INTRINSIC LOCALIZATION OF FRAMES

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ABSTRACT. Several concepts for the localization of a frame are studied. The intrinsic localization of a frame is defined by the decay properties of its Gramian matrix. Our main result asserts that the canonical dual frame possesses the same intrinsic localization as the original frame. The proof relies heavily on Banach algebra techniques, in particular on recent spectral invariance properties for certain Banach algebras of infinite matrices.

Intrinsically localized frames extend in a natural way to Banach frames for a class of associated Banach spaces which are defined by weighted ℓ^p-coefficients of their frame expansions. As an example the time-frequency concentration of distributions is characterized by means of localized (nonuniform) Gabor frames.

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1. INTRODUCTION

The localization of a frame is a new measure for the quality of frames, and its study has created a new direction in frame theory. In principle, frames provide redundant, non-orthogonal series expansions in Hilbert spaces, and are useful whenever the redundancy is a fact of life, as in (nonuniform) sampling problems, in time-frequency analysis, or in $\Sigma\Delta$-modulation. In most cases, however, frame expansions accomplish much more, and they hold in a large class of Banach spaces and help to explain some of the non-linear approximation properties of frame dictionaries. The theory of localized frames was established in order to understand these useful properties on an abstract level and unify many known examples.

The original definition of frame localization in [27] requires an underlying Riesz basis. To be specific, a frame $\{g_n : n \in \mathbb{Z}^d\}$ is localized with respect to the orthonormal basis $\{b_m : m \in \mathbb{Z}^d\}$ if the matrix with entries $\langle g_n, b_m \rangle$ possesses a prescribed decay in $|m - n|$, for instance polynomial decay or exponential decay. The main theorem about localized frames with respect to a Riesz basis asserts that the canonical dual frame possesses the same localization [27]. This statement is at the root of the solution of two problems about nonuniform sampling in shift-invariant spaces and about

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time-frequency molecules [27, Sec. 4,5]. Other applications deal with the characterization of Banach spaces by means of frames, non-linear approximation properties of such frames [1, 24], the convergence of the frame algorithm in Banach space norms, and unstructured wavelet frames [1], frames for α-modulation spaces[16, 20, 21], the density and excess of frames [2], and numerical analysis [6, 36].

An alternative and more general definition for the localization of a frame was proposed in [28]. It does not require the comparison to an auxiliary Riesz basis, and so this new notion of localization was called *intrinsic localization* or *self-localization* of a frame. A frame $\{g_n : n \in \mathbb{Z}^d\}$ is self-localized, if its Gramian matrix with entries $\langle g_n, g_m \rangle$ has a prescribed off-diagonal decay in $|m-n|$. Self-localization helped to give a partial solution of a deep conjecture of Feichtinger: if a frame satisfies the self-localization property $|\langle g_n, g_m\rangle| = \mathcal{O}(|m-n|^{-s}), m, n \in \mathbb{Z}^d$, for $s > d$, then it is a finite union of Riesz sequences [28].

The goal of this paper is to obtain a deeper understanding of self-localized frames. We will introduce and explore several new notions of localized frames. In particular, we will study the dual frame of a self-localized frame and the associated Banach spaces.

To give a flavor of our main result, we formulate the very special case of "polynomial self-localization".

Theorem 1.1. *Let $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\}$ be a frame for the Hilbert space $\mathcal{H}$ with canonical dual frame $\tilde{\mathcal{G}} = \{\tilde{g}_n : n \in \mathbb{Z}^d\}$. If $\mathcal{G}$ is self-localized in the sense that*

$$|\langle g_n, g_m\rangle| \le C(1+|m-n|)^{-s} \qquad m, n \in \mathbb{Z}^d \tag{1}$$

for some $s > d$, then the dual frame is also self-localized and satisfies

$$|\langle \tilde{g}_n, \tilde{g}_m\rangle| \le C'(1+|m-n|)^{-s} \qquad m, n \in \mathbb{Z}^d$$

and also

$$|\langle \tilde{g}_n, g_m\rangle| \le C''(1+|m-n|)^{-s} \qquad m, n \in \mathbb{Z}^d\,.$$

The proof of our main theorem uses a substantial amount of Banach algebra techniques. The key lies in the fact that matrices with polynomial decay off the diagonal as defined in (1) form a Banach algebra. Furthermore, by an important theorem of Jaffard [31] the inverse of a matrix satisfying condition (1) possesses the same off-diagonal decay. This property is called *inverse-closed* and plays a pivotal role in the recent development of the theory of localized frames [1, 30].

This structure lends itself to more general versions of self-localization and of Theorem 1.1. Our most general definition of self-localization requires that the Gramian with entries $\big(\langle g_n, g_m\rangle\big)_{m,n\in\mathbb{Z}^d}$ belongs to a suitable inverse-closed Banach algebra of matrices. The special case of the Sjöstrand algebra plays an important role in the treatise on the density of frames by Balan, Casazza, Heil, and Landau [2].

A relevant theme for our investigation is the interaction between frame theory and Banach algebra theory. While we are mostly interested in frame theory and its applications, some results on Banach algebras may be of independent interest. For example, as a step in the proof of Theorem 1.1, we show that an *inverse-closed*

Banach algebra of matrices is also pseudo-inverse closed, i.e., the Moore-Penrose pseudo-inverse of a matrix possesses the same off-diagonal decay. (See Theorem 3.5 for a precise technical formulation of this result).

Next we study a class of Banach spaces associated to a localized frame $\mathcal{G} = \{g_n : n \in \mathbb{Z}^d\}$. In interpolation theory [3] there are two principles to define abstract spaces: (a) in the *orbit method* a linear space is defined by all series expansions $f = \sum_n c_n g_n$ with coefficients in a given sequence space, (b) in the *coorbit method* we require the coefficient sequence $\langle f, g_n \rangle_{n\in\mathbb{Z}^d}$ to be in some sequence space. For general dictionaries, these classes of spaces may be distinct or not even well-defined, and pathologies may occur. For an intrinsically localized frame $\mathcal{G} = \{g_n\}$ with canonical dual frame $\tilde{\mathcal{G}} = \{\tilde{g}_n\}$ and a suitable class of weighted ℓ^p-spaces, we will prove that the following norms are equivalent and thus generate a well-defined Banach space associated to $\mathcal{G}$: $\|\langle f, g_n\rangle\|_{\ell^p_m}$, $\|\langle f, \tilde{g}_n\rangle\|_{\ell^p_m}$, $\inf\{\|c\|_{\ell^p_m} : f = \sum_n c_n g_n\}$, and $\inf\{\|c\|_{\ell^p_m} : f = \sum_n c_n \tilde{g}_n\}$. In a more technical language, self-localized frames can be extended to Banach frames for a class of associated orbit and coorbit spaces. The subtlety in the proofs lies in the interplay between the decay properties of the Gramian matrix of $\mathcal{G}$ and the growth of the admissible weight functions m.

In addition we will show that these Banach spaces are largely independent of the particular localized frame. We note that the self-localization of the frame is essential for these characterizations; without this hypothesis, there are counter-examples to most of our results [24].

Finally we study the concrete example of (non-uniform) Gabor frames and make explicit the abstract definitions. Even in this example we obtain hitherto unknown results.

As a possible next project one should investigate the relevance of localized frames in numerical analysis. A first attempt in this direction was the Stevenson's extension of the adaptive schemes for the solution of operator equations of Cohen, Dahmen, and DeVore [7, 8] from Riesz bases to frames [36]. We believe that the Banach frame property of localized frames is an important ingredient in this scheme and that it may be a good substitute for the classical Jackson-Bernstein inequalities.

The paper is organized as follows. In Section 2 we collect the relevant facts about Hilbert frames and Banach frames and introduce a natural class of Banach spaces associated to a frame. After a short review of certain Banach algebras of matrices, we define the main object of this paper, namely the concept of intrinsic localization. Section 3 is devoted to the main result: any intrinsically localized frame has intrinsically localized canonical dual. The major tools are Banach algebra techniques and some functional calculus. In Section 4 we show that (non-uniform) Gabor frames are Banach frames for modulation spaces. Our treatment is more transparent than the corresponding result in [27] that used a different concept of localization with respect to a Riesz basis. This section highlights the virtues of intrinsic localization versus localization with respect to a Riesz basis.

2. Intrinsically $\mathcal{A}$-localized frames and associated Banach spaces

We first collect the main concepts associated to frames and Banach algebras and introduce a general concept for the localization of frames.

2.1. **Frames for Hilbert Spaces and Banach Spaces.**

Our main goal is to understand how frames can be used to describe certain Banach spaces and obtain stable decompositions in these Banach spaces.

In the following we use $\mathcal{N}$ and $\mathcal{X} \subseteq \mathbb{R}^d$ as index sets for a frame. Indexing a function by x should indicated that f_x is essentially supported in a neighborhood of $x \in \mathbb{R}^d$. We assume that all index sets are relatively separated, this means that

$$\sup_{k\in\mathbb{Z}^d} \operatorname{card}\left(\mathcal{X} \cap (k+[0,1]^d)\right) := \nu < \infty .$$

Definition 1 ([29, 27]). A *Banach frame* for a separable Banach space B is a sequence $\mathcal{G} = \{g_n\}_{n\in\mathcal{N}}$ in B' with an associated sequence space B_d on $\mathcal{N}$ such that the following properties hold.

(a) The *coefficient operator* $C = C_{\mathcal{G}}$ defined by $Cf = C_{\mathcal{G}}f = (\langle f, g_n\rangle_{n\in\mathcal{N}})$ is bounded from B into B_d.

(b) Norm equivalence:
$$\|f\|_B \asymp \|\langle f, g_n\rangle_{n\in\mathcal{N}}\|_{B_d}. \tag{2}$$

(c) There exists a bounded operator R from B_d onto B, a so-called synthesis or *reconstruction operator*, such that
$$RC_{\mathcal{G}}f = R\left(\langle f, g_n\rangle_{n\in\mathcal{N}}\right) = f.$$

If $B = \mathcal{H}$ is a Hilbert space and $B_d = \ell^2(\mathcal{N})$, then the norm equivalence (2) coincides with the usual definition of a *frame for the Hilbert space* $\mathcal{H}$. For Hilbert frames a particular reconstruction operator can be described as follows [14]: let $D = D_{\mathcal{G}} = C^*_{\mathcal{G}}$ denote the synthesis operator defined by $Dc = \sum_n c_n g_n$, and $S = S_{\mathcal{G}}$ be the frame operator $S = DC = C^*C$. If $\mathcal{G}$ is a frame for $\mathcal{H}$, then S is a positive and boundedly invertible operator on $\mathcal{H}$ and every $f \in \mathcal{H}$ possesses the non-orthogonal expansions (frame reconstructions)

$$f = SS^{-1}f = \sum_n \langle f, S^{-1}g_n\rangle g_n = S^{-1}Sf = \sum_n \langle f, g_n\rangle S^{-1}g_n. \tag{3}$$

The set $\tilde{\mathcal{G}} = S^{-1}\mathcal{G}$ is again a frame for $\mathcal{H}$, the so-called *canonical dual frame*.

In general there exist many possible dual frames $\{\tilde{g}_n\}_{n\in\mathcal{N}} \subseteq \mathcal{H}$ such that

$$f = \sum_{n\in\mathcal{N}} \langle f, \tilde{g}_n\rangle g_n = \sum_{n\in\mathcal{N}} \langle f, g_n\rangle \tilde{g}_n = D_{\tilde{\mathcal{G}}}C_{\mathcal{G}}f \tag{4}$$

with the norm equivalence $\|f\|_{\mathcal{H}} \asymp \|\langle f, g_n\rangle_{n\in\mathcal{N}}\|_2$, but their properties and construction are less obvious.

Our main question concerns the relation between Hilbert and Banach frames and the validity of the frame expansion (3) in different norms. Since Hilbert space theory is substantially easier than Banach space theory, we could also ask under which conditions and for which spaces a Hilbert frame is also a Banach frame. This problem is at the origin of the theory of localized frames in [27] and has also motivated this paper.

2.2. **Gramian Matrices.** The focal point of our analysis of frames will be the *Gramian* matrix of a frame $\mathcal{G}$. Its Gramian is the $\mathcal{N} \times \mathcal{N}$-matrix $A = A(\mathcal{G}, \mathcal{G})$ with entries

$$A_{mn} = \langle g_n, g_m \rangle \qquad \text{for } m, n \in \mathcal{N}\,. \tag{5}$$

The Gramian acts on finite sequences by matrix multiplication, $(Ac)(m) = \sum_{n \in \mathcal{N}} \langle g_n, g_m \rangle c_n$ for $m \in \mathcal{N}$, so we can write the Gramian as

$$A(\mathcal{G}, \mathcal{G}) = C_{\mathcal{G}} D_{\mathcal{G}}\,. \tag{6}$$

For the comparison of two distinct frames $\mathcal{F} = \{f_x : x \in \mathcal{X}\}$ and $\mathcal{G} = \{g_n : n \in \mathcal{N}\}$ (living on different index sets) we will use the cross Gramian of $\mathcal{G}$ and $\mathcal{F}$ defined as

$$A(\mathcal{F}, \mathcal{G}) = C_{\mathcal{F}} D_{\mathcal{G}}\,. \tag{7}$$

This is the $\mathcal{X} \times \mathcal{N}$-matrix with entries $A_{xn} = \langle g_n, f_x \rangle$.

2.3. **Inverse-Closed Banach $*$-Algebras of Matrices.** In the following $\mathcal{A}$ is an involutive Banach algebra of infinite matrices indexed by $\mathcal{N} \subseteq \mathbb{R}^d$ satisfying the following properties:

(A0) $\mathcal{A} \subseteq \mathcal{B}(\ell^2(\mathcal{N}))$, i.e., each $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(\mathcal{N})$.

(A1) If $A \in \mathcal{A}$ is invertible on $\ell^2(\mathcal{N})$, then $A^{-1} \in \mathcal{A}$ as well. In the language of Banach algebras, $\mathcal{A}$ is called *inverse-closed* in $\mathcal{B}(\ell^2(\mathcal{N}))$; as a consequence, $\sigma_{\mathcal{A}}(A) = \sigma(A)$, i.e. the spectrum of A in the algebra $\mathcal{A}$ coincides with the spectrum of the operator A on $\ell^2(\mathcal{N})$.

(A2) If $A \in \mathcal{A}$ and $|b_{kl}| \le |a_{kl}|$ for all $k, l \in \mathcal{N}$, then $B \in \mathcal{A}$ and $\|B\|_{\mathcal{A}} \le \|A\|_{\mathcal{A}}$. We say that $\mathcal{A}$ is *solid*.

In the sequel we will call an involutive Banach algebra $\mathcal{A}$ satisfying properties (A0-2) a *solid spectral matrix algebra*, or for brevity simply a spectral algebra.

Examples 1. Recently a large class of matrix algebras has been proved to satisfy conditions (A0-2).

1. *The Jaffard class* is defined by polynomial decay off the diagonal. Let $\mathcal{A}_s$ be the class of matrices $A = (a_{kl}), k, l \in \mathcal{N}$, such that

$$|a_{kl}| \le C(1 + |k - l|)^{-s} \qquad \forall k, l \in \mathcal{N}. \tag{8}$$

By a Theorem of Jaffard [31] $\mathcal{A}_s$ satisfies the above conditions *provided that* $s > d$ (where $\mathcal{N} \subseteq \mathbb{R}^d$).

2. *Schur-type conditions.* The class $\mathcal{A}_v^1$ consists of all matrices $A = (a_{kl})_{k,l \in \mathcal{N}}$ such that

$$\sup_{k \in \mathcal{N}} \sum_{l \in \mathcal{N}} |a_{kl}|\, v(k - l) < \infty \quad \text{ and } \quad \sup_{l \in \mathcal{N}} \sum_{k \in \mathcal{N}} |a_{kl}|\, v(k - l) < \infty \tag{9}$$

with norm

$$\|A\|_{\mathcal{A}_v^1} = \max\{\sup_k \sum_{l \in \mathcal{N}} |a_{kl}| v(k - l)\,,\, \sup_l \sum_{k \in \mathcal{N}} |a_{kl}| v(k - l)\}\,. \tag{10}$$

If v is *submultiplicative*, i.e., $v(x+y) \leq v(x)v(y)$ and radial and satisfies (1) the growth condition $v(x) \geq C(1+|x|)^\delta$ for some $\delta \in (0,1]$, and (2) the so-called GRS-condition $\lim_{n\to\infty} v(nx)^{1/n} = 1$ for all $x \in \mathbb{R}^d$, then $\mathcal{A}_v^1$ satisfies conditions (A0-2) by [30, Sec. 3].

3. *More general off-diagonal decay.* Let $s > d$, u a submultiplicative weight on $\mathbb{R}^d$ and set $v(x) = u(x)w_s(x) = u(x)(1+|x|)^s$. We define the Banach space $\mathcal{A}_v$ by the norm

$$\|A\|_{\mathcal{A}_v} = \sup_{k,l\in\mathcal{N}} |a_{kl}|v(k-l)\,. \tag{11}$$

Then $\mathcal{A}_v$ is a Banach algebra. If u is radial and satisfies the GRS-condition, then conditions (A0-2) above are satisfied by [30, Sec. 4].

4. *The Sjöstrand class.* We define the class $\mathcal{C}$ as the space of all matrices $A = (a_{kl})_{k,l\in\mathbb{Z}^d}$ such that the norm

$$\|A\|_{\mathcal{C}} := \sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |a_{k,k-l}| \tag{12}$$

is finite. An alternative way to define the norm on $\mathcal{C}$ is

$$\|A\|_{\mathcal{C}} = \inf\{\|\alpha\|_{\ell^1} : |a_{kl}| \leq \alpha(k-l)\}\,.$$

By a result of Sjöstrand [35] this matrix algebra satisfies the basic conditions (A0-2). There are also weighted versions of $\mathcal{C}$ satisfying (A0-2).

Matrices belonging to one of the above algebras extend automatically to bounded operators on a large class of Banach spaces, in particular on certain weighted ℓ^p-spaces. By a weight function m we mean a positive, continuous function on $\mathbb{R}^d$. For technical reasons we always assume that m is v-moderate in the sense that $m(x+y) \leq Cv(x)m(y), \forall x,y \in \mathbb{R}^d$ for some submultiplicative weight v. See [15] and [26, Ch. 11.1].

For Jaffard's algebra $\mathcal{A}_r$ we quote the following statement.

Lemma 2.1. *Assume that $r > s+d$ and let $p_0 = \frac{d}{r-s} < 1$ be the critical index. If $A \in \mathcal{A}_r$, then A is bounded on every ℓ^p_m for $p_0 < p \leq \infty$ and every weight m satisfying the condition $m(x+y) \leq C(1+|x|)^s m(y), x,y \in \mathbb{R}^d$ (m is called s-moderate)* [1].

Similarly we can check the following boundedness properties:

Using a modification of Schur's test, it follows that every $A \in \mathcal{A}_v^1$ yields a bounded operator on ℓ^p_m whenever $1 \leq p \leq \infty$ and m is v-moderate.

In the third example, every $A \in \mathcal{A}_v$ extends to a bounded operator on ℓ^p_m for $1 \leq p \leq \infty$ and every u-moderate weight.

The unweighted Sjöstrand algebra is bounded only on the unweighted ℓ^p spaces for $1 \leq p \leq \infty$.

Definition 2. In view of these examples we call a weight m $\mathcal{A}$-*admissible*, if every $A \in \mathcal{A}$ can be extended to a bounded operator on all ℓ^p_m for $p \in (p_0, \infty] \cup \{1\}$ for some critical index $p_0 \leq 1$.

2.4. **$\mathcal{A}$-localized frames.** We now introduce several related concepts of localization for frames with respect to solid spectral matrix algebras.

Definition 3 (Self-Localization)**.** Let $\mathcal{A}$ be a solid spectral matrix algebra contained in $\mathcal{B}(\ell^2(\mathcal{N}))$ on some relatively separated index set $\mathcal{N} \subset \mathbb{R}^d$. A frame $\mathcal{G}$ is called $\mathcal{A}$-self-localized (or intrinsically $\mathcal{A}$-localized), if its Gramian matrix $A = A(\mathcal{G}, \mathcal{G})$ with entries $A(\mathcal{G}, \mathcal{G})_{mn} = \langle g_n, g_m \rangle, m, n \in \mathcal{N}$, belongs to $\mathcal{A}$. We write $\mathcal{G} \sim_{\mathcal{A}} \mathcal{G}$, if $\mathcal{G}$ is $\mathcal{A}$-self-localized.

If $\mathcal{F}$ and $\mathcal{G}$ are two frames indexed by $\mathcal{N}$, we write $\mathcal{F} \sim_{\mathcal{A}} \mathcal{G}$, if $A(\mathcal{F}, \mathcal{G}) \in \mathcal{A}$.

If we compare two frames which live on different index sets, we will compare them with respect to a solid spectral matrix algebra $\mathcal{A}$ on the index set $\mathbb{Z}^d$.

Definition 4. Given two frames $\mathcal{F} = \{f_x : x \in \mathcal{X}\}$ and $\mathcal{G} = \{g_n : n \in \mathcal{N}\}$, we define the matrix $A^\sharp = A^\sharp(\mathcal{F}, \mathcal{G})$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ with entries

$$(A^\sharp)_{kl} = \max_{x \in k+[0,1]^d} \max_{n \in l+[0,1]^d} |\langle g_n, f_x \rangle| \qquad k, l \in \mathbb{Z}^d . \tag{13}$$

We say that $\mathcal{F}$ is $\mathcal{A}^\sharp$-localized with respect to $\mathcal{G}$, whenever $A^\sharp(\mathcal{F}, \mathcal{G}) \in \mathcal{A}$. We write $\mathcal{F} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$ whenever $A^\sharp(\mathcal{F}, \mathcal{G}) \in \mathcal{A}$.

REMARKS: 1. The matrix algebras in Example 1 are well-defined on any relatively separated index set $\mathcal{X} \subseteq \mathbb{R}^d$. For these examples the concepts of "self-localization" $\mathcal{G} \sim_{\mathcal{A}} \mathcal{G}$ and "$\sharp$-self-localization" $\mathcal{G} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$ are equivalent. For example, let $\mathcal{A}_s(\mathcal{X})$ be Jaffard algebra defined by the decay condition (8) on the index set $\mathcal{X}$. Then $\mathcal{G} \overset{\sharp}{\sim}_{\mathcal{A}_s(\mathbb{Z}^d)} \mathcal{G}$ if and only if $\mathcal{G} \sim_{\mathcal{A}_s(\mathcal{X})} \mathcal{G}$.

Assume that $\mathcal{G} \overset{\sharp}{\sim}_{\mathcal{A}_s(\mathbb{Z}^d)} \mathcal{G}$, i.e., $A^\sharp(\mathcal{G}, \mathcal{G}) \in \mathcal{A}_s(\mathbb{Z}^d)$, and let $k, l \in \mathbb{Z}^d$ and $m \in \mathcal{X} \cap (k + [0,1]^d)$ and $n \in \mathcal{X} \cap (l + [0,1]^d)$. Then by (13)

$$|\langle g_n, g_m \rangle| \le (A^\sharp)_{kl} \le C(1 + |k - l|)^{-s} \le$$
$$\le (1 + |k - m|)^s (1 + |m - n|)^{-s} (1 + |n - l|)^s \le C'(1 + |m - n|)^{-s} ,$$

because $m(x) = (1+|x|)^{-s}$ is s-moderate. Consequently $A(\mathcal{G}, \mathcal{G}) \in \mathcal{A}_s(\mathcal{X})$ or $\mathcal{G} \sim_{\mathcal{A}_s(\mathcal{X})} \mathcal{G}$. The converse is shown similarly.

2. For a frame $\mathcal{F}$, a Riesz basis $\mathcal{G}$, and $A = \mathcal{A}_s$ (the Jaffard algebra) the relation $\mathcal{F} \sim_{\mathcal{A}} \mathcal{G}$ coincides with the original concept of frame localization introduced in [27]. Self-localization was considered in [27, 28, 20]. A weaker concept of localization, using even larger matrix algebras, was used by Balan, Casazza, Heil, and Landau [2] to define the density of frames and relate it to their excess.

The relation $\overset{\sharp}{\sim}_{\mathcal{A}}$ is not an equivalence relation, but it is close to being one. The following lemma shows what is true.

Lemma 2.2. *Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be frames for $\mathcal{H}$ and let $\tilde{\mathcal{F}}$ be an* arbitrary *dual frame of $\mathcal{F}$.*

(a) *If $\mathcal{E} \overset{\sharp}{\sim}_{\mathcal{A}} \tilde{\mathcal{F}}$ and $\mathcal{F} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$, then $\mathcal{E} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$.*

(b) *If $\mathcal{F} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{F}$ and $\tilde{\mathcal{F}} \overset{\sharp}{\sim}_{\mathcal{A}} \tilde{\mathcal{F}}$, then $\mathcal{F} \overset{\sharp}{\sim}_{\mathcal{A}} \tilde{\mathcal{F}}$.*

(c) *If both* $\tilde{\mathcal{F}}_1$ *and* $\tilde{\mathcal{F}}_2$ *are dual frames of* $\mathcal{F}$, $\mathcal{F} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{F}$, *and* $\tilde{\mathcal{F}}_1 \overset{\sharp}{\sim}_{\mathcal{A}} \tilde{\mathcal{F}}_2$, *then* $\mathcal{F} \overset{\sharp}{\sim}_{\mathcal{A}} \tilde{\mathcal{F}}_j$ *for* $j = 1, 2$.

Proof. We show (a), items (b) and (c) are proven similarly. Since $(\mathcal{F}, \tilde{\mathcal{F}})$ is a pair of dual frames, the identity on $\mathcal{H}$ factors as $D_{\tilde{\mathcal{F}}} C_{\mathcal{F}} = \mathrm{I}_{\mathcal{H}}$ by (4). The definition of the cross Gramian (7) then yields the factorization

$$A(\mathcal{E}, \mathcal{G}) = C_{\mathcal{E}} D_{\mathcal{G}} = C_{\mathcal{E}} D_{\tilde{\mathcal{F}}} C_{\mathcal{F}} D_{\mathcal{G}} = A(\mathcal{E}, \tilde{\mathcal{F}}) A(\mathcal{F}, \mathcal{G}) .$$

By taking suprema over cubes, we obtain the entrywise inequality

$$A^{\sharp}(\mathcal{E}, \mathcal{G}) \leq A^{\sharp}(\mathcal{E}, \tilde{\mathcal{F}}) A^{\sharp}(\mathcal{F}, \mathcal{G}) .$$

By hypothesis $A^{\sharp}(\mathcal{E}, \tilde{\mathcal{F}}) \in \mathcal{A}$ and $A^{\sharp}(\mathcal{F}, \mathcal{G}) \in \mathcal{A}$, and thus using the solidity (A1) and the algebra property of $\mathcal{A}$, we have $A^{\sharp}(\mathcal{E}, \mathcal{G}) \in \mathcal{A}$. ■

2.5. **Associated Quasi-Banach spaces.** Motivated by the definition of "associated Banach spaces" in [13, 20, 27], we next introduce a class of spaces attached to every frame $\mathcal{G}$.

Definition 5. Let $\mathcal{G}$ be a frame for $\mathcal{H}$ and $\tilde{\mathcal{G}}$ a dual frame (not necessarily the canonical dual frame). Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be the subspace spanned by the finite linear combinations of elements in $\mathcal{G}$. For $0 < p < \infty$ and m a weight function we define a (quasi-)norm on $\mathcal{H}_0$ by

$$\|f\|_{\mathcal{H}^p_m} = \|(\langle f, \tilde{g}_n \rangle)_{n \in \mathcal{N}}\|_{\ell^p_m} = \|C_{\tilde{\mathcal{G}}} f\|_{\ell^p_m} . \tag{14}$$

Then the space $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ is defined as the norm completion of $\mathcal{H}_0$ with respect to this norm $\| \cdot \|_{\mathcal{H}^p_m}$ for $p < \infty$. For $p = \infty$ we define $\mathcal{H}^\infty_m$ as the completion of $\mathcal{H}_0$ in the $\sigma(\mathcal{H}, \mathcal{H}_0)$-topology, and $\mathcal{H}^0_m$ as the norm completion of $\mathcal{H}_0$ in the $\| \cdot \|_{\mathcal{H}^\infty_m}$-norm.

By definition, $\mathcal{H}^p_m$ is a Banach space for $1 \leq p \leq \infty$ or $p = 0$ and a quasi-Banach space for $0 < p < 1$, and $\mathcal{H}^p_m$ contains $\mathcal{H}_0$ as a dense subspace for $p < \infty$ (a weak*-dense subspace for $p = \infty$).

REMARKS: 1. This definition is clean, but cumbersome. Technically, the elements of $\mathcal{H}^p_m$ are equivalence classes of Cauchy sequences of elements in $\mathcal{H}_0$. In the following we provide several equivalent, but more natural, definitions. To avoid tedious technicalities, we will omit the rigorous discussion of $\mathcal{H}^\infty_m$ and only deal with $\mathcal{H}^0_m$.

2. If $\mathcal{G}$ is a Riesz basis for $\mathcal{H}$, then the theory of these spaces is straightforward and $\mathcal{H}^p_m \simeq \ell^p_m(\mathcal{N})$. For arbitrary pairs of dual frames this definition may be problematic and pathologies may occur. For instance, it may happen that $\mathcal{H}^p_m = \{0\}$ or that $\mathcal{H}^p_m$ does not even contain the finite linear combinations of the form $\sum_n c_n g_n$.

3. The definition of $\mathcal{H}^p_m$ seems to depend on the particular choice of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. For a generic frame $\mathcal{G}$, it can even happen that $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}}) \neq \mathcal{H}^p_m(\tilde{\mathcal{G}}, \mathcal{G})$ [24, Section 3.3].

Our goal is to show that, for reasonable frames, $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ is a well-defined Banach space and independent of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. In our context "reasonable" means that $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are suitably localized.

Proposition 2.3. *Let $\mathcal{A}$ be a solid spectral Banach algebra of matrices on $\mathcal{N}$ (with critical index p_0 and the class of $\mathcal{A}$-admissible weight functions). Assume that $(\mathcal{G},\tilde{\mathcal{G}})$ is a pair of dual frames for $\mathcal{H}$, such that $\tilde{\mathcal{G}} \sim_{\mathcal{A}} \mathcal{G}$. If $\ell^p_m(\mathcal{N}) \subseteq \ell^2(\mathcal{N})$, then $\mathcal{H}^p_m$ can be identified with the subspace*

$$\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}}) \simeq \{f \in \mathcal{H} : \quad f = \sum_{n\in\mathcal{N}} \langle f, \tilde{g}_n\rangle g_n, \quad (\langle f, \tilde{g}_n\rangle)_{n\in\mathcal{N}} \in \ell^p_m(\mathcal{N})\}\,. \tag{15}$$

Proof. Let $\tilde{f} = (f_k)_{k\in\mathbb{N}} \in \mathcal{H}^p_m$, where $(f_k) \subseteq \mathcal{H}_0$ is a Cauchy sequence in the $\|\cdot\|_{\mathcal{H}^p_m}$-norm. Since ℓ^p_m is continuously embedded in ℓ^2 by assumptions, and $\tilde{\mathcal{G}}$ is a frame, we estimate that

$$\|f_k - f_l\|_{\mathcal{H}} \le k_1 \|C_{\tilde{\mathcal{G}}}(f_k - f_l)\|_2 \le k_2 \|C_{\tilde{\mathcal{G}}}(f_k - f_l)\|_{\ell^p_m} = k_2 \|f_k - f_l\|_{\mathcal{H}^p_m}\,.$$

Thus (f_k) is a Cauchy sequence in $\mathcal{H}$ and possess a limit $f \in \mathcal{H}$. Note that f depends only on the equivalence class of $\tilde{f}$, but not on the specific Cauchy sequence (f_k).

Since $\big(C_{\tilde{\mathcal{G}}} f_k\big)$ is a Cauchy sequence in ℓ^p_m, it has a limit $c \in \ell^p_m$. On the other hand, $C_{\tilde{\mathcal{G}}} f_k \longrightarrow C_{\tilde{\mathcal{G}}} f$ in ℓ^2, therefore we conclude that $C_{\tilde{\mathcal{G}}} f = c \in \ell^p_m$. We may now identify $\tilde{f}$ with $f \in \mathcal{H}$ and thus $\mathcal{H}^p_m$ is a well-defined subspace of $\mathcal{H}$. ■

Here is a characterization that works for the full range of $\mathcal{H}^p_m, 1 \le p \le \infty$.

Proposition 2.4. *Under the same assumptions on $\mathcal{A}$ and $\mathcal{G}$ as in Proposition 2.3 the synthesis operator $D_{\mathcal{G}}$ is bounded from $\ell^p_m(\mathcal{N})$* onto *$\mathcal{H}^p_m$ for every $p_0 < p \le \infty$ and $\mathcal{A}$-admissible weight m. Consequently the following two norms are equivalent:*

$$\|f\|_{\mathcal{H}^p_m} \asymp \inf\{\|c\|_{\ell^p_m} : c \in \ell^p_m, \quad f = D_{\mathcal{G}} c\}\,.$$

Proof. Let $f = \sum_{n\in F} c_n g_n = D_{\mathcal{G}} c \in \mathcal{H}_0$ where $F \subseteq \mathcal{N}$ is finite. Then by (7) we have

$$C_{\tilde{\mathcal{G}}} f = C_{\tilde{\mathcal{G}}} D_{\mathcal{G}} c = A(\mathcal{G},\tilde{\mathcal{G}}) c\,.$$

Since by assumption the cross Gramian $A(\mathcal{G},\tilde{\mathcal{G}})$ is bounded on $\ell^p_m(\mathcal{N})$, we find that

$$\|f\|_{\mathcal{H}^p_m} = \|C_{\tilde{\mathcal{G}}} f\|_{\ell^p_m} = \|A(\mathcal{G},\tilde{\mathcal{G}})c\|_{\ell^p_m} \le k\|c\|_{\ell^p_m}\,. \tag{16}$$

For $p_0 < p < \infty$ and $p = 0$, this inequality extends by density to all $c \in \ell^p_m$.

Now assume that $\tilde{f} \in \mathcal{H}^p_m$, this means that $\tilde{f} = (f_k)_{k\in\mathbb{N}}$ for some Cauchy sequence $(f_k) \subseteq \mathcal{H}_0$ in the $\mathcal{H}^p_m$-norm. Then $(C_{\tilde{\mathcal{G}}} f_k)$ is a Cauchy sequence in ℓ^p_m and thus has a limit $c \in \ell^p_m$. Set $f = D_{\mathcal{G}} c$; $f \in \mathcal{H}^p_m$ by the first part of the proof and

$$\|f - f_k\|_{\mathcal{H}^p_m} = \|C_{\tilde{\mathcal{G}}}(f - f_k)\|_{\ell^p_m} = \|c - C_{\tilde{\mathcal{G}}} f_k\|_{\ell^p_m} \longrightarrow 0\,.$$

Consequently $\tilde{f} \simeq f = D_{\mathcal{G}} c \in \mathcal{H}^p_m$ and $D_{\mathcal{G}}$ is onto $\mathcal{H}^p_m$. If $\ell^p_m \subseteq \ell^2$, then the surjectivity of $D_{\mathcal{G}}$ follows also from Proposition 2.3.

The norm equivalence is now standard. Let $\mathcal{M} = \ker D_{\mathcal{G}} \subseteq \ell^p_m$. Since $D_{\mathcal{G}}$ is onto, it is an isomorphism from $\ell^p_m/\mathcal{M}$ onto $\mathcal{H}^p_m$. By the inverse mapping theorem we have for every $f = D_{\mathcal{G}} c \in \mathcal{H}^p_m$ that

$$\|f\|_{\mathcal{H}^p_m} \asymp \|c + \mathcal{M}\|_{\ell^p_m/\mathcal{M}} = \inf\{\|c\|_{\ell^p_m} : c \in \ell^p_m, \quad f = D_{\mathcal{G}} c\}\,.$$

■

REMARK: Proposition 2.4 is reminiscent of the situation in Hilbert spaces. While $\mathcal{H}$ is determined by frame expansions $\sum_{n\in\mathcal{N}} c_n g_n$ with ℓ^2-coefficients, $\mathcal{H}^p_m$ is determined by ℓ^p_m-coefficients. In $\mathcal{H}$ the coefficients of minimum norm are uniquely determined by the canonical dual frame as $\langle f, \tilde{g}_n\rangle$ [14], in $\mathcal{H}^p_m$ we can assert — at least under suitable conditions — that a coefficient sequence with almost minimum norm is given by *some* dual frame.

A slightly more concrete description of $\mathcal{H}^p_m$ for the case $\ell^p_m \not\subseteq \ell^2$ can be obtained from the following observation. Inclusion properties of the associated Banach spaces follow from the corresponding inclusions of the sequence spaces. If m is v-moderate, then it is easy to see that $\ell^1_v \subseteq \ell^p_m \subseteq \ell^0_{1/v} = \{c \in \ell^\infty_{1/v} : \lim_{|n|\to\infty} |c_n| m(n) = 0\}$ for $1 \le p < \infty$. Therefore $\mathcal{H}^p_m \subseteq \mathcal{H}^0_{1/v}$ and we can define $\mathcal{H}^p_m$ as

$$\mathcal{H}^p_m = \{f \in \mathcal{H}^0_{1/v} : C_{\tilde{\mathcal{G}}} f \in \ell^p_m(\mathcal{N})\}\,.$$

The space $\mathcal{H}^0_{1/v}$ serves as a kind of distribution space attached to the frame $\mathcal{G}$ (which is always an abstract object). Thus $\mathcal{H}^p_m$ can be interpreted as a subspace of this large space of distributions. This procedure mimics the standard procedure for defining function spaces.

Finally we investigate the dependence of $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ of the frame $\mathcal{G}$. We need a small technical lemma first.

Lemma 2.5. *Given a sequence $c = (c_n)$ and a matrix $A = (A_{mn})$ on the (relatively separated) index set $\mathcal{N} \subseteq \mathbb{R}^d$, let $c^\sharp$ and $A^\sharp$ be defined on $\mathbb{Z}^d$ $c^\sharp_k = \max_{n\in\mathcal{N}\cap(k+[0,1]^d)} |c_n|$ and $(A^\sharp)_{kl} = \max\{|A_{mn}| : m \in \mathcal{N}\cap(k+[0,1]^d), n \in \mathcal{N}\cap(l+[0,1]^d)\}$. Then*

(a) $(Ac)^\sharp \le A^\sharp c^\sharp$ (as a pointwise inequality), and

(b) $c \in \ell^p_m(\mathcal{N})$ if and only if $c^\sharp \in \ell^p_m(\mathbb{Z}^d)$.

Proof. (a) is easy and omitted. For (b) we use the inequalities $C^{-1}m(k) \le m(k+u) \le Cm(k)$ for all $u \in [0,1]^d$ because m is assumed to be v-moderate. Therefore we have

$$\begin{aligned}
\|c\|^p_{\ell^p_m(\mathcal{N})} &= \sum_{n\in\mathcal{N}} |c_n|^p m(n)^p = \sum_{k\in\mathbb{Z}^d} \sum_{n\in k+[0,1]^d} |c_n|^p m(n)^p \\
&\asymp \sum_{k\in\mathbb{Z}^d} \left(\sum_{n\in k+[0,1]^d} |c_n|^p \right) m(k)^p \\
&\asymp \sum_{k\in\mathbb{Z}^d} \left(\max_{n\in k+[0,1]^d} |c_n| \right)^p m(k)^p = \|c^\sharp\|^p_{\ell^p_m(\mathbb{Z}^d)}\,,
\end{aligned}$$

where we have also used that $\mathcal{N}$ is relatively separated. ■

Proposition 2.6. *Let $\mathcal{A}$ be a solid spectral Banach algebra of matrices on $\mathcal{N}$ (with critical index p_0 and the class of $\mathcal{A}$-admissible weight functions). Assume that $(\mathcal{G}, \tilde{\mathcal{G}})$ and $(\mathcal{F}, \tilde{\mathcal{F}})$ are pairs of dual frames for $\mathcal{H}$.*

If $\tilde{\mathcal{G}} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$, $\tilde{\mathcal{F}} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$ and $\tilde{\mathcal{G}} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{F}$, then $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}}) = \mathcal{H}^p_m(\mathcal{F}, \tilde{\mathcal{F}})$ with equivalent norms.

Proof. The proof is similar to the argument of Proposition 2.4. So we only sketch it. First note that Lemma 2.2 implies that $\tilde{\mathcal{F}} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{F}$. Next, by using Lemma 2.5 in formula (16), one shows that $D_{\mathcal{F}}$ is bounded. Thus $\mathcal{H}_m^p(\mathcal{F}, \tilde{\mathcal{F}})$ is well-defined.

Writing the frame expansion of f as $f = D_{\mathcal{G}} C_{\tilde{\mathcal{G}}} f$, we find that the coefficients with respect to $\tilde{\mathcal{F}}$ are given by

$$C_{\tilde{\mathcal{F}}} f = C_{\tilde{\mathcal{F}}} D_{\mathcal{G}} C_{\tilde{\mathcal{G}}} f = A(\tilde{\mathcal{F}}, \mathcal{G}) C_{\tilde{\mathcal{G}}} f \,,$$

and by Lemma 2.5(a)

$$(C_{\tilde{\mathcal{F}}} f)^\sharp \le A(\tilde{\mathcal{F}}, \mathcal{G})^\sharp (C_{\tilde{\mathcal{G}}} f)^\sharp \,.$$

If $f \in \mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$, then $(C_{\tilde{\mathcal{G}}} f)^\sharp \in \ell_m^p(\mathbb{Z}^d)$ by Lemma 2.5(b). Since $\tilde{\mathcal{F}} \overset{\sharp}{\sim}_{\mathcal{A}} \mathcal{G}$, $A^\sharp(\tilde{\mathcal{F}}, \mathcal{G})$ is bounded on $\ell_m^p(\mathbb{Z}^d)$, and we conclude that

$$\|C_{\tilde{\mathcal{F}}} f\|_{\ell_m^p(\mathcal{N})} \asymp \|(C_{\tilde{\mathcal{F}}} f)^\sharp\|_{\ell_m^p(\mathbb{Z}^d)} \le C \|(C_{\tilde{\mathcal{G}}} f)^\sharp\|_{\ell_m^p(\mathbb{Z}^d)} \,.$$

This means that $f \in \mathcal{H}_m^p(\mathcal{F}, \tilde{\mathcal{F}})$ and thus $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}}) \subseteq \mathcal{H}_m^p(\mathcal{F}, \tilde{\mathcal{F}})$. By reversing the roles of $\mathcal{F}$ and $\mathcal{G}$ we find that $\mathcal{H}_m^p(\mathcal{F}, \tilde{\mathcal{F}}) \subseteq \mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$ and that the norms are equivalent. ■

We can now formulate the definitive theorem about the Banach spaces associated to suitably localized frames. The following theorem extends the structure from the theory of function spaces [17, 18, 22, 29] to the abstract setting of frames expansions.

Theorem 2.7. *Let $\mathcal{A}$ be a solid spectral Banach algebra of matrices on $\mathcal{N}$ (with critical index p_0 and the class of $\mathcal{A}$-admissible weight functions). Assume that $(\mathcal{G}, \tilde{\mathcal{G}})$ is a pair of dual frames and $\tilde{\mathcal{G}} \sim_{\mathcal{A}} \mathcal{G}$.*

Then $\tilde{\mathcal{G}}$ is a Banach frame for $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$ for $1 \le p < \infty$ or $p = 0$. The reconstruction operator from the frame coefficients is given by the standard frame expansion

$$f = \sum_{n \in \mathcal{N}} \langle f, \tilde{g}_n \rangle g_n$$

with unconditional convergence in $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$. The reconstruction also works in the case that $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$ is a quasi-Banach space, i.e., for $p_0 < p < 1$.

Proof. We check the properties postulated in Definition 1. Since $\|f\|_{\mathcal{H}_m^p} = \|C_{\tilde{\mathcal{G}}} f\|_{\ell_m^p}$, properties (a) and (b) are obvious, and the sequence space corresponding to $\mathcal{H}_m^p$ is just $\ell_m^p(\mathcal{N})$. The reconstruction operator is just the synthesis operator $D_{\mathcal{G}}$ and thus the reconstruction is simply the frame expansion. To see this, we note that the identity $f = D_{\mathcal{G}} C_{\tilde{\mathcal{G}}} f = \sum_{n \in \mathcal{N}} \langle f, \tilde{g}_n \rangle g_n$ holds for $f \in \mathcal{H}_0$ because $\mathcal{G}$ is a frame. This identity extends to all of $\mathcal{H}_m^p$ (for $p < \infty$ or $p = 0$) by density. If $f \in \mathcal{H}_m^p$, i.e., $C_{\tilde{\mathcal{G}}} f \in \ell_m^p$, the frame expansion converges unconditionally in the norm of $\mathcal{H}_m^p$ as an easy consequence of estimate (16). So we have proved that $\tilde{\mathcal{G}}$ is a Banach frame for $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$. ■

3. Intrinsically Localized Duals

We have seen that the Banach spaces $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ are meaningful and that they coincide for a large class of frames $\mathcal{G}$. However, in our arguments we have made heavy use of the fact that for the given frame $\mathcal{G}$ there exists some dual frame $\tilde{\mathcal{G}}$ satisfying $\tilde{\mathcal{G}} \sim_{\mathcal{A}} \mathcal{G}$. In this section we show that the theory of Banach frame expansions developed above is non-void. Our main result shows that the localization properties of a frame are preserved by its *canonical* dual frame.

3.1. **Intrinsically $\mathcal{A}$-localized canonical dual frame.** Let $\mathcal{G} = \{g_n\}_{n\in\mathcal{N}}$ be a frame for $\mathcal{H}$ with frame operator S and let A be its Gram matrix with entries $A_{mn} = \langle g_n, g_m\rangle$ for $m, n \in \mathcal{N}$. Since $A = C_{\mathcal{G}} D_{\mathcal{G}}$ by (6), A acts as a bounded operator on $\ell^2(\mathcal{N})$. Since $AC = CDC = CS$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{S} & \mathcal{H} \\ \downarrow C & & \downarrow C \\ \operatorname{ran} C & \xrightarrow{A} & \operatorname{ran} C \end{array} \tag{17}$$

We summarize the properties of A, they are just convenient reformulations of the hypothesis that $\mathcal{G}$ is a frame.

(a) $\ker A = \ker D = (\operatorname{ran} C)^\perp$.
(b) $\operatorname{ran} A = \operatorname{ran} C = (\ker D)^\perp$, in particular, the range of A is closed.
(c) Since $A = CD = CC^*$ and $S = DC = C^*C$, we have $\sigma(A) \cup \{0\} = \sigma(S) \cup \{0\} \subseteq \{0\} \cup [\alpha, \beta]$, see [9]. Thus except for a nontrivial kernel, the Gramian A and the frame operator S have the same spectrum.
(d) By (a) and (b) the restriction of A to $\operatorname{ran} C$ is bijective from $\operatorname{ran} C$ onto $\operatorname{ran} C$. Therefore there exists an operator $B : \operatorname{ran} C \longrightarrow \operatorname{ran} C$, such that $BA = AB = \mathrm{I}_{\operatorname{ran} C}$. Let $A^\dagger$ be the trivial extension of B from $\operatorname{ran} C$ to $\mathcal{H}$, i.e., $A^\dagger h = Bh$ for $h \in \operatorname{ran} C$ and $A^\dagger h = 0$ for $h \in \operatorname{ran} C$. Writing P for the orthogonal projection onto $\operatorname{ran} C$, we have constructed a bounded operator $A^\dagger$, satisfying the relation

$$A^\dagger A = AA^\dagger = P \tag{18}$$

In linear algebra $A^\dagger$ is called the (Moore-Penrose) pseudo-inverse of A.

By construction we have $A^\dagger C = CS^{-1}$ (note that we only need to know $A^\dagger$ on $\operatorname{ran} C$). Written as a commutative diagram we have:

$$\begin{array}{ccc} \mathcal{H} & \xleftarrow{S^{-1}} & \mathcal{H} \\ \downarrow C & & \downarrow C \\ \operatorname{ran} C & \xleftarrow{B} & \operatorname{ran} C \\ \downarrow & & \downarrow \\ \ell^2 & \xleftarrow{A^\dagger} & \ell^2 \end{array} \tag{19}$$

We next express the (cross)-Gramian $A(\tilde{\mathcal{G}},\mathcal{G})$ of $\tilde{\mathcal{G}}$ with respect to $\mathcal{G}$ and the Gramian $A(\tilde{\mathcal{G}},\tilde{\mathcal{G}})$ of $\tilde{\mathcal{G}}$. (This is essentially how frame theory can be formulated in the language of linear algebra, see [5]).

Lemma 3.1. *Let $\mathcal{G}$ be a frame with Gramian $A = A(\mathcal{G},\mathcal{G})$ and pseudo-inverse $A^\dagger$. Then*

$$A(\mathcal{G},\tilde{\mathcal{G}}) = A^\dagger A \tag{20}$$

$$A(\tilde{\mathcal{G}},\tilde{\mathcal{G}}) = (A^\dagger)^2 A \tag{21}$$

Proof. The proof follows by interpretation of the above diagram ($m,n \in \mathcal{N}$):

$$\begin{aligned} A(\mathcal{G},\tilde{\mathcal{G}})_{mn} &= \langle \tilde{g_n}, g_m\rangle = \langle S^{-1}g_n, g_m\rangle \\ &= (CS^{-1}g_n)(m) = (A^\dagger C g_n)(m) \\ &= \sum_{l\in\mathcal{N}} (A^\dagger)_{ml}(Cg_n)(l) = \sum_{l\in\mathcal{N}} (A^\dagger)_{ml}\langle g_n, g_l\rangle \\ &= \sum_{l\in\mathcal{N}} (A^\dagger)_{ml} A_{ln} = (A^\dagger A)_{mn}\,. \end{aligned}$$

Likewise, we have

$$\begin{aligned} A(\tilde{\mathcal{G}},\tilde{\mathcal{G}})_{mn} &= \langle \tilde{g_n}, \tilde{g_m}\rangle = \langle S^{-1}g_n, S^{-1}g_m\rangle \\ &= \langle S^{-2}g_n, g_m\rangle = (CS^{-2}g_n)(m) \\ &= \big((A^\dagger)^2 C g_n\big)(m) = \big((A^\dagger)^2 A\big)_{mn}\,. \end{aligned}$$

■

REMARK: If $\mathcal{G}$ is a Riesz basis then its Gramian A is invertible by definition. If in addition $\mathcal{G}$ is $\mathcal{A}$-self-localized, then by assumption we have $A^{-1} \in \mathcal{A}$. Lemma 3.1 implies immediately that the dual basis $\tilde{\mathcal{G}}$ is $\mathcal{A}$-self-localized and also that $\tilde{\mathcal{G}} \sim_{\mathcal{A}} \mathcal{G}$.

The analogous result for self-localized frames is more difficult to show. Essentially we need to show that a solid spectral matrix algebra is "pseudo-inverse closed". Not surprisingly, the relevant tools to show this fact are Banach algebra techniques. We will use a method of Hulanicki [32] that has become the main tool for dealing with inverse-closed and symmetric Banach algebras; see [19] for further applications.

We denote the spectrum of an algebra element $A \in \mathcal{A}$ by $\sigma_{\mathcal{A}}(A)$ and its spectral radius by $r_{\mathcal{A}}(A)$. As before, we suppress the index when the algebra is $\mathcal{B}(\mathcal{H})$ and simply write $\sigma(A)$ for $\sigma_{\mathcal{B}(\mathcal{H})}(A)$.

Proposition 3.2 (Hulanicki)**.** *Let $\mathcal{S}$ be a (not necessarily closed) $*$-subalgebra of an involutive Banach algebra $\mathcal{A}$. Suppose that there exists a faithful $*$-representation $(\pi,\mathcal{H})$ of $\mathcal{A}$ by bounded operators on a Hilbert space $\mathcal{H}$ such that for all $f = f^* \in \mathcal{S}$*

$$\|\pi(f)\|_{op} = \lim_{n\to\infty} \|f^n\|_{\mathcal{A}}^{1/n} = r_{\mathcal{A}}(f)\,. \tag{22}$$

If $\mathcal{A}$ has an identity e, we also assume that $\pi(e) = \mathrm{I}_{\mathcal{H}}$. Then for each $f = f^* \in \mathcal{S}$ we have

$$\sigma_{\mathcal{A}}(f) = \sigma(\pi(f))\,. \tag{23}$$

For the proof see Hulanicki [32] or [19]. We also need the following lemma from [19] which guarantees the existence of an identity in $\mathcal{A}$.

Lemma 3.3. *Let $\mathcal{C}$ be the $\| \quad \|_{\mathcal{A}}$-closure of some commutative $*$-subalgebra of $\mathcal{A}$. If $\mathrm{I}_{\mathcal{H}}$ is in the operator norm closure of the image of $\mathcal{C}$ under π, then there is some $e \in \mathcal{C}$ with $\pi(e) = \mathrm{I}_{\mathcal{H}}$. It follows that the entire algebra $\mathcal{A}$ also has a unit, namely e.*

We can now prove our main theorem.

Theorem 3.4. *Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ with orthogonal projection P onto $\mathcal{M}$. Assume that $A = A^* \in \mathcal{A}$, $\ker A = \mathcal{M}^\perp$ and that $A : \mathcal{M} \longrightarrow \mathcal{M}$ is invertible. Then the pseudoinverse $A^\dagger$, i.e., the unique element in $\mathcal{B}(\mathcal{H})$ satisfying $A^\dagger A = AA^\dagger = P$ and $\ker A^\dagger = \mathcal{M}^\perp$, is an element of $\mathcal{A}$. In particular $P \in \mathcal{A}$.*

Proof. **Step 1.** We define the subalgebra $\mathcal{S}$ of $\mathcal{A}$ as follows:

$$\mathcal{S} = \{B \in \mathcal{A} : \ker B \supseteq \mathcal{M}^\perp, B : \mathcal{M} \to \mathcal{M}\} = \{B \in \mathcal{A} : B = PBP\} \tag{24}$$

with norm $\|B\|_{\mathcal{S}} = \|B\|_{\mathcal{A}}$. Note that $\mathcal{S}$ is an involutive algebra. Since by hypothesis $\mathcal{S}$ contains A, $\mathcal{S}$ is non-trivial.

Next define a mapping $\pi : \mathcal{S} \longrightarrow \mathcal{B}(\mathcal{M})$ by

$$\pi(B) = B|_{\mathcal{M}} \,. \tag{25}$$

Then π is a $*$-representation of $\mathcal{S}$. If $\pi(B_1) = \pi(B_2)$, then $B_1|_{\mathcal{M}} = B_2|_{\mathcal{M}}$ and $B_1|_{\mathcal{M}^\perp} = B_2|_{\mathcal{M}^\perp} = 0$, so $B_1 = B_2$, and π is faithful.

Step 2. We claim that $\mathcal{S}$ has an identity element E and that E coincides with the projection P. Let $A = A^* \in \mathcal{S}$ as stated in the assumption, and let $\mathcal{C}$ be the closed commutative $*$-subalgebra of $\mathcal{S}$ generated by A. Then $\overline{\pi(\mathcal{C})}$ (with the closure in the operator norm of $\mathcal{B}(\mathcal{M})$) is the C^*-algebra generated by $A|_{\mathcal{M}}$. Since $A|_{\mathcal{M}}$ is invertible in $\mathcal{B}(\mathcal{M})$, $\overline{\pi(\mathcal{C})}$ contains both $(A|_{\mathcal{M}})^{-1}$ and also $\mathrm{I}_{\mathcal{M}}$. The technical Lemma 3.3 implies that $\mathcal{S}$ contains an identity E and that $\pi(E) = \mathrm{I}_{\mathcal{M}}$. By definition of $\mathcal{S}$ this means that $E|_{\mathcal{M}} = \mathrm{I}_{\mathcal{M}}$ and $E|_{\mathcal{M}^\perp} = 0$, in other words, $E = P$ and $P \in \mathcal{S}$.

Step 3. We now apply Hulanicki's Proposition 3.2 to the algebra $\mathcal{S}$ and the representation π. For this we need to verify an identity of spectral radii. By hypothesis, $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\mathcal{H})$, so we know that for all $B = B^* \in \mathcal{A}$

$$\|B\|_{op} = r_{\mathcal{A}}(B) = \lim_{n\to\infty} \|B^n\|_{\mathcal{A}}^{1/n} \,.$$

Since $\mathcal{S}$ and $\mathcal{A}$ have the same norm, we have $r_{\mathcal{S}}(B) = r_{\mathcal{A}}(B)$ for all $B = B^* \in \mathcal{S}$. Since $\ker B \supseteq \mathcal{M}^\perp$ for $B \in \mathcal{S}$, we have $\|\pi(B)\|_{\mathcal{B}(\mathcal{M})} = \|B|_{\mathcal{M}}\|_{\mathcal{B}(\mathcal{M})} = \|B\|_{op}$. Combining these estimates, we have shown that

$$r_{\mathcal{S}}(B) = \|\pi(B)\|_{\mathcal{B}(\mathcal{M})} \qquad \text{for all} \quad B = B^* \in \mathcal{S} \,. \tag{26}$$

By Hulanicki's Proposition 3.2 we conclude that

$$\sigma_{\mathcal{S}}(B) = \sigma(\pi(B)) = \sigma(B|_{\mathcal{M}}) \qquad \forall B = B^* \in \mathcal{S} \,.$$

Step 4. For the special element $A \in \mathcal{S}$ we have $0 \notin \sigma_{\mathcal{B}(\mathcal{M})}(A|_{\mathcal{M}}) = \sigma_{\mathcal{S}}(A)$ and therefore there exists a $A^+ \in \mathcal{S}$ such that $A^+ A = AA^+ = P$. Since $A^+ : \mathcal{M} \to \mathcal{M}$ and $\ker A^+ = \mathcal{M}^\perp$, A^+ must be the Moore-Penrose pseudo-inverse $A^\dagger$ of A. ∎

Our main theorem can also be formulated as follows.

Corollary 3.5. *Assume that $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\mathcal{H})$. If $A \in \mathcal{A}$ has a (Moore-Penrose) pseudo-inverse $A^\dagger$, then also $A^\dagger \in \mathcal{A}$. In other words, if $\mathcal{A}$ is inverse-closed, then it is "pseudo-inverse closed".*

REMARK: An alternative proof can be based on the Riesz functional calculus for Banach algebras as developed in Conway [9], Ch. VII.4. Since $\mathcal{A}$ is inverse-closed in $\mathcal{B}(\mathcal{H})$, the resolvent function $z \longrightarrow (z\mathrm{I} - A)^{-1} \in \mathcal{A}$ is the same on $\mathcal{A}$ as on $\mathcal{B}(\mathcal{H})$ whenever $z \notin \sigma_{\mathcal{B}(\mathcal{H})}(A) = \sigma_{\mathcal{A}}(A)$. This implies that the contour integrals occurring in the Riesz functional calculus make sense simultaneously in $\mathcal{B}(\mathcal{H})$ and in $\mathcal{A}$, and they always define an element in $\mathcal{A}$. It is then easy to see that the projection P and $A^\dagger$ can be defined by functional calculus as follows:

$$P = \mathrm{I} - \frac{1}{2\pi i}\int_\gamma (z\mathrm{I} - A)^{-1}\, dz \tag{27}$$

$$A^\dagger = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\,(z\mathrm{I} - A)^{-1}\, dz\,, \tag{28}$$

where γ is a suitable small circle centered at 0 and Γ is a suitable contour of $\sigma(A)\setminus\{0\}$. These formulas make sense in $\mathcal{A}$ and thus furnish an alternative proof of Theorem 3.4. We can now prove the self-localization of canonical dual frames.

Theorem 3.6. *Assume that $\mathcal{A}$ is a solid spectral matrix algebra. If $\mathcal{G}$ is an $\mathcal{A}$-self-localized frame, then its canonical dual frame $\tilde{\mathcal{G}}$ is also $\mathcal{A}$-self-localized. Likewise $\tilde{\mathcal{G}} \sim_{\mathcal{A}} \mathcal{G}$.*

Proof. For the proof we combine Lemma 3.1 and Theorem 3.4. As a consequence of the frame property, the Gramian A of $\mathcal{G}$ has a pseudo-inverse $A^\dagger$, and by Theorem 3.4 we have $A^\dagger \in \mathcal{A}$.

Now Lemma 3.1 and the algebra property of $\mathcal{A}$ imply that the Gramian of $\tilde{\mathcal{G}}$ is $A(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}) = (A^\dagger)^2 A \in \mathcal{A}$. Similarly $A(\mathcal{G}, \tilde{\mathcal{G}}) = A^\dagger A \in \mathcal{A}$. This means that $\tilde{\mathcal{G}}$ is $\mathcal{A}$-self-localized and that $\tilde{\mathcal{G}} \sim_{\mathcal{A}} \mathcal{G}$. ∎

We formulate the abstract localization theorem explicitly for the most important examples of solid spectral matrix algebras. For simplicity and concreteness we take the weight function to be of the form

$$v(x) = (1 + |x|)^s\, e^{A|x|^\alpha} \log^t(e + |x|)$$

for some parameters $A, s, t \geq 0$, and $0 \leq \alpha < 1$.

Corollary 3.7. *Let $\mathcal{G}$ be a frame for $\mathcal{H}$.*

(a) Let $s > d$. If the Gramian of $\mathcal{G}$ satisfies the condition

$$|\langle g_n, g_m\rangle| \leq Cv(n-m)^{-1}\,, \qquad \forall m, n \in \mathcal{N}$$

then the Gramian of $\tilde{\mathcal{G}}$ also satisfies

$$|\langle \tilde{g}_n, \tilde{g}_m\rangle| \leq C'v(m-n)^{-1} \qquad \forall m, n \in \mathcal{N}$$

and

$$|\langle \tilde{g}_n, g_m \rangle| \le C' v(m-n)^{-1} \qquad \forall m, n \in \mathcal{N}\,.$$

(b) Let either $s > 0$ or if $s = 0$, then $A, \alpha > 0$. If the Gramian of $\mathcal{G}$ satisfies the condition

$$\sup_{m\in\mathcal{N}} \sum_{n\in\mathcal{N}} |\langle g_n, g_m \rangle| v(n-m) < \infty\,,$$

then we also have

$$\sup_{m\in\mathcal{N}} \sum_{n\in\mathcal{N}} |\langle \tilde{g}_n, \tilde{g}_m \rangle| v(m-n) < \infty$$

and

$$\sup_{m\in\mathcal{N}} \sum_{n\in\mathcal{N}} |\langle \tilde{g}_n, g_m \rangle| v(m-n) < \infty\,.$$

(c) If the Gramian of $\mathcal{G}$ satisfies the condition

$$\sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |\langle g_k, g_{k-l} \rangle| < \infty$$

then $\tilde{\mathcal{G}}$ also satisfies

$$\sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |\langle g_k, \tilde{g}_{k-l} \rangle| < \infty$$

and

$$\sum_{l\in\mathbb{Z}^d} \sup_{k\in\mathbb{Z}^d} |\langle \tilde{g}_k, g_{k-l} \rangle| < \infty\,.$$

The proof follows from the fact that the matrix algebras defined in Section 2 all satisfy the required conditions.
REMARK: Statement (c) has been conjectured in an early version of [2]. While writing this paper, Balan and Heil informed us that they also have proved this special case.

Our final theorem complements and refines Theorem 2.7. Since we need Theorem 3.4, it is substantially deeper than the corresponding statement in Theorem 2.7 which follows directly from the definitions.

Theorem 3.8. *Let $\mathcal{A}$ be a solid spectral Banach algebra of matrices on $\mathcal{N}$ (with critical index p_0 and the class of $\mathcal{A}$-admissible weight functions). Assume that $(\mathcal{G}, \tilde{\mathcal{G}})$ is a pair of canonical dual frames and $\mathcal{G} \sim_{\mathcal{A}} \mathcal{G}$. Then*

$$\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}}) = \mathcal{H}^p_m(\tilde{\mathcal{G}}, \mathcal{G})$$

and $\mathcal{G}$ is a Banach frame for $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ for $1 \le p < \infty$ or $p = 0$. The reconstruction operator from the frame coefficients is given by the frame expansion

$$f = \sum_{n\in\mathcal{N}} \langle f, g_n \rangle \tilde{g}_n$$

with unconditional convergence in $\mathcal{H}^p_m(\mathcal{G}, \tilde{\mathcal{G}})$ for $p_0 < p < \infty$.

Proof. On the one hand, the Gramian $A(\mathcal{G},\mathcal{G})$ is in $\mathcal{A}$ by hypothesis, and thus bounded on every admissible ℓ^p_m. On the other hand, $A(\tilde{\mathcal{G}},\tilde{\mathcal{G}})$ is in $\mathcal{A}$ by Theorem 3.6, and again $A(\tilde{\mathcal{G}},\tilde{\mathcal{G}})$ is bounded on every admissible ℓ^p_m.

To prove the identity of the spaces, we use the boundedness of the Gramians and the following factorizations:

$$\begin{aligned} C_{\mathcal{G}} &= C_{\mathcal{G}} D_{\mathcal{G}} C_{\tilde{\mathcal{G}}} = A(\mathcal{G},\mathcal{G}) C_{\tilde{\mathcal{G}}} \\ C_{\tilde{\mathcal{G}}} &= C_{\tilde{\mathcal{G}}} D_{\tilde{\mathcal{G}}} C_{\mathcal{G}} = A(\tilde{\mathcal{G}},\tilde{\mathcal{G}}) C_{\mathcal{G}} \,. \end{aligned}$$

Now assume that $f \in \mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$, i.e., $C_{\tilde{\mathcal{G}}} f \in \ell^p_m$. Then the first factorization implies that

$$\|C_{\mathcal{G}} f\|_{\ell^p_m} \le \|A(\mathcal{G},\mathcal{G})\|_{op} \|C_{\tilde{\mathcal{G}}} f\|_{\ell^p_m} = C_1 \|f\|_{\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})} \,.$$

In other words, we have the inclusion $\mathcal{H}^p_m(\tilde{\mathcal{G}},\mathcal{G}) \subseteq \mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$.

For the converse, assume that $f \in \mathcal{H}^p_m(\tilde{\mathcal{G}},\mathcal{G})$, i.e., $C_{\mathcal{G}} f \in \ell^p_m$. Then the second factorization implies that

$$\|C_{\tilde{\mathcal{G}}} f\|_{\ell^p_m} \le \|A(\tilde{\mathcal{G}},\tilde{\mathcal{G}})\|_{op} \|C_{\mathcal{G}} f\|_{\ell^p_m} = C_2 \|f\|_{\mathcal{H}^p_m(\tilde{\mathcal{G}},\mathcal{G})} \,.$$

Thus $\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}}) \subseteq \mathcal{H}^p_m(\tilde{\mathcal{G}},\mathcal{G})$, and we also proved the norm equivalence. The remaining assertions follow from Theorem 2.7 by interchanging the roles of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. ■

4. Applications to Non-Uniform Gabor Frames

In this section we study the self-localization of Gabor frames. We will formulate explicit conditions for the self-localization of Gabor frames, and we will identify the abstract Banach spaces $\mathcal{H}^p_m$ with concrete function spaces that are well known in time-frequency analysis. As always in time-frequency analysis, the appropriate function spaces are the so-called modulation spaces.

Using the notation of [26], we write

$$T_x f(t) = f(t-x) \text{ and } M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t), \tag{29}$$

for translation and modulation operators and

$$\pi(\lambda) = M_\omega T_x \quad \text{for } \lambda = (x,\omega) \in \mathbb{R}^{2d} \tag{30}$$

for a time-frequency shift.

We consider frames of time-frequency shifts. Let $\mathcal{X}$ be a relatively separated set in the time-frequency plane $\mathbb{R}^{2d}$ and let $g \in L^2(\mathbb{R}^d)$ a fixed non-zero window function. A frame for $L^2(\mathbb{R}^d)$ of the form $\mathcal{G} = \mathcal{G}(g,\mathcal{X}) = \{\pi(\lambda)g : \lambda \in \mathcal{X}\}$ is called a *Gabor frame*. Often one distinguishes uniform Gabor frames, when $\mathcal{X}$ is a lattice, and *non-uniform* or irregular Gabor frames, when $\mathcal{X}$ does not have any structure.

The modulation spaces are defined by means of the short-time Fourier transform. The short-time Fourier transform of a function or distribution f with respect to a window g_0 (for example, the Gaussian $g_0(t) = e^{-\pi|t|^2}$) is defined by

$$V_{g_0} f(\lambda) = \langle f, \pi(\lambda) g\rangle = \int_{\mathbb{R}^d} f(t) \overline{g_0(t-x)} e^{-2\pi i \omega \cdot t} dt. \tag{31}$$

Definition 6. Fix a non-zero C^∞-function g_0 with compact support. For $0 < p, q \leq \infty$ and a weight function m the modulation space $M_m^{p,q}(\mathbb{R}^d)$ is defined as the subspace of all distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ for which the norm

$$\|f|M_m^{p,q}\| := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_{g_0}f(x,\omega)|^p m(x,\omega)^p dx\right)^{q/p} d\omega\right)^{1/q}$$

is finite (with the usual modifications when $pq = \infty$). We denote $M_m^p = M_m^{p,p}$. If $1 \leq p, q \leq \infty$ and m satisfies the condition $m(x+y) \leq C(1+|x|)^s m(y), x, y \in \mathbb{R}^{2d}$ for some $s \geq 0$, then $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space of tempered distributions, for $p, q < 1$ $M_m^{p,q}$ is a quasi-Banach space. The definition does not depend on the particular choice of g_0, and different windows $g_0 \in M_{w_s}^1$ yield equivalent norms [26, Thm. 11.3.7] and [23]. For faster growing weights, $M_m^{p,q}$ may be defined as a subspace of ultradistributions [26, 11.4].

REMARK: If $g \in M_{w_\gamma}^\infty$, for $w_\gamma(\lambda) = (1+|\lambda|)^\gamma$ and $\gamma > 0$, then

$$|\langle\pi(\mu)g, \pi(\lambda)g\rangle| = |V_g(g)(\lambda-\mu)| \leq C\big(1+|\lambda-\mu|\big)^{-\gamma}.$$

Thus if $\mathcal{G} = \{\pi(\lambda)g : \lambda \in \mathcal{X}\}$ is a frame and $g \in M_{w_\gamma}^\infty$, then $\mathcal{G}$ is $\mathcal{A}_\gamma$-self-localized, and by a previous remark also $\mathcal{A}_\gamma^\sharp$-self-localized.

Theorem 4.1. *Assume that $g \in M_{w_\gamma}^\infty \setminus \{0\}$ for some $\gamma = s + 2d + \epsilon > 2d$ and that $\mathcal{G} = \{\pi(\lambda)g : \lambda \in \mathcal{X}\}$ is a frame for $L^2(\mathbb{R}^d)$. Then the following properties hold true:*

(a) Both $\mathcal{G}$ and its canonical *dual frame $\tilde{\mathcal{G}} = \{e_\lambda : \lambda \in \mathcal{X}\}$ are $\mathcal{A}_\gamma$-self-localized, explicitly,*

$$|\langle e_\lambda, e_\mu\rangle| \leq C\big(1+|\lambda-\mu|\big)^{-\gamma} \qquad \forall\lambda,\mu \in \mathcal{X}. \tag{32}$$

(b) The spaces $\mathcal{H}_m^p(\mathcal{G}, \tilde{\mathcal{G}})$ and $M_m^p(\mathbb{R}^d)$ coincide and have equivalent norms, whenever $p_0 = \frac{2d}{2d+\epsilon} < p \leq \infty$ and m is s-moderate. Furthermore, $\mathcal{G}$ is a Banach frame for M_m^p, and the reconstruction operator R in M_m^p coincides with S^{-1}.

Proof. (a) is simply Corollary 3.7 formulated for Gabor frames and so $A(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}) \in \mathcal{A}_\gamma$. Thus the hypotheses of Lemma 2.1 are satisfied and the Gramians $A(\mathcal{G}, \mathcal{G})$ and $A(\tilde{\mathcal{G}}, \tilde{\mathcal{G}})$ are bounded on every ℓ_m^p for $p_0 = \frac{2d}{2d+\epsilon} < p \leq \infty$ and every s-moderate weight m.

To prove (b), we use the boundedness of the analysis and synthesis operators on modulation spaces.

(i) If $g \in M_{w_s}^1$ and $f \in M_m^p$, then $C_{\mathcal{G}}f = \{V_g f(\lambda) : \lambda \in \mathcal{X}\} \in \ell_m^p(\mathcal{X})$ and

$$\|C_{\mathcal{G}}f\|_{\ell_m^p} \leq C\|f\|_{M_m^p}. \tag{33}$$

(ii) If $g \in M_{w_s}^1$ and $c \in \ell_m^p(\mathcal{X})$, then $D_{\mathcal{G}}c = \sum_{\lambda\in\mathcal{X}} c_\lambda \pi(\lambda)g \in M_m^p$ and

$$\|D_{\mathcal{G}}c\|_{M_m^p} \leq C\|c\|_{\ell_m^p}. \tag{34}$$

These statements have been proved in various degrees of generality, see [26], Thm. 12.2.1 and Prop. 11.1.4, or as a special case of [Thm. 6.1(ii)][17]. Since for $\gamma = s + 2d + \epsilon$ we have the inclusion $M_{w_\gamma}^\infty \subseteq M_{w_s}^1$, the estimates (33) and (34) hold for $\mathcal{A}_\gamma$-localized Gabor frames.

We now show the equality of $\mathcal{H}^p_m$ and M^p_m. Assume first that $f \in \mathcal{H}^p_m$, i.e. $(\langle f, e_\lambda\rangle) \in \ell^p_m$. By Proposition 2.4, $f = \sum_{\lambda\in\mathcal{X}} c_\lambda \pi(\lambda) g$ for some $c \in \ell^p_m$, and by (34) therefore $f \in M^p_m$ with the estimate

$$\|f\|_{M^p_m} \leq C \inf\{\|c\|_{\ell^p_m} : f = Dc\} \asymp \|f\|_{\mathcal{H}^p_m}\,.$$

Consequently $\mathcal{H}^p_m \subseteq M^p_m$.

Conversely, assume that $f \in M^p_m$. Writing the frame expansion of the canonical dual frame $e_\lambda = \sum_{\mu\in\mathcal{X}} \langle e_\lambda, e_\mu\rangle \pi(\mu) g$, we find that $\langle f, e_\lambda\rangle = \sum_{\mu\in\mathcal{X}} \langle e_\lambda, e_\mu\rangle \langle f, \pi(\mu) g\rangle$. or in matrix notation

$$C_{\tilde{\mathcal{G}}} f = A(\tilde{\mathcal{G}}, \tilde{\mathcal{G}}) C_{\mathcal{G}} f\,.$$

By (32) the Gramian $A(\tilde{\mathcal{G}}, \tilde{\mathcal{G}})$ is in $\mathcal{A}_\gamma$ and therefore it is bounded on ℓ^p_m for $p_0 < p \leq \infty$ and every s-moderate weight m. If $f \in M^p_m$, then by (33) above, $C_{\mathcal{G}} f \in \ell^p_m$ and so $C_{\tilde{\mathcal{G}}} f \in \ell^p_m$ as well. By definition, this means that $f \in \mathcal{H}^p_m$.

We have shown that $M^p_m = \mathcal{H}^p_m$. The norm equivalence follows from the inverse mapping theorem. The remaining statement were proved in Theorem 2.7 ■

REMARKS: 1. For uniform Gabor frames, Theorem 4.1 was proved in [26, Ch. 13.5.3].

2. A slightly different formulation of Theorem 4.1 for non-uniform Gabor frames was proved in [27, Theorem 5.2] via orthonormal Wilson bases. The concept of self-localization makes the proof of the theorem much more transparent and avoids some technical complications.

3. Statements similar to Theorem 4.1 can also be obtained for the other matrix algebras discussed in Section 2.3. The modifications are left to the reader. Likewise, one could make more explicit the localization properties of other examples of frames. For some frames of interest this will be done in subsequent work.

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