Fast homogenization algorithm based on asymptotic theory and multiscale schemes

Maria Morandi Cecchi
Department of Pure and Applied Mathematics,
University of Padua
Via Belzoni 7, I-35131, Padova, ITALY
Email: mcecchi@math.unipd.it
Tel: +39 049 8275904

Massimo Fornasier
Department MeMoMat
University of Rome "La Sapienza"
Via A. Scarpa, 16/B, I-00161, Roma, ITALY [1]
Email: mfornasi@math.unipd.it
Tel: +39 06 49766631

Abstract

A time-frequency interpretation of the classical asymptotic theory of homogenization of elliptic PDEs with periodic coefficients by means of known multilevel/multiscale numerical schemes is investigated. The knowledge of the relations between these two different approaches is leading to formulate a new fast iterative algorithm for the approximation of homogenized solutions. In fact, the asymptotic homogenization process can be interpreted as a migration to infinity of the frequencies related to microscales contributions and the discovering of those related to the homogenized solution. Recursive low-pass filtering, at different scales/frequency of the periodic coefficients of the operator, selects only the contributions of the homogenized solution which is composed as limit of this procedure. This novel method can be interpreted as a generalized non-stationary subdivision scheme and its convergence and stability is discussed. In particular, stable compositions of the homogenized solution are investigated in relation to the contraction behavior of specific operators generated by reduction processes and Schur complements with respect to wavelets and multiscale bases.

1 Introduction

Given a family of differential operators $\{A^\varepsilon\}_{\varepsilon>0}$ and corresponding solutions $\{u_\varepsilon\}_{\varepsilon>0}$ of $A^\varepsilon u_\varepsilon = f$, one can question whether the solutions u_ε converge to some u_0, solution of a new equation $A^0 u_0 = f$, where A^0 is a suitable (limit) differential operator. This question has been investigated by De Giorgi [11] who introduced the concept of (abstract) Γ-convergence of a family of operators. In the following we discuss the case where u_0 can be synthesized by "partial compositions" of the solutions u_ε's with respect to wavelets or nested finite elements bases in problems of homogenization of elliptic differential operators with periodic coefficients. Consider the (biorthogonal) multiscale basis $\{\psi_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ and a system $\{\phi_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ with the label j indicating the scale level and the label k the space localization,

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in the sense that for any fixed scale j, $\{\phi_{j,k}\}_{k\in\mathbb{Z}}$ is a basis for the scale space $\mathcal{V}_j$ and $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is a basis for $\mathcal{W}_j = \mathcal{V}_{j+1} \ominus \mathcal{V}_j$, where $\{\mathcal{V}_j\}_{j\in\mathbb{N}}$ is a suitable sequence of nested scale spaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset ... \subset \mathcal{V}_n \subset ... \subset H^1$. Denote $\mathcal{L}_0$ the orthogonal projection onto $\mathcal{V}_0$. The spaces $\{\mathcal{V}_j\}_{j\in\mathbb{N}}$ can correspond for example to MRA spaces [4, 5, 6, 7] or to finite elements approximating spaces related to a sequence of nested meshes [16]. Given a corresponding dual $\{\tilde{\psi}_{j,k}\}_{j\in\mathbb{N},k\in\mathbb{Z}}$, we shall show that, starting with a suitable $\varepsilon_0 > 0$ not too small one can express the solution $u_{\varepsilon_{\mathbf{J}}}$ by

$$u_{\varepsilon_{\mathbf{J}}} = \mathcal{L}_0 u_{\varepsilon_{\mathbf{J}}} + \sum_{j,k} \langle u_{\varepsilon_{\mathbf{J}}}, \tilde{\psi}_{j,k}\rangle \psi_{j,k}, \quad \mathbf{J} = 0, 1, ..., \quad \text{where}$$

$$\varepsilon_{\mathbf{J}+1} = \varepsilon_{\mathbf{J}}/2 \quad \text{and} \quad \langle u_{\varepsilon_{\mathbf{J}}}, \tilde{\psi}_{j,k}\rangle \approx \langle u_{\varepsilon_{\mathbf{J}+1}}, \tilde{\psi}_{j,k}\rangle, \quad j \leq \mathbf{J}.$$

So one can compose $u_{\varepsilon_{\mathbf{J}}}$ as

$$u_{\varepsilon_{\mathbf{J}}} \approx \mathcal{L}_0 u_{\varepsilon_{\mathbf{0}}} + \sum_k \left(\langle u_{\varepsilon_{\mathbf{1}}}, \tilde{\psi}_{0,k}\rangle \psi_{0,k} + ... + \langle u_{\varepsilon_{\mathbf{J}-\mathbf{1}}}, \tilde{\psi}_{\mathbf{J}-2,k}\rangle \psi_{\mathbf{J}-2,k} \right) + \sum_{j\geq \mathbf{J}-1} \langle u_{\varepsilon_{\mathbf{J}}}, \tilde{\psi}_{j,k}\rangle \psi_{j,k}.$$

Taking the limit $\mathbf{J} \to \infty$ one has $u_0 = \lim_{\mathbf{J}\to\infty} u_{\varepsilon_{\mathbf{J}}} \approx \mathcal{L}_0 u_{\varepsilon_{\mathbf{0}}} + \sum_{k,j} \langle u_{\varepsilon_j}, \tilde{\psi}_{j,k}\rangle \psi_{j,k}$. Hence, one can (approximatively) solve the limit equation $A^0 u_0 = f$ by calculating

$$\mathcal{H}_j(u_{\varepsilon_{j+1}}) = \sum_k \langle u_{\varepsilon_{j+1}}, \tilde{\psi}_{j,k}\rangle \psi_{j,k}, \quad u_0 \approx \mathcal{L}_0 u_{\varepsilon_0} + \sum_j \mathcal{H}_j(u_{\varepsilon_{j+1}}).$$

Suitable partial information, related to specific frequencies/scales, on intermediate solutions u_ε can be composed in order to generate the limit solution u_0. In the following we present an efficient iterative strategy in order to realize the calculation of $\mathcal{H}_j(u_\varepsilon)$, by localizing the action of the operators A^ε, and the composition/synthesis of u_0, by combining classical asymptotic theory of homogenization and numerical multiscale schemes. Let us introduce, now, the homogenization problem and its mathematical formulation. Interesting physical problems involve differential operators with coefficients highly oscillating. For example composite materials are characterized by containing two or more finely mixed constituents. Usually, the heterogeneities are small compared to the global dimension of the material. So, two (or more) scales characterize the domain, the microscopic one, describing the different compositions, and the macroscopic one, describing the global (effective) behavior as if the material was "homogeneous". The aim of *homogenization* is in fact to deduce the macroscopic properties of a composite by *averaging* the microscopic structure [1, 3, 9]. To model the situation, let $\Omega \subset \mathbb{R}^d$ be a compact set, $\mathcal{A}(x) \in (L^\infty(\Omega))^{d\times d}$ be a $d \times d$ matrix with Π-periodic entries ($\Pi = [0,1)^d$ is the fundamental domain or cell) such that the associated differential operator $A(x) = div(\mathcal{A}(x) \cdot \nabla)$ is elliptic. Let us set $\mathcal{A}^\varepsilon(x) = \mathcal{A}(\varepsilon^{-1}x)$. For $\varepsilon \approx 0$ the coefficients of the operator $A^\varepsilon(x)$ have high frequencies of oscillation and the numerical solution of an equation of the type $A^\varepsilon(x)u_\varepsilon = f$ can be very expensive or inaccurate. The goal of a *homogenization* process is substituting the matrix $\mathcal{A}^\varepsilon(x)$ by a matrix $\mathcal{A}^0$ with constant coefficients such that the solution u_0 of $A^0 u_0 = f$ can be a good approximation of u_ε,

at least in the sense of weak convergence. The classical asymptotic theory for periodic elliptic PDEs [1, 3] allows the calculation of the homogenized matrix $\mathcal{A}^0$ by means of so called auxiliary periodic systems and to express u_ε as asymptotic expansion

$$u_\varepsilon(x) = u_0(x) + \sum_{j=1}^{n} \varepsilon^j u_j(x, \frac{x}{\varepsilon}) + ... \quad \text{or}$$

$$\mathcal{F}u_\varepsilon(w) = \mathcal{F}u_0(w) + \sum_{j=1}^{n} \sum_{k\neq 0} \varepsilon^j \mathcal{F}u_j(w + \frac{k}{\varepsilon}, k) + .., \tag{1}$$

where $u_j = u_j(x, y)$ are Π-periodic in the second variable y and $\mathcal{F}$ is the Fourier transform. The homogenization process can be interpreted as a *migration* to infinity (proportionally to ε^{-1}) of the frequencies related to microscopic scales, and the discovering, at the lower frequencies, of the homogenized solution. For $\varepsilon \to 0$ the process realizes a complete separation of the contributions of the scales, with the vanishing to infinity of those related to the microstructures.

Based on the asymptotic theory many different techniques and related numerical implementations [15] have been introduced. More recent multiresolution techniques [4, 5, 6, 7], based on wavelets and multilevel FEM schemes [16], have defined different strategies to construct possible methods of homogenization with interesting and relatively effective numerical implementations. These approaches have been considered in the literature as two parallel and different ways to treat the problem. We shall present a combined formulation of the classical asymptotic theory and of the numerical multilevel schemes, giving a common interpretation and proposing a novel algorithm for a fast iterative calculation of homogenized solutions.

One can define by the bases the projections $\mathcal{L}_j$ (low-pass filter) onto $\mathcal{V}_j$ and $\mathcal{H}_j = \mathcal{L}_{j+1} - \mathcal{L}_j$ (high-pass filter) onto $\mathcal{W}_j$ and $\mathcal{L}_j^*\mathcal{L}_j + \mathcal{H}_j^*\mathcal{H}_j = \mathcal{I}_{\mathcal{V}_{j+1}}$. Assume the solution u_ε^j in the approximation scale space $\mathcal{V}_j$ and one has $u_\varepsilon^j = \mathcal{L}_{j-1}u_\varepsilon^j \oplus \mathcal{H}_{j-1}u_\varepsilon^j$. Following the Beylkin *et al.* approach [2], one has (for simplicity, we ignore the subscripts)

$$\mathcal{L}Au = (\mathcal{L}A\mathcal{L}^*)\mathcal{L}u + (\mathcal{L}A\mathcal{H}^*)\mathcal{H}u = \mathcal{L}f := f_s \quad \text{and}$$

$$\mathcal{H}Au = (\mathcal{H}A\mathcal{L}^*)\mathcal{L}u + (\mathcal{H}A\mathcal{H}^*)\mathcal{H}u = \mathcal{H}f := f_d.$$

Denoting $\mathcal{T} := \mathcal{L}A\mathcal{L}^*$, $\mathcal{C} := \mathcal{L}A\mathcal{H}^*$, $\mathcal{B} := \mathcal{H}A\mathcal{L}^*$, $\mathcal{D} := \mathcal{H}A\mathcal{H}^*$, one immediately has

$$\begin{cases} \mathcal{T}s + \mathcal{C}d = f_s \\ \mathcal{B}s + \mathcal{D}d = f_d, \end{cases} \quad \text{and} \tag{2}$$

$$(\mathcal{T} - \mathcal{C}\mathcal{D}^{-1}\mathcal{B})s = f_s + \mathcal{C}\mathcal{D}^{-1}f_d, \quad \text{where} \quad s = \mathcal{L}u \in \mathcal{V}_{j-1}, \quad d = \mathcal{H}u \in \mathcal{W}_{j-1}.$$

The application of a Schur complement on the block system in (2) allows to reduce the system and to calculate the lower frequency contribution $\mathcal{L}_{j-1}u_\varepsilon^j$. Because of (1), for ε small enough and j large

enough, one has $\mathcal{L}_{j-l}u_\varepsilon^j \approx \mathcal{L}_{j-l}u_0$ where $\mathcal{L}_{j-l}u_\varepsilon^j$ can be calculate by l iterative reductions of the block system. Although the reductions have been proved to be numerical efficient [4, 5, 16], this strategy requires to start with very fine discretizations and to apply many successive reductions, realizing a possibly inefficient *top-down* scheme. For that, the algorithm we want to present in the following works in the opposite direction, realizing a *down-top* subdivision scheme which is iteratively adding to a previous approximation information related to the homogenized solution only. Let us start with $0 < \varepsilon_0 \leq 1$ not too small and j_0 not too large.

1. Reduce l-times the block system (2) and calculate a first approximation s_0 of the solution u_0.
2. Increase the frequency of the coefficients by $\varepsilon_{\mathbf{J}+1} = \varepsilon_{\mathbf{J}}/2$, moving to infinity the frequencies related to the microscales and discovering a new part of the homogenized solution (Fig. 1);
3. Reduce the new system again $l+1$-times and use the $s_{\mathbf{J}}$ already calculated in order to deduce the detail $d_{\mathbf{J}}$, purely belonging to the homogenized solution;
4. Synthesize the new approximation $s_{\mathbf{J+1}} = s_{\mathbf{J}} + d_{\mathbf{J}}$ and iterate from the point 2).

Since $s_{\mathbf{J}} \approx \mathcal{L}_{\mathbf{J}+\mathbf{j_0}-\mathbf{l}}u_0$ for all $\mathbf{J}$, one has $\lim_{\mathbf{J}\to\infty} s_{\mathbf{J}} \approx u_0$. The scheme is optimal in the sense that it can exploit the numerical advantage of fast reductions, but it also starts working with relatively small matrices, and increasing the dimension only whenever new information is going to be added.
An outline of this paper is as follows: Section 2 introduce the classical asymptotic theory for homogenization of elliptic operators with periodic coefficients, and its frequency interpretation; Section 3, based on the previous classical results, proposes a time-frequency/scale interpretation of the multiscale numerical schemes introduced by Beylkin *et al.* and Morandi Cecchi *et al.* [4, 5, 6, 7, 16]. Section 4 describes the combination of the asymptotic theory and of the numerical multiscale schemes into a novel algorithm. Its numerical behavior and efficiency are discussed.

2 Asymptotic homogenization and frequency interpretation

Let us recall few fundamental results of the classical asymptotic theory [1, 3], which will be useful to formulate our interpretation of the homogenization process and for the comparison with the numerical multilevel schemes. Let $f \in H^{-1}(\Omega)$ and u_ε be the solution of the equation

$$\begin{cases} A^\varepsilon(x)u_\varepsilon = f \quad in \quad \Omega \\ u_\varepsilon = 0 \quad on \quad \partial\Omega, \end{cases} \tag{3}$$

As already mentioned, multiple scales characterize the problem (3), the macroscopic scale x and the microscopic one x/ε, describing the microscillations. Hence, one is led to look for an asymptotic expansion of (the real solution) u_ε of the form

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + ... = \sum_{i=0}^{\infty} \varepsilon^i u_i(x, \frac{x}{\varepsilon}), \tag{4}$$

where $u_j = u_j(x,y)$ are Π-periodic in the second variable y. In particular, u_ε admits the following asymptotic expansion:

$$u_\varepsilon = u_0 - \varepsilon \sum_{k=1}^{d} \chi_k(\frac{x}{\varepsilon}) \frac{\partial u_0}{\partial x_k} + \varepsilon^2 \sum_{k,l=1}^{d} \theta^{k,l}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0}{\partial x_k \partial x_l} + ..., \tag{5}$$

where χ_k and $\theta^{k,l}$ are solutions of special auxiliary periodic problems on Π and u_0 is the solution of the equation

$$\begin{cases} A^0 u_0 = f \quad in \quad \Omega \\ u_0 = 0 \quad on \quad \partial\Omega, \end{cases} \tag{6}$$

where $A^0 = \Gamma\text{-}\lim_{\varepsilon \to 0} A^\varepsilon(x)$ is a limit (in the sense of [11]) elliptic operator with constant coefficients. Moreover, if $f \in C^\infty(\overline{\Omega})$, $\partial\Omega$ is of class C^∞ and, furthermore

$$\chi_k, \theta^{k,l} \in W^{1,\infty}(\Pi), \quad \forall k,l = 1, ..., d,$$

then, there exists a constant C independent of ε such that

$$\| u_\varepsilon - \left(u_0 - \varepsilon \sum_{k=1}^{d} \chi_k(\frac{x}{\varepsilon}) \frac{\partial u_0}{\partial x_k} + \varepsilon^2 \sum_{k,l=1}^{d} \theta^{k,l}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0}{\partial x_k \partial x_l} \right) \|_{H^1(\Omega)} \leq C \varepsilon^{1/2}. \tag{7}$$

For all $u \in H^1$ one can consider the Fourier transform indicated by $\hat{u}$ or by $\mathcal{F}u$. Let us observe that, whenever the domain Ω is regular enough, an equivalent way to express the norm $\| \cdot \|_{H^1}$ can be given by

$$\|u\|_{H^1} \asymp \inf_{v \in H^1(\mathbb{R}^d) \text{ and } v_{|\Omega} \equiv u} \left(\int_{\mathbb{R}^d} |\mathcal{F}v(w)|^2 (1 + |w|^2) dw \right)^{1/2}.$$

Hence, formula (7) can be viewed at the frequency level just applying the Fourier transforms on the functions and evaluating the approximation in the weighted L^2 spaces: $L^2_\eta = \{ f \in L^2 : f\eta \in L^2 \}$, where $\eta(w) = (1 + |w|^2)^{1/2}$ is the corresponding weight function. Let us assume then

$$u_\varepsilon \sim u_0 + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon})$$

where

$$u_1(x, \frac{x}{\varepsilon}) = - \sum_{k=1}^{d} \chi_k(\frac{x}{\varepsilon}) \frac{\partial u_0}{\partial x_k}, \quad u_2(x, \frac{x}{\varepsilon}) = \sum_{k,l=1}^{d} \theta^{k,l}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0}{\partial x_k \partial x_l}$$

and "$\sim$" is approximation in the H^1 norm. One can also assume that the functions $u_j(x,\cdot)$, $j = 1,2$ have vanishing mean on Π. Hence, expanding by Fourier series in the second variables one has

$$u_\varepsilon(x) \sim u_0(x) + \sum_{j=1}^{2} \sum_{k \neq 0} \varepsilon^j \int_\Pi u_j(x,\xi) e^{-2\pi i k(\xi - \frac{x}{\varepsilon})} d\xi = u_0(x) + U_1^\varepsilon(x).$$

Let us apply the Fourier transform to U_1^ε:

$$\mathcal{F}U_1^\varepsilon(w) = \sum_{j=1}^{2}\sum_{k\neq 0}\varepsilon^j \int_{\mathbb{R}^d}\int_\Pi u_j(x,\xi)e^{-2\pi ik\xi}d\xi e^{-2\pi i\frac{kx}{\varepsilon}}e^{-2\pi iwx}dx =$$

interchanging the integrals

$$= \sum_{j=1}^{2}\sum_{k\neq 0}\varepsilon^j \int_\Pi e^{-2\pi ik\xi}\int_{\mathbb{R}^d} u_j(x,\xi)e^{-2\pi i(w+\frac{k}{\varepsilon})x}dxd\xi =$$

$$= \sum_{j=1}^{2}\sum_{k\neq 0}\varepsilon^j \int_\Pi \mathcal{F}_x[u_j(\cdot,\xi)](w+\frac{k}{\varepsilon})e^{-2\pi ik\xi}d\xi =$$

where $\mathcal{F}_x$ is the Fourier transform with respect to the first variable and one has

$$= \sum_{j=1}^{2}\sum_{k\neq 0}\varepsilon^j \mathcal{F}u_j(w+\frac{k}{\varepsilon},k).$$

Hence, up to extending the functions out of Ω, one has

$$\mathcal{F}u_\varepsilon(w) \sim \mathcal{F}u_0(w) + \sum_{j=1}^{2}\sum_{k\neq 0}\varepsilon^j \mathcal{F}u_j(w+\frac{k}{\varepsilon},k), \tag{8}$$

where "$\sim$" is approximation in the L^2_η norm. Under regularity assumptions, all the functions $\hat{u}_j$ can be assumed quite well concentrated around 0 in the frequency domain and hence for $\varepsilon \approx 0$

i) if $w \approx 0$,

$$\hat{u}_j(w+\varepsilon^{-1}k,k) \approx \hat{u}_j(\varepsilon^{-1}k,k) \approx 0,$$

ii) if $w \approx -\varepsilon^{-1}k$ (i.e. quite far form 0),

$$\hat{u}_j(w+\varepsilon^{-1}k,k) \approx \hat{u}_j(0,k).$$

Hence, one can see that close to 0 the contribution in the frequency is given mainly by $\mathcal{F}u_0$ and for the more distant frequencies $\varepsilon^{-1}k$ the contributions are mainly due to the $\mathcal{F}u_j$'s. One can figure this phenomenon as a *migration* to infinity (proportionally to ε^{-1}) of the frequencies related to microscopic scales, and the discovering, at the lower frequencies, of the homogenized solution. For $\varepsilon \to 0$ the process of homogenization realizes a complete separation of the contributions of the scales, with the vanishing to infinity of those related to the microstructures.

Figure 1 shows a solution of a one dimension elliptic differential equation and its Fourier transform. One can observe the different contributions at the frequency level, given by the homogenized solution

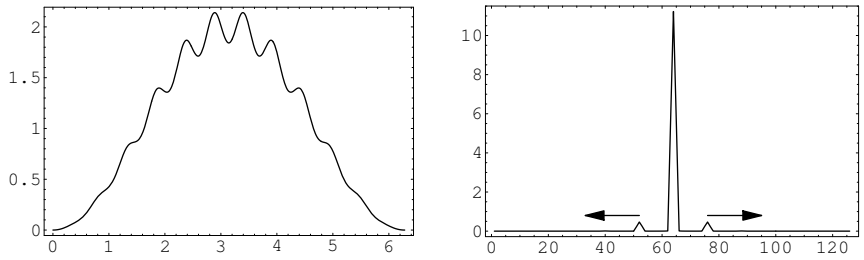

Figure 1: Solution u_ε and its Fourier Transform at $\varepsilon = \frac{1}{6}$. Two scales/frequencies are clearly separated.

u_0 (frequencies localized around zero) and by some other small *waves* moving to infinity for $\varepsilon \to 0$ related to microscopic scales. This observation on the asymptotic behavior of the Fourier transform of solutions in homogenization problems suggests that frequency filtering can be used to implement numerical techniques of homogenization. In fact, discretizations and reductions by means of recursive Schur's complements applied on the global stiffness matrices (assembled in suitable way [16]) actually function as low pass filters as we are going to describe in the following section.

3 Time-frequency/scale analysis and homogenization

This section is devoted to explain why the known numerical multiresolution strategies of homogenization can be effective and their relations with the classical asymptotic homogenization. We want to model the problem (3) by discretization. For example and simplicity, assume $\Omega = [0,1]^d \subset \mathbb{R}^d$ and associated to the differential equation one can construct, by finite differences[2], a finite dimensional approximating linear algebraic system

$$\mathbf{A}_\varepsilon \mathbf{u}_\varepsilon = \mathbf{f}, \tag{9}$$

where $\mathbf{A}_\varepsilon$ is a $2^{dn} \times 2^{dn}$ matrix and $\mathbf{u}_\varepsilon$ is a vector of size 2^{dn}. For n large enough (i.e a sampling rate 2^{-n} quite small) the solution $\mathbf{u}_\varepsilon$, up to interpolation, is, with good accuracy, an approximation of u_ε. By the Whittacker-Shannon theorem one can expect that for $\tau = 2^{-n}$ and n large enough

$$u_\varepsilon(x) \approx \sum_{k=0}^{\mathbf{2^{dn}}} \mathbf{u}_\varepsilon(\tau k) \prod_{i=1}^{d} sinc(\tau_i^{-1} x_i - k_i), \quad \text{where} \quad sinc(x_i) = \frac{sin(\pi x_i)}{\pi x_i}. \tag{10}$$

This approximation is good when the frequency concentration of u_ε is localized in the compact $[-2^n\pi/2, 2^n\pi/2]^d$ and $u_\varepsilon(\tau k) \approx \mathbf{u}_\varepsilon(\tau k)$. By formula (8) one knows that the frequency representation of u_ε can be approximatively decomposed into two parts: the macroscopic scale contribution given essentially by $\hat{u}_0$ and the microscopic scale one, given by $\sum_{j=1}^{\infty} \sum_{k \neq 0} \varepsilon^j \hat{u}_j(w + \varepsilon^{-1}k, k)$. Hence,

[2]This choice is also motivated by the easy reproducibility of the model and consequently of the algorithm that we present in this contribution.

a *conscious selection* of the frequencies can be able to isolate the homogenized solution from the microscopic behavior. For simplicity let $d = 1$. The discrete Haar transform [4] of the vector $\mathbf{u}_\varepsilon$ is an orthogonal change of basis given by

$$s_k^\varepsilon = \frac{1}{\sqrt{2}}\left(\mathbf{u}_\varepsilon(2k-1) + \mathbf{u}_\varepsilon(2k)\right), \quad d_k^\varepsilon = \frac{1}{\sqrt{2}}\left(\mathbf{u}_\varepsilon(2k-1) - \mathbf{u}_\varepsilon(2k)\right), \tag{11}$$

where $k = 1, ..., 2^{n-1}$. Hence, the discrete Haar transform can be represented by the 2^n-square matrix

$$\mathbf{M}_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 1 & 1 & 0 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \\ 1 & -1 & 0 & 0 & ... & ... & ... \\ 0 & 0 & 1 & -1 & 0 & 0 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{pmatrix}. \tag{12}$$

Let us denote the top half of $\mathbf{M}_n$ by $\mathbf{L}_n$ and the bottom half by $\mathbf{H}_n$. The operators associated to the matrices $\mathbf{L}_n$ and $\mathbf{H}_n$ function as *low-pass* and *high-pass* filters respectively. Following the arguments of Beylkin and Brewster [5], let us apply these operators to the algebraic equation (9) obtaining (do not consider the subscripts for simplicity)

$$\begin{aligned} \mathbf{LAu} &= (\mathbf{LAL}^T)\mathbf{Lu} + (\mathbf{LAH}^T)\mathbf{Hu} = \mathbf{Lf} := \mathbf{f}_s \\ \mathbf{HAu} &= (\mathbf{HAL}^T)\mathbf{Lu} + (\mathbf{HAH}^T)\mathbf{Hu} = \mathbf{Hf} := \mathbf{f}_d. \end{aligned} \tag{13}$$

Denoting

$$\mathbf{T} := \mathbf{LAL}^T, \quad \mathbf{C} := \mathbf{LAH}^T, \quad \mathbf{B} := \mathbf{HAL}^T \quad \mathbf{D} := \mathbf{HAH}^T, \tag{14}$$

one immediately has

$$\begin{cases} \mathbf{Ts} + \mathbf{Cd} = \mathbf{f}_s \\ \mathbf{Bs} + \mathbf{Dd} = \mathbf{f}_d, \end{cases} \tag{15}$$

By (15) one can deduce the reduced system $(\mathbf{T} - \mathbf{CD}^{-1}\mathbf{B})\mathbf{s} = \mathbf{f}_s + \mathbf{CD}^{-1}\mathbf{f}_d$. The number of unknowns has been reduced by half and the solution $\mathbf{s}$ is actually representing the frequency contribution of u_ε concentrated in the compact $[-2^{n-1}\pi/2, 2^{n-1}\pi/2]$. Then, the applied reduction strategy, equivalent to the calculation of the Schur's complement, is realizing a low-pass filtering of the solution. Let us write then $\mathbf{A}_\varepsilon^1 := \mathbf{T} - \mathbf{CD}^{-1}\mathbf{B}$ and $\mathbf{f}^1 := \mathbf{f}_s + \mathbf{CD}^{-1}\mathbf{f}_d$. One can iterate this procedure on the new system

$$\mathbf{A}_\varepsilon^1 \mathbf{u}_\varepsilon^1 = \mathbf{f}^1, \tag{16}$$

deducing systems always smaller and smaller with solutions representing lower frequencies. The idea which we have presented in the simple case of finite differences discretization can be generalized by using wavelets (bi)orthogonal expansions and the multiresolution analysis [5, 6, 7] but also by means of multilevel FEM schemes [16] leading to powerful strategies and computational advantages by means

of fast successive reduction steps. In fact, one can consider a sequence multilevel/MRA nested spaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset ... \subset \mathcal{V}_n \subset ... \subset H^1$, $\mathcal{W}_j = \mathcal{V}_{j+1} \ominus \mathcal{V}_j$ and corresponding projection operators $\mathcal{L}_j$ (low-pass filter) onto $\mathcal{V}_j$ and $\mathcal{H}_j = \mathcal{L}_{j+1} - \mathcal{L}_j$ (high-pass filter) onto $\mathcal{W}_j$. Thus, the reduction procedure can be directly generalized as described in the Introduction.

One of the interesting computational advantages of this strategy is the possibility of compressing the

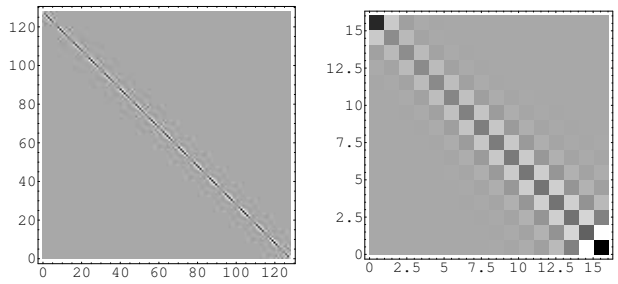

Figure 2: Band conservation under reduction

operator matrix (whenever one uses wavelets with enough vanishing moments) [5, 17] and by the fact that the reduction does not affect the possible band-structure of the matrix (Fig. 2).

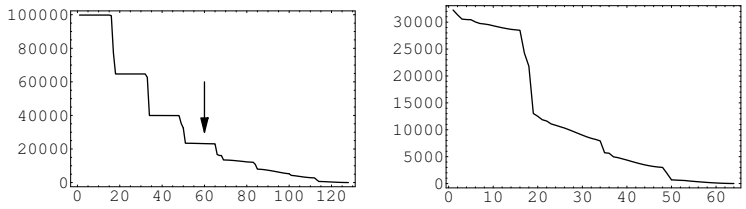

Figure 3: Lower eigenvalues conservation

Consequent fast inversions and multiplication matrix-matrix, matrix-vector can be implemented. Moreover, the reduction is preserving also the lower part of its spectrum [5, 6, 7] (Fig. 3). This property has been used to have a fast calculation of lower eigenvalues [6]. On the other hand, the strategy presented above has some side effects and inefficiencies.

- The numerical reduction algorithm presented in the previous section for a discrete algebraic approximation of (3) assumes that ε and the sampling rate $\tau = 2^{-n}$ (or the size of the mesh) can be quite small. Otherwise there is not enough separation of the frequencies related to the microscales and those related essentially to the homogenized solution. Even if the computation of the reduction can be realized in an efficient way exploiting the conservation of the band-structure of the matrix, one must start the reduction process from systems and matrices very large.

- It is not possible to know a priori if, for a given small scale ε, a good separation of the frequencies is already happened or whether the result of the reductions is still affected by frequencies related to microscales. Moreover, one cannot know when the process of reduction should stop, having detected a good approximation of the homogenized solution.

4 An optimized approximating algorithm

We want to construct an improvement of the previous strategy, by combining the reduction algorithm and the special frequency behavior of the asymptotic homogenization process described by (8). This novel algorithm will exploit the computational advantage of the reductions, but it will start from low frequencies and small matrices and it overcomes those mentioned inefficiencies. The block system (15) can be used in the converse way, to achieve a reconstruction of the details (higher frequencies) once the lower frequencies are known. In fact, if one knows the lower frequency contribution $\mathbf{s}$ one has

$$\mathbf{d} = \mathbf{C}^{-1}\mathbf{f}_s - \mathbf{C}^{-1}\mathbf{T}\mathbf{s}, \tag{17}$$

$$\mathbf{d} = \mathbf{D}^{-1}\mathbf{f}_d - \mathbf{D}^{-1}\mathbf{B}\mathbf{s}. \tag{18}$$

The scheme we shall propose moves into iterative short reduction and reconstruction schemes starting from low oscillating coefficients (and small discretization matrices) to higher frequencies, composing at each step new parts of the homogenized solution and producing a *down-top* scheme. The new iteration is based in solving a new reduced block algebraic system by means of equations (17-18) where the unknowns are the details to be composed with the lower frequencies contribution coming from the previous iteration.
Let us detail now formally how the algorithm described in the Introduction works. Assume $0 < \varepsilon_0 \leq 1$, not too small. The coefficients of the differential operator in (3) do not oscillate too much and, up to regularity conditions on the data f, the solution u_{ε_0} should be quite well concentrated in the frequency domain. Hence, one can find a minimal sampling rate $\tau_0 = 2^{-m_0}$ and consider the (relatively small) associated discrete algebraic system

$$\mathbf{A}^0_{\varepsilon_0}\mathbf{u}^0_{\varepsilon_0} = \mathbf{f}^0_0. \tag{19}$$

Algorithm 4.1. *(Iterative subdivision)*

1. *By reduction applied l-times, one has the system*

$$\mathbf{A}^l_{\varepsilon_0}\mathbf{s}^l_{\varepsilon_0} = \mathbf{f}^l_0, \quad \exists l \geq 1, \tag{20}$$

where $\mathbf{s}^l_{\varepsilon_0}$ is the filtered solution of size 2^{n_0}, $n_0 = m_0 - l$.

2. *Denote $\varepsilon_{\mathbf{J}} = \varepsilon_{\mathbf{J}-1}/2$ and $m_{\mathbf{J}} = m_{\mathbf{J}-1} + 1$. One has the new system*

$$\mathbf{A}^0_{\varepsilon_{\mathbf{J}}}\mathbf{u}^0_{\varepsilon_{\mathbf{J}}} = \mathbf{f}^0_{\mathbf{J}}. \tag{21}$$

3. *Applying* $l_{\mathbf{J}} = l + 1$ *reduction steps one can calculate the details by means of (17) or (18), for example*

$$\mathbf{d}_{\varepsilon_{\mathbf{J}}}^{l_{\mathbf{J}}} = \mathbf{D}^{-1}\mathbf{f}_d^{l_{\mathbf{J}}} - \mathbf{D}^{-1}\mathbf{B}\mathbf{s}_{\varepsilon_{\mathbf{J}-1}}^{l_{\mathbf{J}-1}}. \tag{22}$$

4. *Observe that* $2^{n_{\mathbf{J}}-1} := size(\mathbf{d}_{\varepsilon_{\mathbf{J}}}^{l_{\mathbf{J}}}) = size(\mathbf{s}_{\varepsilon_{\mathbf{J}-1}}^{l_{\mathbf{J}-1}})$ *and one can up date the approximation by*

$$\mathbf{s}_{\varepsilon_{\mathbf{J}}}^{l_{\mathbf{J}}} := \mathbf{M}_{n_{\mathbf{J}}}^{-1}(\mathbf{s}_{\varepsilon_{\mathbf{J}-1}}^{l_{\mathbf{J}-1}}, \mathbf{d}_{\varepsilon_{\mathbf{J}}}^{l_{\mathbf{J}}}). \tag{23}$$

Iterate from point 2).

One can stop at the iteration $\mathbf{J} = 0, 1, \ldots$ and set $\mathbf{u_0} = \mathbf{s}_{\varepsilon_{\mathbf{J}}}^{l_{\mathbf{J}}}$ [3].

At every step $\mathbf{J}$ one can deduce from the previous step $\mathbf{J}-1$ a higher frequency part (a new detail) of the homogenized solution. In fact, by reducing the scale $\varepsilon_{\mathbf{J}}$ at every step one can ensure that the detail deduced is *not* belonging to the microscale components of the solution as their frequency contributions are moving to infinity with the same rate as the filter bandwidth. The advantage of this algorithm with respect to known schemes as in [5, 6, 7, 16] is to isolated at each different scale only that part of the homogenized solution $\mathbf{u}_0$ with an optimal sampling rate and to move by short reduction/reconstructions operations. In the following section we want to discuss the numerical behavior of the algorithm, with emphasis on the conditions for the convergence.

4.1 Homogenization as generalized non stationary subdivision scheme

Subdivision schemes constitute a large class of fast recursive methods for computing curves and surfaces. They seem to have their origin in the geometrical problem of smoothing the corners of a given polyhedral surface [12]. An extensive study of such subdivision schemes and the corresponding notion of convergence is given by Cavaretta, Dahmen, Micchelli [8]. For related work in geometrical modeling see also Dyn, Gregory, Levin [13]. The convergence of the algorithm described in the previous section can be related to a generalized formulation of subdivision scheme. Given sequences of invertible operators $\{D_k\}_{k\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ and a sequence of data $\{f_d^k\}_{k\in\mathbb{N}}$, one can define for all $k \in \mathbb{N}$ the subdivision operator given by:

$$S_k : s_{k-1} \mapsto M^{-1}(s_{k-1}, D_k^{-1} f_d^k - D_k^{-1} B_k s_{k-1}), \quad s_k = S_k s_{k-1}. \tag{24}$$

This generalized formulation of subdivision scheme appears as the right tool to study the convergence of the algorithm described in 4.1. On the other hand, the analysis of conditions of convergence and stability of this kind of scheme is a difficult and interesting task [10].

In the Beylkin scheme and assuming to work with a fixed scale ε for the coefficients of the equation (3), the reconstructing subdivision scheme (24) is exact and converging [5]: in fact, any initial s_0 can

[3] The numerical experiments have been realized on the base of the C++ linear algebra and multilevel libraries in IGPMLib.2.1[2, 17]

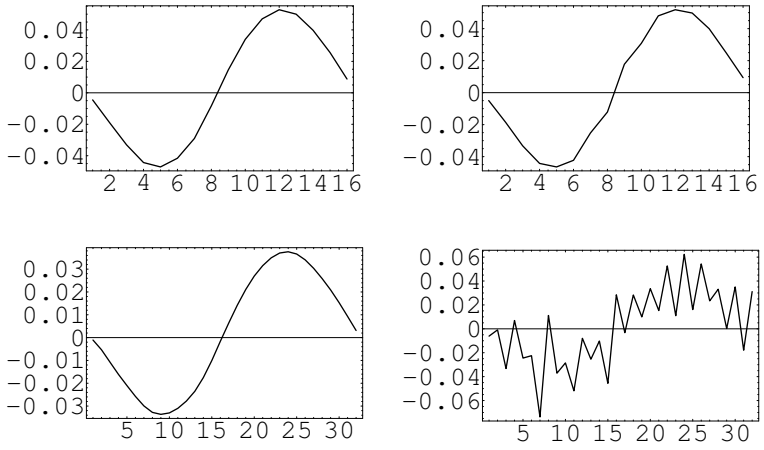

Figure 4: Application of the subdivision algorithm 4.1 on a periodic elliptic differential equation in 1D. In the left column a stable and converging iteration of the homogenization scheme by using equation (18) is presented. The right column shows the unstable iteration of the scheme using equation (17).

be considered as the reduced lower frequency part of a solution u_ε and the set $\{f_d^k\}_{k\in\mathbb{N}}$ as the sequence of the reduced part of the initial data f. On the other side, the convergence of the scheme generated by the homogenization algorithm with variable scale ε proposed in 4.1 is strongly dependent on the choice of the reconstruction equation in (22). In fact, even if equations (17) and (18) are equivalent in the case of fixed scale, in the variable scale case, at each step, the lower frequencies are *only approximatively* the same of the homogenized solution, because of formula (8). In order not to make exploding the initial errors one have to choose the right *contracting* equations, controlling the approximation. The reason of the different convergence phenomena shown in Fig. 4 is essentially due to the contracting behavior of equation (18), meanwhile equation (17) is an expanding equation. In fact, the initial number of reduction steps has to be chosen to achieve the contracting property of the operator $D^{-1}B$. At every reduction step the spectrum of this operator is reduced and its maximal eigenvalue, essentially coinciding with the operator norm, is going to be smaller then 1 (Fig. 5).

The contracting property of the subdivision operator generated by successive reductions is able to control the initial approximation error:

$$\tilde{\mathbf{d}} = \mathbf{D^{-1}f_d} - \mathbf{D^{-1}B\tilde{s}} = \mathbf{D^{-1}f_d} - \mathbf{D^{-1}Bs} - \mathbf{D^{-1}B}\delta_\mathbf{s},$$

where $\tilde{s} = s + \delta_s$. Since $\mathbf{d} = \mathbf{D}^{-1}\mathbf{f}_d - \mathbf{D^{-1}Bs}$ then the error on the new detail is given by

$$\|\tilde{\mathbf{d}} - \mathbf{d}\| = \|\mathbf{D^{-1}B}\delta_\mathbf{s}\| < \|\delta_\mathbf{s}\|.$$

At the *practical level* this is in general sufficient to ensure the convergence and the stability of the

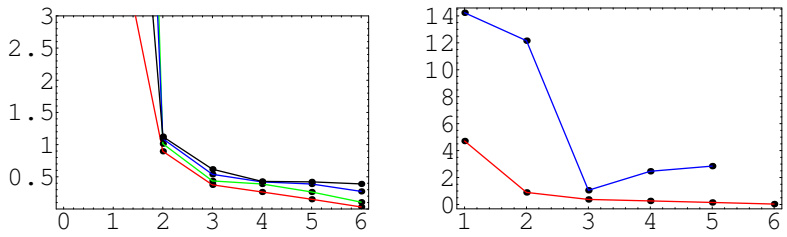

Figure 5: On the left side, the norms of the operator $D^{-1}B$ are presented with respect to successive reduction steps and at different scales ε. After few reductions $|||D^{-1}B||| < 1$. On the right side, a comparison of the norms of the operators $D^{-1}B$ (red) and $C^{-1}T$ (blue) are shown with respect to successive reductions and at fixed scale.

scheme (Fig. 4) to smooth curves. On the other hand, at each step, one can also apply an iterative procedure to correct the error and to up-date the approximation, as we discuss in the following section.

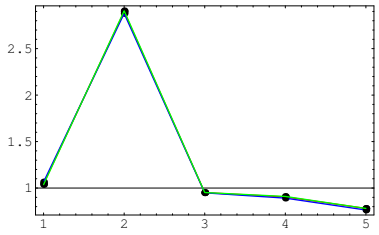

Figure 6: The norms of $|||\mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}|||$ are presented at different scales and successive reductions.

4.2 Convergence and iterative corrections

The equation (17-18) can be rewritten in the following way:

$$\mathbf{d} = \mathbf{D}^{-1}\mathbf{f}_d - \mathbf{D}^{-1}\mathbf{B}\mathbf{s}, \quad \mathbf{s} = \mathbf{T}^{-1}\mathbf{f}_s - \mathbf{T}^{-1}\mathbf{C}\mathbf{d}. \tag{25}$$

One has the following

Proposition 4.2. *If* $|||\mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}||| \leq \gamma < 1$, *then the iterative scheme:*

$$\mathbf{d_k} = \mathbf{D}^{-1}\mathbf{f}_d - \mathbf{D}^{-1}\mathbf{B}\mathbf{s_k}, \quad \mathbf{s_{k+1}} = \mathbf{T}^{-1}\mathbf{f}_s - \mathbf{T}^{-1}\mathbf{C}\mathbf{d_k}, \tag{26}$$

is convergent for any initial choice of $\mathbf{s_0}$. *In particular*

$$\mathbf{d} = \lim_{k\to\infty} \mathbf{d_k}, \quad \mathbf{s} = \lim_{k\to\infty} \mathbf{s_k} \tag{27}$$

Proof. Just observe that by (26) one has

$$\mathbf{d_{k+1}} = \mathbf{D}^{-1}\mathrm{f}_d - \mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathrm{f}_s + \mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}\mathbf{d_k}. \tag{28}$$

In the same way, by (25), one has

$$\mathbf{d} = \mathbf{D}^{-1}\mathrm{f}_d - \mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathrm{f}_s + \mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}\mathbf{d}. \tag{29}$$

Thus one obtains

$$\|\mathbf{d_{k+1}} - \mathbf{d}\| = \|\mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}(\mathbf{d_k} - \mathbf{d})\| \leq |||\mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}|||\|\mathbf{d_k} - \mathbf{d}\| \leq \gamma^k\|\mathbf{d_0} - \mathbf{d}\|.$$

Since $\gamma < 1$, one has $\mathbf{d_k} \to \mathbf{d}$, for $k \to \infty$ and, as a consequence, $\mathbf{s_k} \to \mathbf{s}$, for $k \to \infty$. □

In fact, in the numerical experiments we have developed, a sufficient number of reduction steps ensures that $|||\mathbf{D}^{-1}\mathbf{B}||| < 1$, but also that $|||\mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}||| < 1$ (Fig. 6). Hence, even if the new detail calculated by (22) is just an approximation, one can correct the error by means of the iterative scheme (26). Therefore, one can combine steps of the subdivision scheme (24) and few corrections

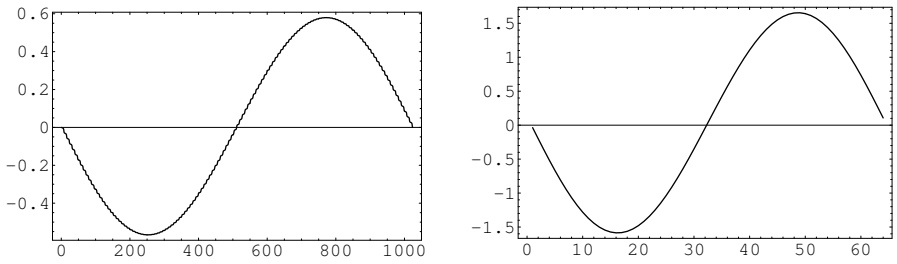

Figure 7: On the left the solution $\mathbf{u}_\varepsilon$ and on the right the corresponding homogenized after 4 reductions.

by means of (26), ensuring convergence to u_0 and stability. Based on these numerical evidences, an analytic description of the behavior of $|||\mathbf{D}^{-1}\mathbf{B}|||$ and $|||\mathbf{D}^{-1}\mathbf{B}\mathbf{T}^{-1}\mathbf{C}|||$ depending on the number of reductions is still matter of investigation and it appears a quite challenging problem. Generalizations to the multidimensional case and to operators with coefficients highly oscillating, but not periodic are also expected.

4.3 Comparison with the Beylkin's scheme on a simple case

Figure 7 shows u_ε on the left for $\varepsilon \approx 0$. The microscales contribution is present, but it is not much relevant to give a macroscopic description of the solution. The filtered solution after 4 reduction steps

(16), at lower dimension, is shown on the right. The computational cost to calculate a homogenized solution by successive reductions as in the top-down Beylkin's scheme is given by 513 time-units, for 4 reduction steps:

Operator calculation	*Reduced system calculation*
351	97
46	12
5	1
1	0

Tab. 1. Beylkin reduction scheme: 4 reduction and 513 t.u. cost.

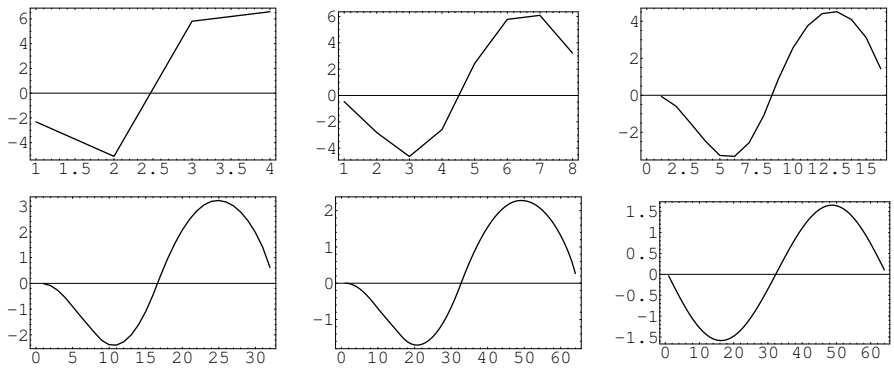

Figure 8: Iterative compositions (without correction steps) of the homogenized solution by means of 4.1.

The computational cost to calculate the homogenized solution by means of down-top subdivision algorithm 4.1 is given by 72 t.u. in 5 compositions requiring 3 reductions:

Operator calculation	*Reduced system calculation*
0	0
0	0
0	0
6	1
52	13

Tab. 1. Reduction/composition scheme: 5 composition, each after 3 reductions, and 72 t.u. cost.

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