rho := g - g_\rho$

$$\sup_{\omega\in\mathbb{R}} \quad \sigma^\alpha_{g_\rho,a}(\omega) \leq O\left(\|(h_t)_\rho\|_1 \cdot \|(h_t)_\rho\|_{W(C^0, l^1_{(s;\alpha)})}\right) := B_{g_\rho,a}$$

$$\sup_{\omega\in\mathbb{R}} \quad \sigma^\alpha_{g^\rho,a}(\omega) \leq O\left(\|h^\rho_t\|_1 \cdot \|h^\rho_t\|_{W(C^0, l^1_{(s;\alpha)})}\right) := B_{g^\rho,a} \qquad (63)$$

i.e. (AD3) is valid. Moreover, for $a = a(\rho) = c_0^{-1}\rho^{-1}$, one achieves (AD4) as follows:

$$a^{-1} \cdot B_{g^\rho,a} = O\left(\rho \cdot \|h^\rho_t\|_1 \cdot \|h^\rho_t\|_{W(C^0, l^1_{(s;\alpha)})}\right) = O(\rho^{-1}) \to 0 \ \text{ for } \ \rho \to \infty.$$

Remark 6. If $\widehat{g} \in W(C^0, l^\infty_t)$ then $|\widehat{g}(\omega)| \lesssim h_t(\omega)$. Moreover $\mathcal{F}\left(W(\mathcal{F}L^2_t, l^1)\right) \subset W(\mathcal{F}L^1, l^2_t) \subset W(\mathcal{F}L^1, l^\infty_t) \cap L^2_t$. Hence, if $t \geq 2 + (s;\alpha)$, $g \in W(\mathcal{F}L^2_t, l^1)$ and $\widehat{g}(\omega) \neq 0$ on Ω then g is α-admissible with respect to (P_α, S_α).

Remark 7. Assume $t \geq 2 + (s;\alpha)$, $g \in W(\mathcal{F}L^2_t, l^1)$ and $\widehat{g}(\omega) \neq 0$ on Ω. Then g is α-admissible with respect to (P'_α, S'_α) (in the sense that (AD1-4) are valid substituting (P'_α, S'_α) to (P_α, S_α)) whenever $P'_\alpha \asymp P_\alpha$ and $S'_\alpha \asymp S_\alpha$. In fact, following the proof of the previous lemma, it will be sufficient to check that $h_t(\frac{\omega - P_\alpha(j)}{S_\alpha(j)}) \asymp h_t(\frac{\omega - P'_\alpha(j)}{S'_\alpha(j)})$ uniformly with respect to w and j. Hence, in this case, we will say that "g is α-admissible" instead of "g is α-admissible with respect to (P_α, S_α)".

Example 1. By the previous remark it is not hard to show that all $g \in \mathcal{S}(\mathbb{R})$ such that $\hat{g}(\omega) \neq 0$ on Ω are α-admissible for any $\alpha \in [0,1)$ and $s \geq 0$. Obviously the Gauss function $g(x) = e^{-\pi x^2}$ is an ideal candidate for an α-admissible function for all $s \geq 0$ and $\alpha \in [0,1)$.

6. The proof of Theorem 1

Denote $\Omega_0 = [0,1]$. Let $\mathbb{W}_0 = H^s_{\Omega_0}$ be the closed subspace of $H^s(\mathbb{R})$ of the band-limited functions f such that $\mathrm{spec}(f) \subseteq \Omega_0$ and $b > 0$ be a fixed positive constant. Assume, for example for $j > 0$ (for $j < 0$ is analogous, up to mirroring),

$$D_j^\alpha = M_{P_\alpha(j)} D_{S_\alpha(j)^{-1}}, \quad \mathbb{W}_j = D_j^\alpha(\mathbb{W}_0) = H^s_{\Omega_j^\alpha}, \quad D_0 = I_{\mathbb{W}_0}.$$

Since $\mathcal{O}^\alpha$ is an admissible covering of the frequency domain, by application of the Fourier transform identification of H^s with L^2_s, one easily has that $(\mathbb{W}_0, \{D_j^\alpha\}_{j\in\mathbb{Z}})$ is a decomposition of H^s in the sense of [30, Definition 2]. Moreover, for any BAPU $\{\varphi_j^\alpha\}_{j\in\mathbb{Z}} \subset \mathcal{S}$ associated to $\mathcal{O}^\alpha$ one can consider the *system of bounded quasi projectors* $\mathcal{P}^\alpha = \{\mathcal{P}_j^\alpha\}_{j\in\mathbb{Z}}$, in the sense of [30, Definition 4], $\mathcal{P}_j^\alpha$ mapping H^s into $\mathbb{W}_j$, given by

$$\mathcal{P}_j^\alpha(f) = f * \mathcal{F}^{-1}\varphi_j^\alpha, \quad \text{for all } f \in H^s. \tag{64}$$

In fact $\langle \mathcal{P}_j f, g\rangle = \langle \varphi_j^\alpha \cdot \hat{f}, \hat{g}\rangle = \langle \hat{f}, \varphi_j^\alpha \cdot \hat{g}\rangle = \langle f, \mathcal{P}_j g\rangle$. Hence $\mathcal{P}_j^\alpha = (\mathcal{P}_j^\alpha)^*$. $\hat{f} = 1 \cdot \hat{f} = \left(\sum_j \varphi_j^\alpha\right)\hat{f} = \sum_j \mathcal{F}(\mathcal{P}_j f)$ implies $\sum_j \mathcal{P}_j = I_{\mathcal{H}}$.

$$\sum_j \|\mathcal{P}_j^\alpha f\|^2 = \sum_j \int_{\mathbb{R}} \varphi_j^\alpha(\omega)^2 |\hat{f}(\omega)|^2 (1+|\omega|)^s d\omega =$$

$$= \sum_j \int_{\mathbb{R}} \varphi_j^\alpha(\omega)^2 \widehat{D_j^\alpha}\chi_{\Omega_0}(\omega) |\hat{f}(\omega)|^2 (1+|\omega|)^s d\omega \leq$$

$$\leq \int_{\mathbb{R}} K(\omega) \cdot |\hat{f}(\omega)|^2 (1+|\omega|)^s d\omega \leq \left(\max_{\omega\in\mathbb{R}} K(\omega)\right) \|f\|^2_{H^s}, \tag{65}$$

where $\widehat{D_j^\alpha} = T_{\mathrm{sgn}(j)\left((1+(1-\alpha)\cdot b\cdot|j|)^{\frac{1}{1-\alpha}}-1\right)} D_{b\cdot(1+(1-\alpha)\cdot b\cdot(|j|+1))^{\frac{\alpha}{1-\alpha}}} = \mathcal{F}D_j^\alpha$ and $K(\omega) = \sum_j \widehat{D_j^\alpha}\chi_{\Omega_0}(\omega)$.
Finally, $\mathcal{F}\mathcal{P}_j^\alpha(\pi_{W_j}(f)) = \varphi_j^\alpha \cdot \widehat{D_j^\alpha}\chi_{\Omega_0} \cdot \hat{f} = \varphi_j^\alpha \cdot \hat{f} = \mathcal{F}\mathcal{P}_j^\alpha(f)$. Since (AD1) and by Proposition 1 (or Theorem 4), one has that, for all $\rho > 1$ and $0 < a = a(\rho) = c_0^{-1}\rho^{-1}$ small enough, $\{\psi_k^0 = T_{ak}g_\rho\}_{k\in\mathbb{Z}}$ is a local family of atoms for $\mathbb{W}_0$, where g_ρ is a band-limited approximation of g in the sense of Definition 7 (AD3). Observe now, that the assumptions on D_j^α of [30, Theorem 1] hold if one takes as lower constant $\alpha_j = \frac{1}{2^s}\min_{\omega\in\Omega_j^\alpha}(1+|\omega|^2)^s$

and upper constant $\beta_j = \max_{\omega \in \Omega_j^\alpha}(1+|\omega|^2)^s$. In fact, for all $f \in \mathbb{W}_0$

$$\begin{aligned}
\|D_j^\alpha(f)\|_{H^s}^2 &= \int_{\Omega_j^\alpha} |\widehat{D_j^\alpha(f)}(\omega)|^2 (1+|\omega|^2)^s d\omega \\
&\leq \left(\max_{\xi \in \Omega_j^\alpha}(1+|\xi|^2)^s\right) \cdot \int_{\Omega_j^\alpha} |\widehat{D_j^\alpha(f)}(\omega)|^2 d\omega \\
&= \left(\max_{\xi \in \Omega_j^\alpha}(1+|\xi|^2)^s\right) \int_{\Omega_0} |\hat{f}(\omega)|^2 d\omega \\
&\leq \left(\max_{\omega \in \Omega_j^\alpha}(1+|\omega|^2)^s\right) \cdot \|f\|_{H^s}^2 .
\end{aligned}$$

In the same way:

$$\begin{aligned}
\frac{1}{2^s} \cdot \left(\min_{\xi \in \Omega_j^\alpha}(1+|\xi|^2)^s\right) \quad \int_{\Omega_0} |\hat{f}(\omega)|^2 (1+|\omega|^2)^s d\omega \\
&\leq \left(\min_{\xi \in \Omega_j^\alpha}(1+|\xi|^2)^s\right) \int_{\Omega_0} |\hat{f}(\omega)|^2 d\omega \\
&\leq \int_{\Omega_j^\alpha} |\widehat{D_j^\alpha(f)}(\omega)|^2 (1+|\omega|^2)^s \\
&= \|D_j^\alpha(f)\|_{H^s}^2 .
\end{aligned}$$

Moreover, making use of th fact that

$$\max_{\omega \in \Omega_j^\alpha}(1+|\omega|^2)^s \asymp \min_{\omega \in \Omega_j^\alpha}(1+|\omega|^2)^s \asymp (1+(1-\alpha)b \cdot |j|)^{\frac{2s}{1-\alpha}} ,$$

one achieves by application of [30, Theorem 1] that for all $f \in H^s$

$$f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{\alpha,a,b,\rho}^{j,k}(f)(g_\rho)_{\alpha,a,b}^{j,k}, \tag{66}$$

$$\|f\|_{H^s}^2 \asymp \sum_{k,j} |c_{\alpha,a,b,\rho}^{j,k}(f)|^2 \left(1+(1-\alpha)\cdot b \cdot |j|\right)^{\frac{2s}{1-\alpha}} . \tag{67}$$

Due to condition (AD2), consider the well defined linear bounded operator

$$S_\rho f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{\alpha,a,b,\rho}^{j,k}(f) g_{\alpha,a,b}^{j,k}. \tag{68}$$

By Lemma 4 and (AD3)

$$\begin{aligned}
\|(I-S_\rho)f\|_{H^s}^2 &= \|\sum_{k \in \mathbb{Z}} c_{\alpha,a,b,\rho}^{j,k}(f)(g^\rho)_{\alpha,a,b}^{j,k}\|_{H^s}^2 \\
&\leq a^{-1} \cdot B_{g^\rho,a} \cdot \sum_{k,j} |c_{\alpha,a,b,\rho}^{j,k}(f)|^2 \left(1+(1-\alpha)\cdot b \cdot |j|\right)^{\frac{2s}{1-\alpha}} \\
&\leq a^{-1} \cdot B_{g^\rho,a} \cdot C \cdot \|f\|_{H^s}^2 .
\end{aligned}$$

By the proof of [30, Theorem 1] and of Proposition 1 and by Remark 2, the constant C can be chosen uniformly with respect to ρ (maybe at the cost to increase the density of the translation sampling points). Moreover, by (AD4) $a^{-1} \cdot B_{g^\rho, a} \to 0$ and therefore one has

$$|||I - S_\rho|||_{H^s \to H^s} \leq \eta < 1, \text{ for } \rho > \rho_0 > 0. \tag{69}$$

Hence S_ρ is invertible by using the Neumann series expansion $S_\rho^{-1} = \sum_{n=0}^\infty (I - S_\rho)^n$, and

$$f = S_\rho S_\rho^{-1} f = \sum_{k \in \mathbb{Z}} c^{j,k}_{\alpha,a,b,\rho}(S_\rho^{-1} f) g^{j,k}_{\alpha,a,b}. \tag{70}$$

Furthermore, $|||S_\rho^{-1}|||_{H^s \to H^s} \leq \frac{1}{1-\eta}$, and one has

$$\sum_{k,j} |c^{j,k}_{\alpha,a,b,\rho}(S_\rho^{-1} f)|^2 \left(1 + (1-\alpha) \cdot b \cdot |j|\right)^{\frac{2s}{1-\alpha}} \asymp \|f\|^2_{H^s}. \tag{71}$$

This concludes the proof.

Remark 8. Theorem 1 holds for $\alpha \to 1$ (and for exponential coverings), but, in this case, one should modify the admissibility condition which becomes a pure wavelet type condition, and the proof also should be adapted. Since Theorem 4 works for irregular nodes $\{y_k\}_{k \in \mathbb{Z}}$ and Theorem 1 does not depend on the particular choice of equivalent covering $\mathcal{O}^\alpha$, up to suitable modifications of formulas (56-57), Theorem 1 can be extended to *irregular* (Gabor and wavelet) cases, where the frame is of the type $\{M_{P'_\alpha(j)} T_{y_k S'_\alpha(j)^{-1}} D_{S'_\alpha(j)^{-1}} g\}_{j,k \in \mathbb{Z}}$ [32]. In fact, the α-admissibility does not depend on (small) perturbations of $P_\alpha(j)$ and $S_\alpha(j)$ in the sense of Remark 7 nor does Theorem 1 depend on the particular choice of equivalent position and size functions. The multidimensional case can be treated in an analogous way by suitable geometrical decomposition into coronas of the frequency space centered in $P_\alpha(j)$ and with width $S_\alpha(j)$ or by suitable *stretching* of multidimensional regular coverings by means of the function $P_\alpha(\omega)$, see for example [18,14,35,47,46, 34]

6.1. *Computing and approximating duals*

If $g \in H^s$ is an α-admissible band-limited function then $\{T_{y_k} g\}_{k \in \mathbb{Z}}$ is a local family of atoms for $\mathbb{W}_0$ for any sufficiently small $\delta > 0$, and one considers $\{\tilde{\psi}^0_k\}_{k \in \mathbb{Z}}$ as its (local) dual. Then, by [30, Theorem 1] (see also [31]), for any BAPU $\{\varphi^\alpha_j\}_{j \in \mathbb{Z}}$ associated to $\mathcal{O}^\alpha$ and for all $f \in H^s$

$$f = \sum_{j,k \in \mathbb{Z}} \langle f, \left(((D^\alpha_j)^{-1})^* \tilde{\psi}^0_k\right) * \mathcal{F}^{-1} \varphi^\alpha_j \rangle_{L^2} D^\alpha_j T_{y_k} g. \tag{72}$$

Hence, calculating the local dual, maybe using the [LDA]-algorithm (Remark 3), one can completely analyze and recover any H^s-function by means

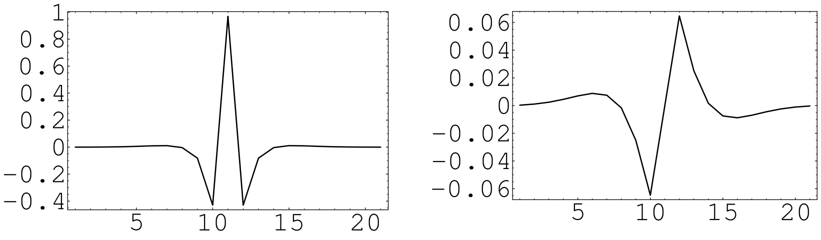

Fig. 1. Real and Imaginary part of a band-limited *analyzing function*

of flexible Gabor-wavelet frames (for all $\alpha \in [0,1)$!). One example of band-limited analyzing function g is given by

$$\hat{g}(x) = \begin{cases} e^{\frac{1}{1-|\frac{x}{1+\varepsilon}|^2}}, & |x| < 1+\varepsilon \\ 0, & \text{otherwise} \end{cases} \tag{73}$$

which is an α-admissible function (Fig. 1) for every choice of $\alpha \in [0,1)$.

Moreover, a dual frame for $\{g^{j,k}_{\alpha,a,b}\}_{j,k\in\mathbb{Z}}$ can be provided by

$$\tilde{g}^{j,k}_{\alpha,a,b} = (D^\alpha_j \tilde{\psi}^0_k) * \mathcal{F}^{-1}\varphi^\alpha_j. \tag{74}$$

Hence (74) expresses a dual in a *quasi coherent* form (with respect to the same family of operators $\{D^\alpha_j\}_{j\in\mathbb{Z}}$). If there exists Φ_0 such that $S_\alpha(j)^{-1/2} \cdot D^\alpha_j \Phi_0 = \mathcal{F}^{-1}\varphi^\alpha_j$, then the dual frame is coherent with respect to the local construction $(\mathbb{W}_0, \{D^\alpha_j\}_j)$ and can be written as:

$$\tilde{g}^{j,k}_{\alpha,a,b} = D^\alpha_j \tilde{\psi}^*_k, \tag{75}$$

where $\tilde{\psi}^*_k = \tilde{\psi}^0_k * \Phi_0$. Moreover, in the regular case (Proposition 1), for all $f \in H^s$, one has, in the sense of a unconditional convergent expansion,

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, D^\alpha_j T_{\delta\cdot k} \tilde{\psi}^*_0 \rangle D^\alpha_j T_{\delta\cdot k} g. \tag{76}$$

When $\alpha = 0$ or $\alpha = 1$ then there exist always *coherent* BAPUs (regular and e^b-adic resp.) such that $S_\alpha(j)^{-1/2} \cdot D^\alpha_j \Phi_0 = \mathcal{F}^{-1}\varphi^\alpha_j$, but, in general, one cannot expect that this is true for arbitrary $\alpha \in (0,1)$, because no group structure is available anymore.

If the function g is not band-limited, one can consider for ρ large enough the dual given by

$$\tilde{g}^{j,k}_{\alpha,a,b,\rho} = (S^{-1}_\rho)^* \left((D^\alpha_j \tilde{\psi}^0_k) * \mathcal{F}^{-1}\varphi^\alpha_j \right), \tag{77}$$

where $\tilde{\psi}^0_k$ is a dual of the local family of atoms generated by a band-limited approximation g_ρ of g and S_ρ is the operator defined in (68). Since

$$(S^{-1}_\rho)^* = \sum_{n=0}^{\infty} (I - S^*_\rho)^n, \tag{78}$$

one has that

$$|||I - (S_\rho^{-1})^*|||_{H^s \to H^s} \le \frac{\eta}{1-\eta}, \tag{79}$$

where η is as in formula (69). Moreover, because of Definition 4.i)

$$\begin{aligned}
\|(\tilde{g}_\rho)_{\alpha,a,b}^{j,k} - \tilde{g}_{\alpha,a,b,\rho}^{j,k}\|_{H^s} &= \|(I - (S_\rho^{-1})^*)\left((D_j^\alpha \tilde{\psi}_k^0) * \mathcal{F}^{-1}\varphi_j^\alpha\right)\|_{H^s} \\
&\le \frac{\eta}{1-\eta}\|(D_j^\alpha \tilde{\psi}_k^0) * \mathcal{F}^{-1}\varphi_j^\alpha\|_{H^s} \\
&\le \frac{\eta}{1-\eta}\|\varphi_j^\alpha\|_{\mathcal{F}L_s^1}\|D_j^\alpha \tilde{\psi}_k^0\|_{H^s} \\
&\le \frac{C_0\eta}{1-\eta}(1+(1-\alpha)b\cdot|j|)^{\frac{s}{1-\alpha}}\|\tilde{\psi}_k^0\|_{H^s}.
\end{aligned} \tag{80}$$

Hence, up to choosing $\eta > 0$ small and, as a consequence, ρ large enough, a dual for the L^2-frame (i.e. for s=0) $\{g_{\alpha,a,b}^{j,k}\}_{j,k}$ can be approximated by a dual of the type (74) of a band-limited approximation g_ρ of g.

Corollary 1. *Under the assumptions of Theorem 1 and using the notations of its proof, for $f \in L^2$ the following conditions are equivalent*

i) $f \in H^s$;

ii) $\sum_{j,k} |\langle f, \left((D_j^\alpha \tilde{\psi}_k^0) * \mathcal{F}^{-1}\varphi_j^\alpha\right)\rangle_{L^2}|^2 \, (1+(1-\alpha)\cdot b\cdot|j|)^{\frac{2s}{1-\alpha}} < \infty$;

iii) $\sum_{j,k} |\langle f, \left((S_\rho^{-1})^*(D_j^\alpha \tilde{\psi}_k^0) * \mathcal{F}^{-1}\varphi_j^\alpha\right)\rangle_{L^2}|^2 \, (1+(1-\alpha)\cdot b\cdot|j|)^{\frac{2s}{1-\alpha}} < \infty$.

6.2. L^2-Sobolev spaces as α-modulation spaces

Theorem 3 by Holschneider and Nazaret [42] completes the analysis of the characterization of H^s by means of the (continuous) flexible Gabor-wavelet transform V_g^α and it represents the continuous version of Theorem 1. In particular, condition (22) corresponds to the (AD2)-condition and the reconstruction formula (24) to formulas (7-8).

Fractional L^2-Sobolev spaces are just particular instances of a more general class of spaces.

Definition 8. *Related to the covering $\mathcal{O}^\alpha$ one can fix an $\mathcal{O}^\alpha$-BAPU $\{\varphi_j^\alpha\}_{j\in\mathbb{Z}}$ in $\mathcal{S}(\mathbb{R})$ and, for $1 \le p,q < \infty$, $s \in \mathbb{R}$ and $\alpha \in [0,1)$ ($\alpha = 1$ as limit case), one can define the spaces*

$$M_{p,q}^{s,\alpha}(\mathbb{R}) = \{f \in S' : \left(\sum_{j\in\mathbb{Z}} \|\mathcal{P}_j^\alpha f\|_p^q \, (1+(1-\alpha)b\cdot|j|)^{\frac{s\cdot q}{1-\alpha}}\right)^{1/q} < \infty\}, \tag{81}$$

where $\mathcal{P}_j^\alpha$ is defined as in (64). Endowed with the norm

$$\|f|M_{p,q}^{s,\alpha}\| := \left(\sum_{j\in\mathbb{Z}} \|\mathcal{P}_j^\alpha f\|_p^q \, (1+(1-\alpha)b\cdot|j|)^{\frac{s\cdot q}{1-\alpha}}\right)^{1/q}, \tag{82}$$

they are Banach spaces. The usual modifications apply for $p \cdot q = \infty$.

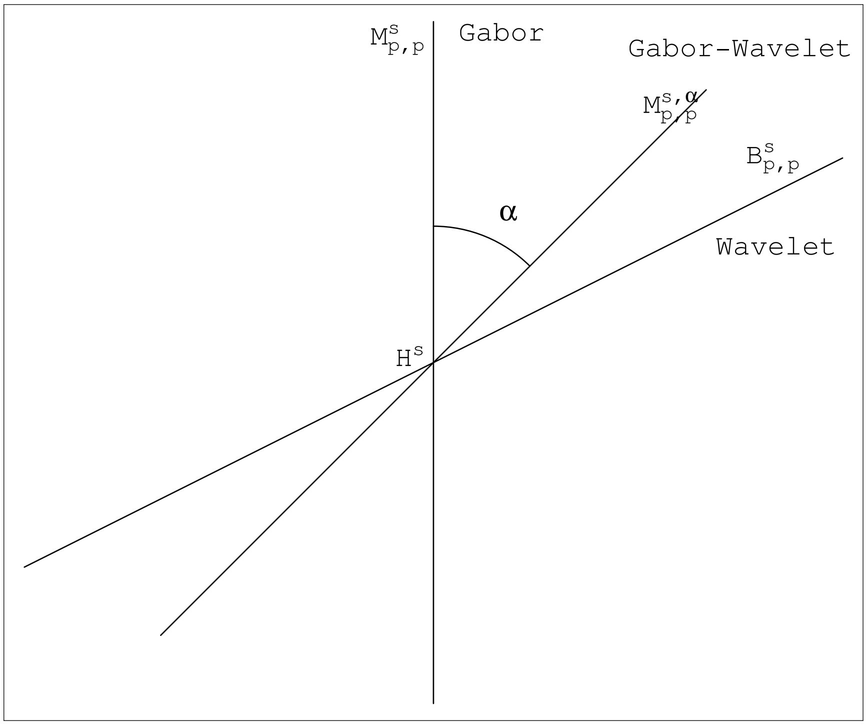

Fig. 2. α-modulation spaces

One can prove that $M^{s,\alpha}_{p,q}(\mathbb{R})$ does not depend on the particular value of $b > 0$, nor on the particular $\mathcal{O}^\alpha$-BAPU chosen. These spaces, introduced by Gröbner and Feichtinger in [18,14,35], are called α-modulation spaces and they appear to be the right spaces where to study flexible Gabor-wavelet analysis as generalized *coorbit spaces* related to the *Stone-Von-Neumann representation* (19) (restricted on a suitable *homogeneous space*, see [1,43, 44,8]) as it has been discussed in [29, Chapter 5], and it is detailed in the subsequent paper [32]. In particular, note that

$$M^{s,\alpha}_{2,2} = H^s, \quad \text{for all } s \in \mathbb{R} \text{ and } \alpha \in [0,1]. \tag{83}$$

Hence L^2-Sobolev spaces are α-modulation spaces for any exponent $s \in \mathbb{R}$ and any $\alpha \in [0,1]$. Moreover, for $\alpha = 0$ these spaces are just modulation spaces $M^s_{p,q}$ [16,37] (Gabor analysis) and for $\alpha \to 1$ are just Besov-Triebel spaces $B^s_{p,q}$ [47,46,34] (wavelet analysis).

7. Conclusion

Theorem 1 can be seen as a *unified* approach to Gabor and wavelet frames and introduces new classes of intermediate frames. It gives unified sufficient conditions for the existence of such frames which are satisfied by a quite large class of interesting functions. The construction allows some flexibility in the choice of coefficients for the decompositions as the duals depend essentially on the choice of suitable BAPUs. They are typically nice smooth functions which are well localized in time and in frequency and show a "quasi-coherent" behavior. The calculation of this kind of duals is also possible in the irregular case and can be done iteratively by means of the proposed algorithm.

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