# BANACH FRAMES FOR $\alpha$-MODULATION SPACES

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Abstract. This paper is concerned with the characterization of $\alpha$-modulation spaces by Banach frames, i.e., stable and redundant non-orthogonal expansions, constituted of functions obtained by a suitable combination of translation, modulation and dilation of a mother atom. In particular, the parameter $\alpha \in [0,1]$ governs the dependence of the dilation factor on the frequency. The result is achieved by exploiting intrinsic properties of localization of such frames. The well-known Gabor and wavelet frames arise as special cases ($\alpha = 0$) and limiting case ($\alpha \to 1$), to characterize respectively modulation and Besov spaces. This intermediate theory contributes to a further answer to the theoretical need of a common interpretation and framework between Gabor and wavelet theory and to the construction of new tools for applications in time-frequency analysis, signal processing, and numerical analysis.

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## 1. Introduction

The theory of *frames*, or stable redundant non-orthogonal expansions in Hilbert spaces, introduced by Duffin and Schaeffer [15], plays an important role in *wavelet theory* [11, 12, 13] as well as in *Gabor (time-frequency) analysis* [34, 24, 25] for functions in $L^2(\mathbb{R}^d)$. Besides traditional and relevant applications of frames in signal processing, image processing, data compression, pattern matching, sampling theory, communication and data transmission, recently the use of frames also in numerical analysis for the solution of operator equation is investigated [47, 8]. Therefore, not only the characterization by frames of functions in $L^2(\mathbb{R}^d)$ is relevant but also that of (smoothness) Banach function spaces is crucial to have a correct formulation of effective and stable numerical schemes. The concept of *Banach frame* as an extension of atomic decompositions in *coorbit spaces* [20, 21] has been already introduced in [33]. Moreover this classical theory of Feichtinger and Gröchenig has shown in particular that Gabor and wavelet $L^2$-frames can in fact extend to Banach frames for *modulation* [17, 34, 35, 30] and *(homogeneous) Besov spaces* [31, 50, 51] respectively. As a further answer to the theoretical need of a common interpretation and framework between Gabor and wavelet theory, the author has recently proposed [18] the construction of frames, which allows to ensure that certain families of Schwartz functions (atoms) on $\mathbb{R}$ obtained by a suitable combination of translation, modulation and dilation

$$\begin{aligned} &T_x(f)(t) = f(t-x), \\ &M_\omega(f)(t) = e^{2\pi i\omega\cdot t} f(t), \\ &D_a(f)(t) = |a|^{-1/2} f(t/a), \quad x,\omega,t \in \mathbb{R}, a \in \mathbb{R}_+, \end{aligned}$$

form Banach frames for the family of $L^2$-Sobolev spaces of any order. In this construction a parameter $\alpha \in [0,1)$ governs the dependence of the dilation factor on the frequency parameter. The well-known Gabor and wavelet frames (also valid for the same scale of Hilbert spaces that constitutes an intersection of the modulation and Besov space families) arise as special cases ($\alpha = 0$) and limiting case ($\alpha \to 1$) respectively. Thus, let us call these families $\alpha$-Gabor-wavelet frames. In contrast to those limiting cases it is no longer possible to use group theoretic arguments nor the coorbit space theory can be applied anymore to extend the $L^2$-frame to a Banach frame. A similar approach was proposed by Hogan and Lakey [39] to construct *coherent frames* generated by representations of extensions of the Heisenberg group by dilation. Other contributions due to Weiss *et al.* [37, 38, 43] developed characterizations of a large class of mixed decompositions in $L^2$ as an attempt of a unified approach to Gabor, wavelet, and more general wave packet frames.

New tools for extending an $L^2$-frame to Banach frames have been introduced by Gröchenig. The key concept in [35] is the localization properties of the frame with respect to an auxiliary Riesz basis. The localization has been measured by polynomial or sub-exponential off-diagonal decay of the cross Gramian matrix of the frame and the Riesz basis. The main result in [35] asserts that a localized frame has canonical dual with the same localization properties and that the frame extends to a Banach frame for the Banach spaces for which the reference auxiliary Riesz basis is a unconditional basis. Inspired by this work, the author [29, Chapter 5] showed that the extension of a frame to Banach frames does not depend on localization properties with respect to any auxiliary Riesz basis, but it can be formulated also as an intrinsic property of the frame. In particular, if the frame is *intrinsically or self-localized*, i.e. if its Gramian matrix has a suitable off-diagonal decay, and there exists a corresponding *dual frame* with the same property then the frame extends in fact to a Banach frame for a suitable class of Banach spaces. Based on a rather tricky and technical construction of an intrinsically localized dual frame, this principle has been applied in [29, Chapter 5] to extend $\alpha$-Gabor-wavelet $L^2$-frames to atomic decompositions for $\alpha$-*modulation spaces.* These Banach (smoothness) function spaces have been introduced independently by Gröbner [32] and Paivärinta/Somersalo [46] as an "intermediate" family between modulation and Besov spaces. They appears also as particular cases of the spaces introduced by Holschneider and Nazaret in [41, Section 4.2], and Hogan and Lakey in [40, Section 4.5], by retract or pull back methods based on generalized *Fourier-Bros-Iagolnitzer transforms* [5] (or *flexible Gabor-wavelet transforms* as they are called in [18, 29]). Characterizations of $\alpha$-modulation spaces by *brushlet unconditional basis* have been given by Nielsen and Borup [45] and the mapping properties of pseudodifferential operators in Hörmander classes on $\alpha$-modulation spaces have been studied by Holschneider and Nazaret [41] and Borup [4], as generalizations of classical results of Cordoba and Fefferman [7].

In this paper we shall present a Banach frame characterization of $\alpha$-modulation spaces, following and generalizing the intrinsic localization strategy already proposed in [29, Chapter 5]. We will show that the argument can be simplified, in particular avoiding the construction of a specific and suitable auxiliary intrinsically localized dual, by recent results by Gröchenig and the author [30]: in fact any intrinsically localized frame has the *canonical dual* endowed with the same localization property. This principle, already known for frames localized to Riesz bases [35], was conjectured in [29, Remark 5.3.8] and then shown in a particular case independently also by Balan *et al.* in [3].

The paper is organized as follows. Section 2 recalls the concept of frames in Hilbert and Banach spaces. In particular, the intrinsic localization of frame theory is discussed as a method to extend frames in Hilbert spaces to Banach frames, by illustrating the relevant results in [30]. In Section 3 we present $\alpha$-modulation spaces as a generalization of modulation and inhomogeneous Besov spaces and the localization principles applied to $\alpha$-Gabor-wavelet frames to characterize them. We conclude with few remarks and a characterization of $\alpha$-modulation spaces by pull back of certain weighted $L^{p,q}$ spaces (mixed norm Lebesgue space) by the *flexible Gabor-wavelet transform* introduced in [41, 18, 29].

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1.1. **Notations.** We denote with $L^p(\mathbb{R}^d)$ the Lebesgue space of measurable functions on $\mathbb{R}^d$ that are $p$-integrable and with $L^p_m(\mathbb{R}^d)$ the Lebesgue space of measurable functions $f$ such that $fm \in L^p(\mathbb{R}^d)$. Similarly are defined the spaces $\ell^p_m(\mathbb{Z}^d)$ of weighted $p$-summable sequences. The space $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz functions and its dual $\mathcal{S}'(\mathbb{R}^d)$ is the space of temperate distributions. We denote with $\mathcal{F}$ the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$. For positive quantities $F$ and $G$, we will write $F \lesssim G$ whenever $F(x) \leq C \cdot G(x)$ for some universal constant $C > 0$ and for all variable $x$. When $F \lesssim G$ and $G \lesssim F$ then we will write $F \asymp G$.

## 2. Intrinsically localized frames in Banach spaces

2.1. **Frames in Hilbert and Banach spaces.** In this section we recall the concept of frames, how they can be used to define certain associated Banach spaces, and how to obtain stable decompositions in these Banach spaces.

A subset $\mathcal{G} = \{g_n\}_{n \in \mathbb{Z}^d}$ of a separable Hilbert space $\mathcal{H}$ is called *frame* for $\mathcal{H}$ if

$$A\|f\|^2 \leq \sum_{n \in \mathbb{Z}^d} |\langle f, g_n \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{H}, \tag{1}$$

for some constants $0 < A \leq B < \infty$.

Equivalently, we could define a frame by the requirement that the corresponding *analysis operator* $C = C_{\mathcal{G}}$ defined by $Cf = (\langle f, g_n \rangle)_n$ is bounded from $\mathcal{H} \to \ell^2(\mathbb{Z}^d)$ or that the *synthesis operator* $D = D_{\mathcal{G}} = C^*, D\mathbf{c} = \sum_n c_n g_n$, is bounded from $\ell^2(\mathbb{Z}^d) \to \mathcal{H}$. This means that frame operator $S = DC$ is a boundedly invertible (positive and self-adjoint) on $\mathcal{H}$. The set $\tilde{\mathcal{G}} = S^{-1}\mathcal{G}$ is again a frame for $\mathcal{H}$. This so-called *canonical dual frame* plays an important role in the reconstruction of $f \in \mathcal{H}$ from the frame coefficients and in non-orthogonal expansions, because we have

$$f = SS^{-1}f = \sum_n \langle f, S^{-1}g_n \rangle g_n = S^{-1}Sf = \sum_n \langle f, g_n \rangle S^{-1} g_n. \tag{2}$$

Since in general a frame is overcomplete, the coefficients in this expansion are in general not unique (unless $\mathcal{G}$ is a Riesz basis, we have $\ker(D) \neq \{0\}$) and there exist many possible dual frames $\{\tilde{g}_n\}_{n \in \mathbb{Z}^d}$ in $\mathcal{H}$ such that

$$f = \sum_{n \in \mathbb{Z}^d} \langle f, \tilde{g}_n \rangle g_n$$

with the norm equivalence $\|f\|_{\mathcal{H}} \asymp \|\langle f, g_n \rangle\|_2$. More information on frames can be found in the book [6]. The concept of frame can be extended to Banach spaces as follows:

**Definition 1** (Gröchenig [33, 35]). A *Banach frame* for a separable Banach space $B$ is a sequence $\mathcal{G} = \{g_n\}_{n \in \mathbb{Z}^d}$ in $B'$ with an associated sequence space $B_d$ such that the following properties hold.

(a) The *coefficient operator* $C$ defined by $Cf = (\langle f, g_n \rangle_{n \in \mathbb{Z}^d})$ is bounded from $B$ into $B_d$.

(b) Norm equivalence:

$$\|f\|_B \asymp \|\langle f, g_n \rangle_{n \in \mathbb{Z}^d}\|_{B_d}.$$

(c) There exists a bounded operator $R$ from $B_d$ onto $B$, a so-called *synthesis or reconstruction operator*, such that

$$R\left(\langle f, g_n \rangle_{n \in \mathbb{Z}^d}\right) = f.$$

In the following we discuss under which (sufficient) conditions and for which Banach spaces a Hilbert frame is also a Banach frame. This problem has motivated the theory of localized frames recently introduced by Gröchenig [35, 36, 2, 30].

2.2. **Intrinsic localization of frames.** We want to recall here the concept of mutual localization of two frames measured by their (cross-)Gramian matrix belonging to a Banach $*$-algebra $\mathcal{A}$ of matrices which is inverse-closed in $\mathcal{B}(\ell^2)$. The theory of localized frames with respect to an algebra has been introduced in [35, 36] and developed in [30]. In particular in [30] it has been shown that a localized frame can extend to a Banach frame in a natural way for a large family of Banach spaces together with its canonical dual.

In this paper we shall work with the *Jaffard algebra* [42] which is defined as the class of matrices $A = (a_{kl}), k, l \in \mathbb{Z}^d$, such that

$$|a_{kl}| \lesssim (1 + |k - l|)^{-s} \qquad \forall k, l \in \mathbb{Z}^d, \quad s > d.$$

Let us denote the Jaffard algebra $\mathcal{A} := \mathcal{A}_s$ and one can show [42, 35] that

(A0) $\mathcal{A} \subseteq \mathcal{B}(\ell^2(\mathbb{Z}^d))$, i.e., each $A \in \mathcal{A}$ defines a bounded operator on $\ell^2(\mathbb{Z}^d)$.
(A1) If $A \in \mathcal{A}$ is invertible on $\ell^2(\mathbb{Z}^d)$, then $A^{-1} \in \mathcal{A}$ as well. In the language of Banach algebras, $\mathcal{A}$ is called inverse-closed in $\mathcal{B}(\ell^2(\mathbb{Z}^d))$;
(A2) $\mathcal{A}$ is solid: i.e., if $A \in \mathcal{A}$ and $|b_{kl}| \leq |a_{kl}|$ for all $k, l \in \mathbb{Z}^d$, then $B \in \mathcal{A}$ as well.

We refer to [36] for further information where a characterization of a large class of algebras with properties (A0-2) is presented. Let us denote $w_s(x) = (1 + |x|)^s$, for $s > d$, the polynomially growing submultiplicative and radial symmetric weight function on $\mathbb{R}^d$. A weight $m$ on $\mathbb{R}^d$ is called $s$-moderate if $m(x + y) \leq w_s(x)m(y)$. In particular, if $m$ is $s$-moderate then $m^{-1}$ is also $s$-moderate and $m(x) \lesssim w_s(x)$ for all $x \in \mathbb{R}^d$. By Schur test, any $A \in \mathcal{A}$ extends to a bounded operator from $\ell^p_m$ to $\ell^p_m$, for $1 \leq p \leq \infty$ (see also [35, Lemma 2.3]). By means of the algebra $\mathcal{A}$, we can now state the general localization principle.

Given two frames $\mathcal{G} = \{g_n\}_{n \in \mathbb{Z}^d}$ and $\mathcal{F} = \{f_x\}_{x \in \mathbb{Z}^d}$ for the Hilbert space $\mathcal{H}$, the (cross-) Gramian matrix $A = A(\mathcal{G}, \mathcal{F})$ of $\mathcal{G}$ with respect to $\mathcal{F}$ is the $\mathbb{Z}^d \times \mathbb{Z}^d$-matrix with entries

$$a_{nx} = \langle g_n, f_x \rangle.$$

A frame $\mathcal{G}$ for $\mathcal{H}$ is called $\mathcal{A}$-*localized* with respect to another frame $\mathcal{F}$ if $A(\mathcal{G}, \mathcal{F}) \in \mathcal{A}$. In this case we write $\mathcal{G} \sim_{\mathcal{A}} \mathcal{F}$. If $\mathcal{G} \sim_{\mathcal{A}} \mathcal{G}$, then $\mathcal{G}$ is called $\mathcal{A}$-*self-localized* or *intrinsically* $\mathcal{A}$-*localized.*

**Theorem 2.1** (Fornasier, Gröchenig [30]). *Any $\mathcal{A}$-self-localized frame $\mathcal{G}$ has always $\mathcal{A}$-self-localized canonical dual.*

2.3. **Associated Banach Spaces.** In this subsection, we want to show that $\mathcal{A}$-self-localized frames can characterize suitable families of Banach spaces in a natural way. In the following we assume $s > d$ and $m$ is an $s$-moderate weight.

Let $(\mathcal{G},\tilde{\mathcal{G}})$ be a pair of dual $\mathcal{A}$-self-localized frames for $\mathcal{H}$. Assume $\ell^p_m(\mathbb{Z}^d)\subset\ell^2(\mathbb{Z}^d)$. Then the Banach space $\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$ is defined to be

$$\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}}) := \{f\in\mathcal{H}: \quad f=\sum_{n\in\mathbb{Z}^d}\langle f,\tilde g_n\rangle g_n, \quad (\langle f,\tilde g_n\rangle)_{n\in\mathbb{Z}^d}\in\ell^p_m(\mathbb{Z}^d)\} \tag{3}$$

with the norm $\|f\|_{\mathcal{H}^p_m}=\|(\langle f,\tilde g_n\rangle)_{n\in\mathbb{Z}^d}\|_{\ell^p_m}$ and $1\le p\le\infty$. Since $\ell^p_m(\mathbb{Z}^d)\subset\ell^2(\mathbb{Z}^d)$, $\mathcal{H}^p_m$ is a dense subspace of $\mathcal{H}$. If $\ell^p_m(\mathbb{Z}^d)$ is not included in $\ell^2(\mathbb{Z}^d)$ and $1\le p<\infty$ then we define $\mathcal{H}^p_m$ to be the completion of the subspace $\mathcal{H}_0$ of all finite linear combinations in $\mathcal{G}$ with respect to the norm $\|f\|_{\mathcal{H}^p_m}=\|(\langle f,\tilde g_n\rangle)_{n\in\mathbb{Z}^d}\|_{\ell^p_m}$. If $p=\infty$ then we take the weak*-completion of $\mathcal{H}_0$ to define $\mathcal{H}^\infty_m$.

*REMARK:* The definition of $\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$ does not depend on the particular $\mathcal{A}$-self localized dual chosen, and any other $\mathcal{A}$-self-localized frame $\mathcal{F}$ which is localized to $\mathcal{G}$ generates in fact the same spaces. In particular one has the following

**Proposition 2.2** (Fornasier, Gröchenig [30]). *Assume that $(\mathcal{G},\tilde{\mathcal{G}})$ is a pair of dual $\mathcal{A}$-self-localized frames for $\mathcal{H}$. The the following characterizations hold.*

(a) *The space $\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$ as in* (3) *can be equivalently defined as*

$$\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}}) = \{f\in\mathcal{H}: \quad f=\sum_{n\in\mathbb{Z}^d}c_n g_n, \quad (c_n)_{n\in\mathbb{Z}^d}\in\ell^p_m(\mathbb{Z}^d)\},$$

*and* $\|f\|_{\mathcal{H}^p_m}\asymp\inf\{\|c\|_{\ell^p_m}: c\in\ell^p_m, \quad f=F^*c\}$.

(b) *$\mathcal{F}$ is an other $\mathcal{A}$-self-localized frame for $\mathcal{H}$ with an $\mathcal{A}$-self-localized dual $\tilde{\mathcal{F}}$. If $\mathcal{F}\sim_{\mathcal{A}}\mathcal{G}$, then they generate the same family of Banach spaces $\mathcal{H}^p_m(\mathcal{F},\tilde{\mathcal{F}})=\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$ with equivalent norms.*

**Theorem 2.3** (Fornasier, Gröchenig [30]). *Assume that $\mathcal{G}$ is an $\mathcal{A}$-self-localized frame for $\mathcal{H}$. Then $\mathcal{G}$ and its canonical dual frame $\tilde{\mathcal{G}}$ are Banach frames for $\mathcal{H}^p_m(\mathcal{G},\tilde{\mathcal{G}})$.*

## 3. $\alpha$-MODULATION SPACES

3.1. **$\alpha$-modulation spaces as decomposition spaces.** In this section we want to recall the definition of $\alpha$-modulation spaces based on decomposition methods, without introducing them in full generality. For major details we refer to [32, 19, 16]. In fact the spaces depend on a parameter $\alpha\in[0,1]$ which is a "tuning tool" to perform a suitable *segmentation* (decomposition) of the frequency domain as an *intermediate* geometry between those of modulation [17, 34] and Besov [31, 50, 51] spaces.

**Definition 2.** A countable set $\mathcal{I}$ of intervals $I\subset\mathbb{R}$ is called an *admissible covering* of $\mathbb{R}$ if

(a) $\mathbb{R}=\bigcup_{I\in\mathcal{I}}I$, and
(b) $\#\{I\in\mathcal{I}: x\in I\}\le 2$ for all $x\in\mathbb{R}$.

Furthermore, if there exists a constant $0\le\alpha\le1$ such that $|I|\asymp(1+|\xi|)^\alpha$ for all $I\in\mathcal{I}_\alpha$, and all $\xi\in I$, then $\mathcal{I}_\alpha$ is called an *$\alpha$-covering.*

For an $\alpha$-covering $\mathcal{I}_\alpha$ one can identify the constituting intervals by means of two maps. The *position map* $p_\alpha$ from $\mathbb{Z}$ to $\mathbb{R}$, $p_\alpha : j \to p_\alpha(j)$, and the *size map* $s_\alpha$ from $\mathbb{Z}$ to $\mathbb{R}_+$, $s_\alpha : j \to s_\alpha(j)$, so that the map from $\mathbb{Z}$ to $\mathcal{I}_\alpha$, $j \to p_\alpha(j) + [0, s_\alpha(j)]$ is a bijection.

**Example 1** (Fornasier, Feichtinger [18]). For $b > 0$ and $\alpha \in [0, 1)$ an explicit example of $\alpha$-covering has been constructed in [18], by choosing as position and size functions

$$p_\alpha(j) = \mathrm{sgn}(j)\left((1 + (1-\alpha)\cdot b \cdot |j|)^{\frac{1}{1-\alpha}} - 1\right) \tag{4}$$

and

$$s_\alpha(j) = b \cdot (1 + (1-\alpha)\cdot b \cdot (|j| + 1))^{\frac{\alpha}{1-\alpha}}, \tag{5}$$

respectively. In particular, for $\alpha \to 1$ one has

$$\mathcal{I}_1 = \{\mathrm{sgn}(j)(e^{b|j|} - 1) + [0, e^{b(|j|+1)}]\}_{j\in\mathbb{Z}}$$

is an $\alpha$-covering of exponential type and for $b = \ln(2)$ is dyadic.

Without loss of generality we can assume that, associated to an admissible $\alpha$-covering $\mathcal{I}_\alpha$, one can construct [16, Theorem 4.2] a corresponding *bounded admissible partition of the unity* (BAPU) $\Psi^\alpha = \{\psi_I^\alpha\}_{I\in\mathcal{I}}$ in $\mathcal{S}(\mathbb{R})$, i.e.

(p1) $\sup_{I\in\mathcal{I}} \|\psi_I^\alpha\|_{\mathcal{F}L^1} < \infty$,
(p2) $\mathrm{supp}(\psi_I^\alpha) \subset I$ for all $I \in \mathcal{I}_\alpha$, and
(p3) $\sum_{I\in\mathcal{I}} \psi_I^\alpha(\xi) = 1$ for all $\xi \in \mathbb{R}$.

Furthermore we define the *segmentation operator* $\mathcal{P}_I^\alpha$ by

$$\mathcal{P}_I^\alpha(f) := \mathcal{F}^{-1}(\psi_I^\alpha \mathcal{F} f), \quad I \in \mathcal{I}, \quad \text{for all } f \in \mathcal{S}'(\mathbb{R}). \tag{6}$$

**Definition 3** ($\alpha$-modulation spaces, Gröbner [32]). Given $1 \le p, q \le \infty$, $s \in \mathbb{R}$ and $0 \le \alpha \le 1$, let $\mathcal{I}_\alpha$ be an $\alpha$-covering of $\mathbb{R}$ and let $\Psi^\alpha$ be a corresponding bounded admissible partition of the unity. Then we define the *$\alpha$-modulation space* $M_{p,q}^{s,\alpha}(\mathbb{R})$ for $q < \infty$ as the set of temperate distributions $f \in \mathcal{S}'(\mathbb{R})$ satisfying

$$\|f\|_{M_{p,q}^{s,\alpha}} := \left(\sum_{I\in\mathcal{I}_\alpha} \|\mathcal{P}_I^\alpha(f)\|_p^q (1 + |\omega_I|)^{sq}\right)^{1/q} < \infty, \tag{7}$$

with $\omega_I \in I$ for all $I \in \mathcal{I}_\alpha$. For $q = \infty$ the definition is adapted substituting the $\ell^q$-norm with the sup-norm over $I \in \mathcal{I}_\alpha$. Let us denote $M_p^{s,\alpha} := M_{p,p}^{s,\alpha}$.

*REMARK:* It is not difficult to check that the definition of $M_{p,q}^{s,\alpha}(\mathbb{R})$ does not depend on the particular choice of $\{\omega_I\}_{I\in\mathcal{I}_\alpha}$. As a canonical choice we can assume $\omega_I = p_\alpha(j)$, for $I = p_\alpha(j) + [0, s_\alpha(j)]$. Moreover, since two $\alpha$-coverings are equivalent in the sense of [19, Definition 3.3], the definition of $M_{p,q}^{s,\alpha}(\mathbb{R})$ does not depend on the particular choice of $\mathcal{I}_\alpha$ [19, Theorem 3.7] nor on $\{\mathcal{P}_I^\alpha\}_{I\in\mathcal{I}_\alpha}$ [19, Theorem 2.3 (B)]. In particular, from formula (4), we can assume without loss of generality that $p_\alpha(j) \asymp \mathrm{sgn}(j)\left((1 + (1-\alpha)\cdot b\cdot |j|)^{\frac{1}{1-\alpha}} - 1\right)$ and $p_\alpha(j) = 0$.

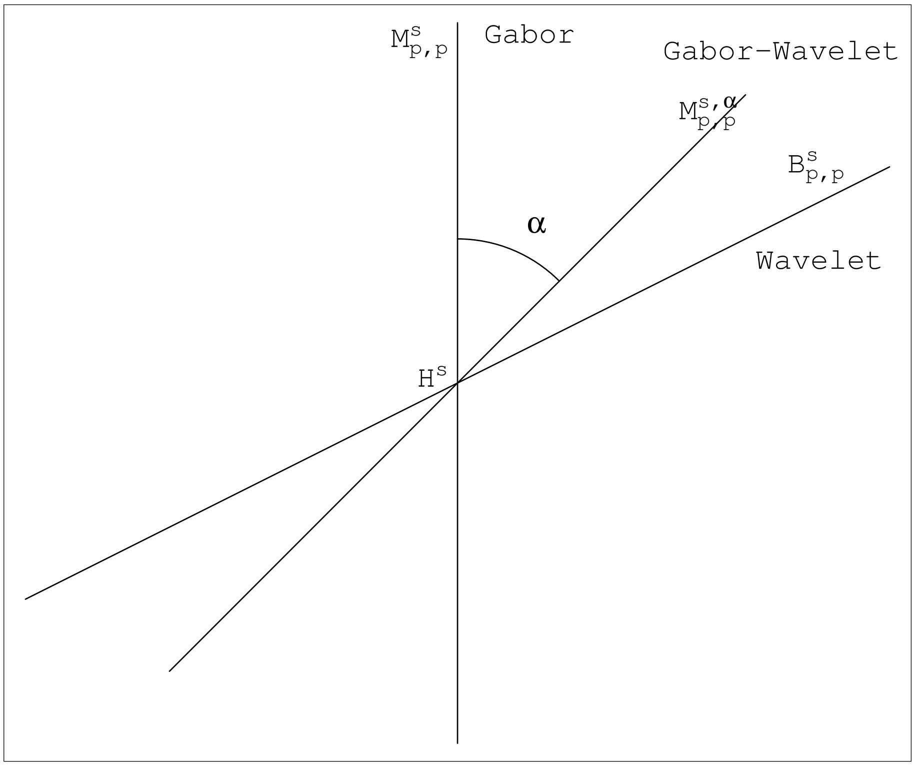


FIGURE 1. $\alpha$-modulation spaces

**Examples 1.** *Modulation spaces.* For $\alpha = 0$ the space $M^{s,0}_{p,q}(\mathbb{R})$ coincides with the modulation space $M^s_{p,q}(\mathbb{R})$. We refer to [17, 34] for major details on such spaces. They are naturally related to Gabor (time-frequency) frames, as we illustrate in the following.

The combination of modulation and translation operators

$$\pi(\lambda) = M_\omega T_x \quad \text{for } \lambda = (x, \omega) \in \mathbb{R}^2 \tag{8}$$

is called a time-frequency shift. Let $\mathcal{X}$ be a *relatively separated* set in the time-frequency plane $\mathbb{R}^2$ and let $g \in L^2(\mathbb{R})$ a fixed analyzing function. If the sequence $\mathcal{G}(g, \mathcal{X}) = \{\pi(\lambda)g\}_{\lambda \in \mathcal{X}}$ is a frame for $L^2(\mathbb{R})$ then it is called *Gabor frame* if $\mathcal{X}$ is a regular lattice, *non-uniform or irregular Gabor frame* otherwise. If $g \in \mathcal{S}(\mathbb{R})$ generates an irregular Gabor frame $\mathcal{G} = \mathcal{G}(g, \mathcal{X})$ then for any $s > 1$ $\mathcal{G}$ is intrinsically $\mathcal{A}_s$-localized and, by Theorem 2.1, it has intrinsically $\mathcal{A}_s$-self-localized canonical dual $\tilde{\mathcal{G}} = \{\tilde{e}_\lambda\}_{\lambda \in \mathcal{X}}$. Moreover, it is shown in [35, 30] that $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are Banach frames for suitable classes of modulation spaces. This means that

- the frame expansions

$$f = \sum_{\lambda \in \mathcal{X}} \langle f, \tilde{e}_\lambda \rangle \pi(\lambda) g = \sum_{\lambda \in \mathcal{X}} \langle f, \pi(\lambda) g \rangle \tilde{e}_\lambda, \tag{9}$$

  converge unconditionally in $M^s_p(\mathbb{R})$;

- the modulation space $M_p^s(\mathbb{R})$ can be characterized by the frame coefficients as follows:

$$\|f\|_{M_p^s} \asymp \|(\langle f, \tilde{e}_\lambda\rangle)_\lambda\|_{\ell^p_{w_s}(\mathcal{X})} \asymp \|(\langle f, \pi(\lambda)g)_\lambda\|_{\ell^p_{w_s}(\mathcal{X})} \quad (10)$$

Therefore the spaces $\mathcal{H}^p_{w_s}(\mathcal{G}, \tilde{\mathcal{G}})$ and $M_p^s(\mathbb{R})$ coincide with equivalent norms.

*Inhomogeneous Besov spaces.* For $\alpha \to 1$ the space $M^{s,1}_{p,q}(\mathbb{R})$ coincides with the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R})$. Refer to [31, 50, 51] for major details on these classical spaces. It is well known [44] that inhomogeneous Besov spaces can be characterized by expansions of *wavelet frames* of the type

$$\mathcal{G} = \{T_k\varphi\}_{k\in\mathbb{Z}} \bigcup \{D_{2^{-j}} T_k \psi\}_{j\in\mathbb{N}, k\in\mathbb{Z}},$$

where $\varphi$ is a smooth refinable function and $\psi$ is a smooth wavelet function with enough vanishing moments.

An application of the intrinsic localization of frame theory to characterize Besov space requires a different measure of localization. In particular, one should work with exponentially localized frames [35, 2] as we will see also in the following. Therefore we postpone this limiting case to be discussed elsewhere.

3.2. **Banach frames for $\alpha$-modulation spaces.** Assume $\alpha \in [0,1)$ and that $(p_\alpha, s_\alpha)$ is a pair of position and size functions. Given the family

$$\mathcal{G} := \mathcal{G}_\alpha(g, p_\alpha, s_\alpha, a) = \{M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} g\}_{j\in\mathbb{Z}, k\in\mathbb{Z}} \quad a > 0, \quad (11)$$

we want to illustrate under which (sufficient) conditions on the function $g$ one can ensure that if $\mathcal{G}$ is a frame for $L^2(\mathbb{R})$ then $\mathcal{G}$ extends also to a Banach frame for a suitable class of Banach spaces. We want also to show that this class of Banach spaces is in fact constituted by $\alpha$-modulation spaces. To this end, we discuss the properties of localization of $\mathcal{G}$ and then we apply the principles illustrated in the previous section.

*REMARK:* For $\alpha = 0$ the size function $s_\alpha(j) \asymp (1 + |p_\alpha(j)|)^0 = \text{const}$ and the position function $p_\alpha$ describes a relatively separated set. Therefore, for $\alpha = 0$ the frame $\mathcal{G}$ is a Gabor frame. For $\alpha \to 1$, the dilation factor is controlled by $s_\alpha(j) \asymp (1 + |p_\alpha(j)|)$. Therefore, since $\frac{p_\alpha(j)}{s_\alpha(j)} \asymp \text{const}$, the frame $\mathcal{G} = \{e^{2\pi i \frac{p_\alpha(j)}{s_\alpha(j)} ak} D_{s_\alpha(j)^{-1}} T_{ak}(e^{2\pi i \frac{p_\alpha(j)}{s_\alpha(j)} \cdot} g)\}_{j\in\mathbb{Z}, k\in\mathbb{Z}}$ is just a slight modification of a wavelet type frame.

**Lemma 3.1.** *For any $s > 1$ and $0 < \delta \le 1$*

$$\int_{\mathbb{R}} (1 + |x-n|)^{-s} (\delta + |x-m|)^{-s} dx \lesssim (\delta + |n-m|)^{-s}, \quad \textit{for all } m, n \in \mathbb{Z}. \quad (12)$$

*Proof.* The lemma can be proved with similar arguments as [35, Lemma 2.2]. ∎

**Lemma 3.2.** *Assume $a > 0$, $\gamma_f, \gamma_t > 1$, $\alpha \in [0,1)$, and let $(p_\alpha, s_\alpha)$ be a pair of position and size functions.*

*Let* $\{g_\ell\}_{\ell\in\mathbb{Z}} \subset L^1(\mathbb{R})\cap C(\mathbb{R})$ *such that*

$$|g_\ell(x)\mathcal{F}g_\ell(\omega)| \lesssim (1+|x|)^{-\gamma_t}(1+|\omega|)^{-\gamma_f}, \quad x,\omega\in\mathbb{R}, \tag{13}$$

*uniformly with respect to* $\ell\in\mathbb{Z}$. *Then, for* $\frac{\alpha}{1-\alpha}+\gamma_t \le \gamma_f$ *and* $\gamma=\gamma_t/2$,

(a) *one has*

$$\begin{aligned}&|\langle M_{p_\alpha(j)}D_{s_\alpha(j)^{-1}}T_{ak}g_0, M_{p_\alpha(i)}D_{s_\alpha(i)^{-1}}T_{ah}g_i\rangle|\\ &\lesssim (1+|(j,k)-(i,h)|)^{-\gamma}, \quad \text{for all } i,j,h,k\in\mathbb{Z}.\end{aligned} \tag{14}$$

(b) *for a suitable system of segmentation operators* $\{\mathcal{P}_j^\alpha\}_{j\in\mathbb{Z}}$ (6) *associated to a* BAPU $\Psi^\alpha=\{\psi_j^\alpha\}_{j\in\mathbb{Z}}$, *one has*

$$\begin{aligned}&|\langle \mathcal{P}_j^\alpha M_{p_\alpha(j)}D_{s_\alpha(j)^{-1}}T_{ak}g_j, M_{p_\alpha(i)}D_{s_\alpha(i)^{-1}}T_{ah}g_i\rangle|\\ &\lesssim (1+|(j,k)-(i,h)|)^{-\gamma}, \quad \text{for all } i,j,h,k\in\mathbb{Z}.\end{aligned} \tag{15}$$

*and*

$$\begin{aligned}&|\langle \mathcal{P}_j^\alpha M_{p_\alpha(j)}D_{s_\alpha(j)^{-1}}T_{ak}g_j, \mathcal{P}_i^\alpha M_{p_\alpha(i)}D_{s_\alpha(i)^{-1}}T_{ah}g_i\rangle|\\ &\lesssim (1+|(j,k)-(i,h)|)^{-\gamma}, \quad \text{for all } i,j,h,k\in\mathbb{Z}.\end{aligned} \tag{16}$$

*Proof.* Let us start showing (a), and, in particular, the case $j\ge i\ge 0$; the other cases can be shown with similar arguments.

$$\begin{aligned}&|\langle M_{p_\alpha(j)}T_{a\cdot s_\alpha(j)^{-1}\cdot k}D_{s_\alpha(j)^{-1}}g_0, M_{p_\alpha(i)}T_{a\cdot s_\alpha(i)^{-1}\cdot h}D_{s_\alpha(i)^{-1}}g_i\rangle|\\ &= |\langle M_{aks_\alpha(j)^{-1}}T_{p_\alpha(j)}D_{s_\alpha(j)}\mathcal{F}g_0, M_{ahs_\alpha(i)^{-1}}T_{p_\alpha(i)}D_{s_\alpha(i)}\mathcal{F}g_i\rangle|\\ &= \left|\int_{\mathbb{R}}\left(T_{p_\alpha(j)}D_{s_\alpha(j)}\mathcal{F}g_0(\omega)\right)\overline{\left(T_{p_\alpha(i)}D_{s_\alpha(i)}\mathcal{F}g_i(\omega)\right)}e^{2\pi i\left(a(ks_\alpha(j)^{-1}-hs_\alpha(i)^{-1})\right)\omega}d\omega\right|.\end{aligned} \tag{17}$$

Step 1. (Frequency localization)

From (17) one has an estimation of (14) in the frequency domain:

$$|\langle M_{p_\alpha(j)}D_{s_\alpha(j)^{-1}}T_{ak}g_0, M_{p_\alpha(i)}D_{s_\alpha(i)^{-1}}T_{ah}g_i\rangle| \le \int_{\mathbb{R}}\left|T_{p_\alpha(j)}D_{s_\alpha(j)}\mathcal{F}g_0(\omega)T_{p_\alpha(i)}D_{s_\alpha(i)}\mathcal{F}g_i(\omega)\right|d\omega$$

$$\begin{aligned}&\lesssim \left(\frac{1}{s_\alpha(j)s_\alpha(i)}\right)^{1/2}\int_{\mathbb{R}}\left(1+\left|\frac{\omega-p_\alpha(j)}{s_\alpha(j)}\right|\right)^{-\gamma_f}\left(1+\left|\frac{\omega-p_\alpha(i)}{s_\alpha(i)}\right|\right)^{-\gamma_f}d\omega\\ &= \left(\frac{1}{s_\alpha(j)s_\alpha(i)}\right)^{1/2}\int_{\mathbb{R}}\left(1+\left|\frac{\omega}{s_\alpha(j)}-\frac{p_\alpha(j)}{s_\alpha(j)}\right|\right)^{-\gamma_f}\\ &\times \left(1+\left|\frac{\omega}{s_\alpha(j)}\frac{s_\alpha(j)}{s_\alpha(i)}-\frac{s_\alpha(j)}{s_\alpha(i)}\frac{s_\alpha(i)}{s_\alpha(j)}\frac{p_\alpha(i)}{s_\alpha(i)}\right|\right)^{-\gamma_f}d\omega\\ &= \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2}\int_{\mathbb{R}}\left(1+\left|\omega-\frac{p_\alpha(j)}{s_\alpha(j)}\right|\right)^{-\gamma_f}\left(1+\frac{s_\alpha(j)}{s_\alpha(i)}\left|\omega-\frac{p_\alpha(i)}{s_\alpha(j)}\right|\right)^{-\gamma_f}d\omega\\ &\lesssim \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2}\int_{\mathbb{R}}\left(1+\left|\omega-\frac{p_\alpha(j)}{s_\alpha(j)}\right|\right)^{-\gamma_f}\left(1+\left|\omega-\frac{p_\alpha(i)}{s_\alpha(j)}\right|\right)^{-\gamma_f}d\omega.\end{aligned} \tag{18}$$

By an application of Lemma 3.1

$$(18) \lesssim \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2}\left(1+\left|\frac{p_\alpha(j)}{s_\alpha(j)}-\frac{p_\alpha(i)}{s_\alpha(j)}\right|\right)^{-\gamma_f} \lesssim \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2}\left(1+\left|\frac{p_\alpha(j)}{s_\alpha(j)}-\frac{p_\alpha(i)}{s_\alpha(i)}\right|\right)^{-\gamma_f}.$$

Observing that $\frac{1+|y|}{1+|x|} \leq (1+|x-y|)$ for all $x, y \in \mathbb{R}$, one has

$$\begin{aligned} & |\langle M_{p_\alpha(j)} D_{s_\alpha(j)^{-1}} T_{ak} g_0, M_{p_\alpha(i)} D_{s_\alpha(i)^{-1}} T_{ah} g_i\rangle| && (19)\\ \lesssim\ & (1+|j-i|)^{\frac{\alpha}{2(1-\alpha)}}(1+|j-i|)^{-\gamma_f} && (20)\\ =\ & (1+|j-i|)^{\frac{\alpha}{2(1-\alpha)}-\gamma_f}. && (21)\end{aligned}$$

Step 2. (Time localization)

From (17) one has an estimation of (14) also in the time domain:

$$\begin{aligned} & |\langle M_{p_\alpha(j)} D_{s_\alpha(j)^{-1}} T_{ak} g_0, M_{p_\alpha(i)} D_{s_\alpha(i)^{-1}} T_{ah} g_i\rangle| \\ \leq\ & (D_{s_\alpha(j)^{-1}}|g_0|) * (D_{s_\alpha(i)^{-1}}|g_i|)(a(ks_\alpha(j)^{-1}-hs_\alpha(i)^{-1})) \\ \lesssim\ & \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2}\int_{\mathbb{R}}(1+|s_\alpha(i)y-x|)^{-\gamma_t}\left(1+|\frac{s_\alpha(j)}{s_\alpha(i)}x|\right)^{-\gamma_t}dx,\end{aligned}$$

where $y = a(ks_\alpha(j)^{-1} - hs_\alpha(i)^{-1})$. By Lemma 3.1 , one has

$$\begin{aligned} & \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2}\int_{\mathbb{R}}(1+|s_\alpha(i)y-x|)^{-\gamma_t}\left(1+|\frac{s_\alpha(j)}{s_\alpha(i)}x|\right)^{-\gamma_t}dx && (22)\\ \lesssim\ & \left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{1/2-\gamma_t}\int_{\mathbb{R}}\left(\frac{s_\alpha(i)}{s_\alpha(j)}+|s_\alpha(i)y-x|\right)^{-\gamma_t}(1+|x|)^{-\gamma_t}dx && (23)\\ \lesssim\ & (1+|j-i|)^{\frac{\alpha}{2(1-\alpha)}}\left(\frac{s_\alpha(j)}{s_\alpha(i)}\right)^{-\gamma_t}\left(\frac{s_\alpha(i)}{s_\alpha(j)}+|s_\alpha(i)y|\right)^{-\gamma_t}. && (24)\end{aligned}$$

Assume $h \geq 0$ (the other case can be treated in similar way).

- If $h \geq k$ and $k \geq 0$ then $|ks_\alpha(i)/s_\alpha(j) - h| \geq |k-h|$. Therefore

$$\left(\frac{s_\alpha(i)}{s_\alpha(j)}+|s_\alpha(i)y|\right)^{-\gamma_t} \lesssim \left(\frac{s_\alpha(i)}{s_\alpha(j)}+a|k-h|\right)^{-\gamma_t}. \qquad (25)$$

- If $h \geq k$ and $k < 0$ then $|ks_\alpha(i)/s_\alpha(j) - h| \geq s_\alpha(i)/s_\alpha(j)|k-h|$, and one has

$$\left(\frac{s_\alpha(i)}{s_\alpha(j)}+|s_\alpha(i)y|\right)^{-\gamma_t} \lesssim \left(\frac{s_\alpha(i)}{s_\alpha(j)}+a\frac{s_\alpha(i)}{s_\alpha(j)}|k-h|\right)^{-\gamma_t}. \qquad (26)$$

- If $h < k$ and $ks_\alpha(j)^{-1} - hs_\alpha(i)^{-1} \leq 0$, then

$$\begin{aligned} |(k-h)(s_\alpha(j)^{-1}+s_\alpha(i)^{-1})| &= |(ks_\alpha(j)^{-1}-hs_\alpha(i)^{-1})+(ks_\alpha(i)^{-1}-hs_\alpha(j)^{-1})| && (27)\\ &\leq |ks_\alpha(j)^{-1}-hs_\alpha(i)^{-1}|. && (28)\end{aligned}$$

Hence, one has

$$\left(\frac{s_\alpha(i)}{s_\alpha(j)} + |s_\alpha(i)y|\right)^{-\gamma_t} \lesssim \left(\frac{s_\alpha(i)}{s_\alpha(j)} + a|k-h|(s_\alpha(i)/s_\alpha(j)+1)\right)^{-\gamma_t} \quad (29)$$

$$\lesssim \left(\frac{s_\alpha(i)}{s_\alpha(j)} + a|k-h|\right)^{-\gamma_t}. \quad (30)$$

- If $h < k$ and $ks_\alpha(j)^{-1} - hs_\alpha(i)^{-1} > 0$, then

$$|h-k| \geq |h\frac{s_\alpha(j)}{s_\alpha(i)} - k| = s_\alpha(j)|hs_\alpha(i)^{-1} - ks_\alpha(j)^{-1}| \geq |hs_\alpha(i)^{-1} - ks_\alpha(j)^{-1}|.$$

This implies

$$\left(\frac{s_\alpha(i)}{s_\alpha(j)} + |s_\alpha(i)y|\right)^{-\gamma_t} = \left(\frac{s_\alpha(i)}{s_\alpha(j)} + \frac{s_\alpha(i)}{s_\alpha(j)}|s_\alpha(j)y|\right)^{-\gamma_t} \quad (31)$$

$$\lesssim \left(\frac{s_\alpha(i)}{s_\alpha(j)} + a\frac{s_\alpha(i)}{s_\alpha(j)}|h-k|\right)^{-\gamma_t}. \quad (32)$$

Hence, from (25)-(32) one immediately has

$$|\langle M_{p_\alpha(j)} D_{s_\alpha(j)^{-1}} T_{ak} g_0, M_{p_\alpha(i)} D_{s_\alpha(i)^{-1}} T_{ah} g_i\rangle| \lesssim (1+|j-i|)^{\frac{\alpha}{2(1-\alpha)}} (1+a|h-k|)^{-\gamma_t}. \quad (33)$$

Step 3. (Time-frequency localization)
By combining formulae (21) and (33), one has

$$\begin{aligned} &|\langle M_{p_\alpha(j)} T_{a\cdot s_\alpha(j)^{-1}\cdot k} D_{s_\alpha(j)^{-1}} g_0, M_{p_\alpha(i)} T_{a\cdot s_\alpha(i)^{-1}\cdot h} D_{s_\alpha(i)^{-1}} g_i\rangle|^2 \\ \lesssim\ & (1+|j-i|)^{\frac{\alpha}{(1-\alpha)}-\gamma_f} (1+a|h-k|)^{-\gamma_t}. \end{aligned}$$

For $\gamma_f \geq \frac{\alpha(1-\gamma_t)+\gamma_t}{1-\alpha}$,

$$\begin{aligned} &|\langle M_{p_\alpha(j)} T_{a\cdot s_\alpha(j)^{-1}\cdot k} D_{s_\alpha(j)^{-1}} g_0, M_{p_\alpha(i)} T_{a\cdot s_\alpha(i)^{-1}\cdot h} D_{s_\alpha(i)^{-1}} g_i\rangle|^2 \\ \lesssim\ & (1+|j-i|)^{-\gamma_t} (1+a|h-k|)^{-\gamma_t} \\ \lesssim\ & (1+|j-i|+a|h-k|)^{-\gamma_t} \\ \lesssim\ & (1+|j-i|+|h-k|)^{-\gamma_t}. \end{aligned}$$

We want to show now (b).

Observe that

$$\mathcal{F}(\mathcal{P}_j^\alpha M_{p_\alpha(j)} T_{a\cdot s_\alpha(j)\cdot k} D_{s_\alpha(j)^{-1}} g_j) = \psi_j^\alpha T_{p_\alpha(j)} M_{aks_\alpha(j)^{-1}} D_{s_\alpha(j)} \mathcal{F} g_j,$$

Without loss of generality, by similar arguments as in [16, Theorem 4.2] we can assume $\psi_j^\alpha = s_\alpha(j)^{1/2} T_{p_\alpha(j)} D_{s_\alpha(j)} \varphi_j^\alpha$, with

$$\varphi_j^\alpha(x)\mathcal{F}\varphi_j^\alpha(\omega) \lesssim (1+|x|)^{-\gamma_t}(1+|\omega|)^{-\gamma_f}$$

for all $j \in \mathbb{Z}$ and $x, \omega \in \mathbb{R}$. Therefore

$$\mathcal{F}(\mathcal{P}_j^\alpha M_{p_\alpha(j)} T_{a\cdot s_\alpha(j)\cdot k} D_{s_\alpha(j)^{-1}} g_j) = T_{p_\alpha(j)} M_{aks_\alpha(j)^{-1}} D_{s_\alpha(j)} (\varphi_j^\alpha \mathcal{F} g_j).$$

If $|\mathcal{F}g_j(\omega)| \lesssim (1+|\omega|)^{-\gamma_f}$, then $|\varphi_j^\alpha \mathcal{F}g_j(\omega)| \lesssim (1+|\omega|)^{-\gamma_f}$, uniformly with respect to $j \in \mathbb{Z}$. Moreover, by Lemma 3.1, one has

$$|\mathcal{F}^{-1}\left(\varphi_j^\alpha \mathcal{F}g_j\right)(x)| \lesssim (1+|x|)^{-\gamma_t},$$

uniformly with respect to $j \in \mathbb{Z}$. At this point one can conclude the proof of (b) by an application of (a). ∎

Assume $s > 0$ and $\alpha \in [0,1)$. We say that $g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ is $(s;\alpha)$-*localized*, if, for some $\gamma_t > 2\left(1+\frac{s}{1-\alpha}\right)$ and $\gamma_f \geq \gamma_t + \frac{\alpha}{1-\alpha}$,

$$|g(x)\mathcal{F}g_\ell(\omega)| \lesssim (1+|x|)^{-\gamma_t}(1+|\omega|)^{-\gamma_f}, \quad x,\omega \in \mathbb{R}. \tag{34}$$

**Theorem 3.3.** *Let $\alpha \in [0,1)$, $s \in \mathbb{R}$, and $0 < a < 1$. Assume that $g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ is $(|s|;\alpha)$-localized, $\mathcal{F}g(\omega) \neq 0$ for $\omega \in \Omega_0 = [-1,1]$ and*

$$\mathcal{G} := \mathcal{G}_\alpha(g, p_\alpha, s_\alpha, a) = \{M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} g\}_{j\in\mathbb{Z},k\in\mathbb{Z}}, \tag{35}$$

*is a frame for $L^2(\mathbb{R})$. Then,*

(a) *$\mathcal{G}$ and its canonical dual frame $\tilde{\mathcal{G}}$ are $\mathcal{A}_{\gamma_t/2}$-self-localized;*

(b) *the frame $\mathcal{G}$ and its canonical dual frame $\tilde{\mathcal{G}}$ extend to Banach frames for $\alpha$-modulation spaces in the sense that*

$$\mathcal{H}^p_{m_{s,\alpha}}(\mathcal{G},\tilde{\mathcal{G}}) = M_p^{s+\alpha(1/p-1/2),\alpha}(\mathbb{R}), \tag{36}$$

*for all $p \in [1,\infty]$, where $m_{s,\alpha}(j,k) = (1+(1-\alpha)|j|)^{\frac{s}{1-\alpha}}$.*

*Proof.* Consider functions $\varphi_+ \in C_c^\infty(\mathbb{R})$, $\mathrm{supp}(\varphi_+) = [-\varepsilon, 1+\varepsilon]$, and $\varphi_- \in C_c^\infty(\mathbb{R})$, $\mathrm{supp}(\varphi_-) = [-\varepsilon-1, \varepsilon]$, for $\varepsilon > 0$ small enough, with $\varphi_+ \equiv 1$ on $[0,1]$ and $\varphi_- \equiv 1$ on $[-1,0]$. Let us write $g^{\varphi_\star} = \mathcal{F}^{-1}(\varphi_\star \mathcal{F}g)$, $\star \in \{+,-\}$. If $f \in M_p^{s+\alpha(1/p-1/2),\alpha}(\mathbb{R})$ then, for $j \in \mathbb{Z}$, $\mathcal{P}_j^\alpha(f)$ is an $L^p(\mathbb{R})$ band-limited function and, by classical theorems on series expansions of band-limited functions (see also [23]), one has

$$\mathcal{P}_j^\alpha(f) = \sum_{k\in\mathbb{Z}} \langle f, M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} \tilde{g}^{\varphi_{\mathrm{sgn}(j)}}\rangle M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} g^{\varphi_{\mathrm{sgn}(j)}}, \tag{37}$$

where $\tilde{g}^{\varphi_{\mathrm{sgn}(j)}}$ is a suitable band-limited dual function, and

$$s_\alpha(j)^{\frac{2-p}{2}} \cdot \|\mathcal{P}_j^\alpha(f)\|_p^p \asymp \sum_{k\in\mathbb{Z}} |\langle f, M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} \tilde{g}^{\varphi_{\mathrm{sgn}(j)}}\rangle|^p. \tag{38}$$

In particular, by an application of [28, Theorem 14 and Corollary 17], the systems

$$\mathcal{G}^\varphi := \{M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} g^{\varphi_{\mathrm{sgn}(j)}}\}_{j\in\mathbb{Z},k\in\mathbb{Z}} \text{ and } \tilde{\mathcal{G}}^\varphi := \{\mathcal{P}_j^\alpha M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} \tilde{g}^{\varphi_{\mathrm{sgn}(j)}}\}_{j\in\mathbb{Z},k\in\mathbb{Z}} \tag{39}$$

constitute a dual pair $(\mathcal{G}^\varphi, \tilde{\mathcal{G}}^\varphi)$ of frames for $L^2(\mathbb{R})$. By Lemma 3.2 (a),(b) $\mathcal{G}^\varphi$ and $\tilde{\mathcal{G}}^\varphi$ are $\mathcal{A}_{\gamma_t/2}$-self-localized. Therefore, it makes sense to define the abstract Banach space $\mathcal{H}^p_{m_{s,\alpha}}(\mathcal{G}^\varphi, \tilde{\mathcal{G}}^\varphi)$ and, by Definition 3, one has the equivalence of norms:

$$\|f\|_{M_p^{s+\alpha(\frac{1}{p}-\frac{1}{2}),\alpha}} \asymp \left\| \left(\langle f, \mathcal{P}_j^\alpha M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} \tilde{g}^{\varphi_{\mathrm{sgn}(j)}}\rangle\right)_{j,k\in\mathbb{Z}} \right\|_{\ell^p_{m_{s,\alpha}}(\mathbb{Z}^2)} = \|f\|_{\mathcal{H}^p_{m_{s,\alpha}}(\mathcal{G}^\varphi,\tilde{\mathcal{G}}^\varphi)}. \tag{40}$$

It is not difficult to see that the space of linear combinations of elements of $\mathcal{G}^\varphi$ is in fact dense in $M_p^{s+\alpha(\frac{1}{p}-\frac{1}{2}),\alpha}(\mathbb{R})$, and hence one has $\mathcal{H}^p_{m_{s,\alpha}}(\mathcal{G}^\varphi,\tilde{\mathcal{G}}^\varphi) = M_p^{s+\alpha(\frac{1}{p}-\frac{1}{2}),\alpha}(\mathbb{R})$. Since $g$ is $(|s|;\alpha)$-localized, by Lemma 3.2 (a) the frame $\mathcal{G}$ is $\mathcal{A}_{\gamma_t/2}$-self-localized, and, in particular, is $A_{\gamma_t/2}$-localized with respect to $\mathcal{G}^\varphi$. By an application of Theorem 2.1 the canonical dual frame $\tilde{\mathcal{G}}$ is also $\mathcal{A}_{\gamma_t/2}$-self-localized. Proposition 2.2 (b) implies immediately that $\mathcal{H}^p_{m_{s,\alpha}}(\mathcal{G},\tilde{\mathcal{G}}) = \mathcal{H}^p_{m_{s,\alpha}}(\mathcal{G}^\varphi,\tilde{\mathcal{G}}^\varphi) = M_p^{s+\alpha(\frac{1}{p}-\frac{1}{2}),\alpha}(\mathbb{R})$. One concludes by Theorem 2.3. ∎

**Corollary 3.4.** *Let $\alpha \in [0,1)$ and assume that $g \in \mathcal{S}(\mathbb{R})$ is such that $\mathcal{F}g(\omega) \neq 0$ for $\omega \in \Omega_0 = [-1,1]$. Then, there exists $0 < a < 1$ such that*

$$\mathcal{G} := \mathcal{G}_\alpha(g, p_\alpha, s_\alpha, a) = \{M_{p_\alpha(j)} D_{s_\alpha^{-1}(j)} T_{ak} g\}_{j\in\mathbb{Z},k\in\mathbb{Z}}, \tag{41}$$

*is a frame for $L^2(\mathbb{R})$ and $\mathcal{G}$ and its canonical dual $\tilde{\mathcal{G}}$ are Banach frames for the $\alpha$-modulation space $M_p^{s,\alpha}(\mathbb{R})$, for all $p \in [1,\infty]$ and $s \in \mathbb{R}$.*

*Proof.* If $g \in \mathcal{S}(\mathbb{R})$ is such that $\mathcal{F}g(\omega) \neq 0$ for $\omega \in \Omega_0 = [-1,1]$ then $g$ is $\alpha$-admissible in the sense of [18, Definition 7]. By an application of [18, Theorem 1] there exists $0 < a < 1$ such that $\mathcal{G}$ is a frame for $L^2(\mathbb{R})$. Clearly $g$ is $(|s|;\alpha)$-localized for any $s \in \mathbb{R}$ and then one concludes by Theorem 3.3. ∎

*REMARKS:* 1. The assumption $\mathcal{F}g \neq 0$ on $\Omega_0 = [-1,1]$ is technical and it is essentially a non-vanishing condition. We expect that it can be removed.

2. Theorem 3.3 is a generalization of [34, Theorem 13.5.3] and [35, Theorem 5.2] (see also [30]), corresponding to the case $\alpha = 0$, where Gabor frame characterizations of modulation spaces have been given. We conjecture that Theorem 3.3 can be formulated for the case $\alpha \to 1$ to characterize inhomogeneous Besov spaces $B_p^{s-1/p-1/2}(\mathbb{R})$. Since $\lim_{\alpha\to 1} m_{s,\alpha}(j,k) = e^{s|j|}$, we expect that the extension of our theory to the case $\alpha \to 1$ should involve *exponentially localized* frames [35]. Interesting results in this direction have been presented by Cordero and Gröchenig in [2] for the wavelet frame characterization of *homogeneous* Besov spaces.

3. A first formulation of the intrinsic localization of frame theory for the frame characterization of $\alpha$-modulation spaces has been presented in [29, Chapter 5]. The proof [29, Thereom 5.3.5] is again based on Lemma 3.2 and on the specific construction of an auxiliary intrinsically localized dual of $\mathcal{G}$ (in general not coinciding with the canonical one!). In fact, the intrinsic localization of the canonical dual has been conjectured in [29, Remark 5.3.8] and only later it has been proven in [30, 3]. Therefore, Theorem 3.3 and the use of the canonical dual in this paper simplify much the argument, avoiding the construction of an auxiliary and maybe more "artificial" localized dual.

4. Theorem 3.3 extends to the frame characterization of $M_{p,q}^{s,\alpha}(\mathbb{R})$ for $p \neq q$, just considering $\ell^{p,q}_{m_{s,\alpha}}$ spaces instead of $\ell^p_{m_{s,\alpha}}$ and observing that $\ell^{p,q}_{m_{s,\alpha}}(\mathbb{Z}^2) * \ell^1_{m_{|s|,\alpha}}(\mathbb{Z}^2) \subset \ell^{p,q}_{m_{s,\alpha}}(\mathbb{Z}^2)$.

5. Lemma 3.2 is strongly dependent on the particular geometry of the $\alpha$-covering determined by $(p_\alpha, s_\alpha)$ on the real line. We expect that the approach illustrated in this paper can be useful also for a frame characterization of $M^{s,\alpha}_{p,q}(\mathbb{R}^d)$ for $d > 1$, with major technical difficulties.

3.3. **$\alpha$-modulation spaces and time-frequency transforms.** In several relevant contributions, for example [1, 5, 7, 18, 26, 29, 39, 40, 41, 48, 49], an "intermediate" time-frequency transform between wavelet and short time Fourier transform is considered.

Assume $\alpha \in [0,1]$ and $c > 0$. For any $g \in L^2(\mathbb{R})\backslash\{0\}$ and for $f \in L^2(\mathbb{R})$ we define the *flexible Gabor-wavelet transform* (or *$\alpha$-transform*) as

$$V_g^\alpha(f)(x,\omega) := \langle f, T_x M_\omega D_{c(1+|\omega|)^{-\alpha}} g\rangle \tag{42}$$

$$= \int_{\mathbb{R}} f(t)\overline{T_x M_\omega D_{c(1+|\omega|)^{-\alpha}} g(t)}\,dt, \quad x,\omega \in \mathbb{R}. \tag{43}$$

The transform can naturally extend to distributions. For $\alpha = 0$ the transform $V_g^\alpha$ coincides with the well-known short time Fourier transform, while for $\alpha = 1$ it is a slight modification of the wavelet transform. In particular, the intermediate case $\alpha = 1/2$ is the Fourier-Bros-Iagolnitzer transform [5]. In [41, Theorem 4.4] Holschneider and Nazaret proved a characterization of $L^2$-Sobolev spaces by pull back techniques based on $\alpha$-transforms. For a suitable choice of $g \in \mathcal{S}\backslash\{0\}$ (for example the Gaussian) one has

$$f \in H^s(\mathbb{R}) \text{ if and only if } V_g^\alpha(f) \in L^2_{w_{s,f}}(\mathbb{R}^2), \tag{44}$$

where $w_{s,f}(x,\omega) = (1+|\omega|)^s$, $x,\omega \in \mathbb{R}$. In particular the following equivalence of norms holds

$$\|f\|_{H^s(\mathbb{R})} \asymp \|V_g^\alpha(f)\|_{2,w_{s,f}}, \text{ for all } f \in H^s(\mathbb{R}). \tag{45}$$

Inspired by this characterization, they introduce a more general class of Banach spaces [41, Definition 4.7]. For a suitable choice of a Banach function space $B$ on the time-frequency plane $\mathbb{R}^2$ one can define the space of distributions on $\mathbb{R}$ given by

$$\mathbb{B}(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : \quad V_g^\alpha(f) \in B\}, \tag{46}$$

endowed with the retract norm

$$\|f\|_{\mathbb{B}(\mathbb{R})} = \|V_g^\alpha(f)\|_B. \tag{47}$$

A similar approach can be found in [40, Section 4.6] where generalizations of modulation spaces are introduced by Hogan and Lakey.

We want to observe here that, for the choice of $B$ as a certain weighted $L^{p,q}$ space, the corresponding $\mathbb{B}(\mathbb{R})$ space is an $\alpha$-modulation space. In fact, since $f \in M^{s,\alpha}_{p,q}(\mathbb{R})$ if and only if $\mathcal{F}f \in D(\mathcal{I}_\alpha, \mathcal{F}L^p, \ell^q_{w_s})$, the *decomposition space* subordinate to the covering $\mathcal{I}_\alpha$, with local component $\mathcal{F}L^p$, and global component $\ell^q_{w_s}(\mathcal{I}_\alpha)$ (see [16, 19] for details), by an application of [16, Theorem 4.3] one can show the following

**Theorem 3.5.** *Assume $s \in \mathbb{R}$, $\alpha \in [0,1]$, and $1 \le p,q < \infty$. For a suitable band-limited $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$*

$$M_{p,q}^{s+\alpha(1/q-1/2),\alpha}(\mathbb{R}) = \{f \in \mathcal{S}'(\mathbb{R}) : V_g^\alpha(f) \in L_{w_{s,f}}^{p,q}(\mathbb{R}^2)\}. \tag{48}$$

*Moreover the norm of $M_{p,q}^{s+\alpha(1/q-1/2),\alpha}(\mathbb{R})$ can be equivalently expressed by*

$$\|f\|_{M_{p,q}^{s+\alpha(1/q-1/2),\alpha}(\mathbb{R})} \asymp \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |V_g^\alpha(f)(x,\omega)|^p dx \right)^{q/p} (1+|\omega|)^{sq} d\omega \right)^{1/q}, \tag{49}$$

*for all $f \in M_{p,q}^{s+\alpha(1/q-1/2),\alpha}(\mathbb{R})$. For $p \cdot q = \infty$ the usual modifications apply.*

A detailed discussion on the relations between continuous and discrete characterization of $\alpha$-modulation spaces will be given elsewhere in the context of recent generalizations of the coorbit space theory [9, 10].

3.4. **Equivalence of frames and $\alpha$-modulation spaces.** It is not difficult to show that $\sim_{\mathcal{A}}$ is an equivalence relation on the set of the intrinsically $\mathcal{A}$-localized frames. In particular, Proposition 2.2 (b) establishes that equivalent frames arise equivalent associated Banach spaces. Therefore, the "differences" between associated Banach spaces can be considered a "measure" of the different analysis that two frames perform. The results in this paper can be interpreted as a qualitative study of the "degree of difference" of the analysis performed by Gabor and wavelet frames (Fig. 1).

Let us conclude recalling in the following some of the relevant results related to inclusions of $\alpha$-modulations spaces, investigated by Gröbner [32]:

**Theorem 3.6.** *If $1 \le p,q \le \infty$, $s \in \mathbb{R}$ and $0 \le \alpha_1 < \alpha_2 \le 1$ then*

$$M_{p,q}^{s',\alpha_2}(\mathbb{R}) \subset M_{p,q}^{s,\alpha_1}(\mathbb{R}), \quad s' = s + \frac{(\alpha_2 - \alpha_1)}{q} \tag{50}$$

$$M_{p,q}^{s,\alpha_1}(\mathbb{R}) \subset M_{p,q}^{s',\alpha_2}(\mathbb{R}), \quad s' = s - (1-1/q)(\alpha_2 - \alpha_1). \tag{51}$$

*In particular, for $\alpha_2 = 1$ and $\alpha_1 = 0$,*

$$B_{p,q}^{s+1/q}(\mathbb{R}) \subset M_{p,q}^{s}(\mathbb{R}). \tag{52}$$

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