Adaptive Frame Methods for Elliptic Operator Equations *

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Abstract

This paper is concerned with the development of adaptive numerical methods for elliptic operator equations. We are especially interested in discretization schemes based on frames. The central objective is to derive an adaptive frame algorithm which is guaranteed to converge for a wide range of cases. As a core ingredient we use the concept of Gelfand frames which induces equivalences between smoothness norms and weighted sequence norms of frame coefficients. It turns out that this Gelfand characteristic of frames is closely related to their localization properties. We also give constructive examples of Gelfand wavelet frames on bounded domains. Finally, an application to the efficient adaptive computation of canonical dual frames is presented.

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1 Introduction

The analysis of adaptive numerical schemes for operator equations is a field of enormous current interest. Recent developments, for instance in the finite element context,

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indeed indicate their promising potential [1, 2, 4, 6, 33, 53]. Moreover, it has also turned out that adaptive schemes based on *wavelets* have several important advantages. The wavelet methodology differs from other conventional schemes in so far as direct use of bases is made which span appropriate complements between successive approximation spaces. The basic idea of adaptive wavelet schemes can be described as follows. By using the fact that weighted sequence norms of wavelet expansions are equivalent to Sobolev norms, efficient and reliable error estimators based on the wavelet expansion of the residual can be derived. By catching the bulk of the residual coefficients, these error estimators lead to adaptive refinement strategies which are guaranteed to converge for a wide range of problems. Indeed, by combining these ideas with the compression properties of wavelets, in [18] a first *implementable* convergent adaptive scheme for symmetric elliptic problems has been derived. Moreover, it has turned out that a judicious variant of this approach produces an asymptotically optimal algorithm [14]. Generalizations to nonsymmetric and nonlinear problems also exist [16, 15]. Moreover, by using adaptive variants of the classical Uzawa algorithm, saddle point problems can be handled [19, 20], and the applicability of the resulting algorithms to practical problems has been demonstrated in [3]. Nevertheless, the efficiency of all these approaches is still limited by a serious bottleneck. Usually, the operator under consideration is defined on a bounded domain $\Omega \subset \mathbb{R}^d$ or on a closed manifold, and therefore the construction of a wavelet basis with specific properties on this domain or on the manifold is needed. Although there exist by now several construction methods such as, e.g., [27, 28], none of them seems to be fully satisfying in the sense that some serious drawbacks such as stability problems cannot be avoided. One way out could be to use a fictitious domain method [51], however, then the compressibility of the problem might be reduced.

Motivated by these difficulties, we therefore suggest to use a slightly weaker concept, namely *frames.* In general, a sequence $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ is a frame for the Hilbert space $\mathcal{H}$ if

$$A_{\mathcal{F}}\|f\|_{\mathcal{H}}^2 \leq \sum_{n\in\mathcal{N}} \left|\langle f, f_n\rangle_{\mathcal{H}}\right|^2 \leq B_{\mathcal{F}}\|f\|_{\mathcal{H}}^2, \quad \text{for all } f \in \mathcal{H},$$

for suitable constants $0 < A_{\mathcal{F}} \leq B_{\mathcal{F}} < \infty$, see Section 3 and [11, 30] for further details. Every element of $\mathcal{H}$ has an expansion with respect to the frame elements, but in contrast to stable multiscale bases, its representation is not necessarily unique. Therefore frame expansions may contain some redundancy.

On the one hand, because of the redundancy of a frame, orthonormal and biorthogonal representations of functions by means of Riesz bases have been preferred and considered to be a maybe more useful concept since the overcompleteness of a frame has been interpreted as "low compression rate", "larger amount of data" and "undetermined representation".

On the other hand, the redundancy of a frame proved to play an important role in practical problems where stability and error tolerance are fundamental as, for example, denoising, pattern matching, or irregular sampling problems [36] with recent applications in $\Sigma\Delta$ quantization [32]. Moreover, since one is working with a weaker concept, the concrete *construction* of a frame is usually much simpler when compared to stable multiscale bases. Consequently, there is some hope that the frame approach might simplify the geometrical construction on bounded domains and manifolds significantly, and that this important advantage compensates some drawbacks such as singularity problems in discretizing operators. Moreover, the redundancy of a frame can give the freedom to implement further properties, which would be mutually exclusive in the Riesz basis case, e.g., both high smoothness and small support. Potentially this would allow faster and more accurate computation of the stiffness matrix entries associated, for example, to differential operators with smooth coefficients by Gauss quadrature methods, and sparser matrix representations of operators.

The potential of frames in numerical analysis is an almost unexplored field. One of the first interesting attempts to use frames for numerical simulation is [52], being a pioneering approach to the application of wavelet frames to the adaptive solution of operator equations. The results presented in this paper are very much inspired by these developments. However, for several reasons, we work with a different setting. Instead of using a frame for the solution space H^s, we start with a frame for L_2 which, similar to the classical wavelet setting, gives rise to norm equivalences for H^s as well as for its dual with respect to corresponding $\ell_{2,2^s}$ sequence spaces. Therefore we introduce the new concept of *Gelfand frames*, inducing norm equivalences for Gelfand triples $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ of Banach spaces with respect to corresponding sequence spaces $(\mathcal{B}_d, \ell_2, \mathcal{B}'_d)$. Gelfand frames appear to us to be a more natural generalization of the well–established concept of unconditional bases in smoothness spaces. We show that, by employing recent results on Banach frames [17, 35, 36, 42, 44, 47], the analysis of [52] and [14] carries over to the Gelfand frame case, i.e., we derive an adaptive frame algorithm which is guaranteed to converge for a wide range of problems. This is the main result of this paper.

To read the content in the correct light, let us add the following general remarks. We want to emphasize that we neither claim to rediscover the whole world of adaptive numerical schemes nor to give the frame analysis a shake–up. It is clear that many of the building blocks used in this paper have already been established before. Nevertheless, having in mind the fact that adaptive numerical analysis and frame theory have developed almost independently in the last years, we think that it is fruitful to bring these two different fields together and to show that many approaches investigated so far fit together quite nicely. Especially, we show that many concepts of modern frame theory such as localization of frames can in fact be very well exploited for numerical purposes.

We also want to deliberately point at the following fact here. The research presented in this paper was mainly motivated by the numerical treatment of elliptic operator equations. In this context, wavelet frames would be the most natural choice. However, for the following reason, our approach can also be applied to different settings involving other kinds of frames. The applicability of adaptive numerical schemes with guaranteed convergence essentially relies on compressibility properties. But it turns out that for a large class of operator equations, it is indeed possible to design (Gelfand) frames fitting the particular problems at hand in the sense that the system matrices arising by the discretization indeed exhibit the nice compressibility properties. One example would be the discretization of pseudodifferential operators by brushlet systems, see Section 4.3. In the course of this paper, more details and examples will be given to illustrate and support this general leitmotif.

This paper is organized as follows. In Section 2, we introduce the scope of problems we shall be concerned with. The whole analysis is based on the concept of Banach frames. Therefore, in Section 3, we briefly recall the definition and the basic properties of Banach and Gelfand frames as far as they are needed for our purposes. Section 4 contains the main result of this paper. We show that, based on a Richardson iteration, a convergent and implementable adaptive frame algorithm can be derived. The whole analysis relies on certain norm equivalences the Gelfand frame has to satisfy, and on the scheme proposed by Stevenson [52] for the adaptive *pseudo–inversion* of infinite matrices. The norm equivalences are closely related to the localization properties of the underlying frame. These relationships are discussed in detail in Section 5. Then, in Section 6, we give an outline how Gelfand wavelet frames on domains can be constructed. It turns out that a relatively simple approach by using an overlapping partition of the domain already works. Our result relies on a general concept of *exponential localization* with respect to an additional biorthogonal wavelet basis. The underlying metric to measure such localization is a modified version of the well–known Lemarié metric as, e.g., described in [43]. Section 7 illustrates another application of our theory. When working with frames, the computation of the canonical dual, based on the inversion of the *frame operator*, is always a nontrivial problem and usually the known numerical algorithms converge quite slowly. However, it turns out that this problem exactly fits into our setting so that we are able to derive an adaptive numerical scheme with optimal order of convergence to compute a dual frame. Finally, some technical lemmata, especially concerning the properties of the generalized Lemarié metric, are collected and proved in the appendix.

2 The Scope of Problems

We shall be concerned with linear operator equations

$$\mathcal{L}u = f, \tag{1}$$

where we will assume $\mathcal{L}$ to be a boundedly invertible operator from some Hilbert space H into its normed dual H', i.e.,

$$\|\mathcal{L}u\|_{H'} \sim \|u\|_H, \quad u \in H. \tag{2}$$

Here '$a \sim b$' means that both quantities can be uniformly bounded by some constant multiple of each other. Likewise, '$\lesssim$' indicates inequalities up to constant factors. We write out such constants explicitly only when their value matters. When $\mathcal{L}$ is assumed to be boundedly invertible, then (1) has a unique solution u for any $f \in H'$. In the sequel, we shall mainly focus on the important special case where

$$a(v, w) := \langle \mathcal{L}v, w \rangle \tag{3}$$

defines a *symmetric* bilinear form on H, $\langle \cdot, \cdot \rangle$ corresponding to the dual pairing of H and H'. We will always assume that $a(\cdot, \cdot)$ is *elliptic* in the sense that

$$a(v, v) \sim \|v\|_H^2, \tag{4}$$

which is easily seen to imply (2).

Typical examples are second order elliptic boundary value problems on a Lipschitz domain $\Omega \subset \mathbb{R}^d$ such as the Poisson equation

$$\begin{aligned} -\triangle u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{5}$$

In this case, $H = H_0^1(\Omega)$, $H' = H^{-1}(\Omega)$, and the corresponding bilinear form is given by

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w dx. \tag{6}$$

Thus H typically is a Sobolev space. Therefore we will from now on always assume that H and H', together with $L_2(\Omega)$, form a *Gelfand triple*, i.e.,

$$H \subset L_2(\Omega) \subset H' \tag{7}$$

with continuous and dense embeddings.

More general, one also may assume that $\mathcal{L}$ is an operator with global Schwartz kernel

$$(\mathcal{L}v)(x) = \int_\Omega K(x, y) v(y)\, dy,$$

where for $d + t + |\alpha| + |\beta| > 0$

$$\left|\partial_x^\alpha \partial_y^\beta K(x, y)\right| \lesssim \|x - y\|_{\mathbb{R}^d}^{-(d+t+|\alpha|+|\beta|)}. \tag{8}$$

Here Ω denotes a domain contained in $\mathbb{R}^d$ or a closed d–dimensional manifold and t a suitable parameter, $\|\cdot\|_{\mathbb{R}^d}$ denotes the Euclidean norm. Assumption (8) covers a wide range of cases, including pseudodifferential operators as well as Calderón–Zygmund operators, cf. [25]. Nevertheless the adaptive numerical scheme we discuss later works on more general (pseudodifferential) operators, provided that *compressibility properties* of the corresponding discretization matrices hold.

We are interested in solving (1) approximately with the aid of a suitable numerical scheme. One candidate would clearly be the Galerkin method. There one picks some finite dimensional space $S \subset H$ and searches for $u_S \in S$ such that

$$\langle \mathcal{L} u_S, v \rangle = \langle f, v \rangle, \qquad u \in S, \tag{9}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard L_2–inner product. A resonable choice for S would, e.g., be a finite element space. However, in this paper we are mainly interested in (wavelet) frames. Then, one possible strategy would be the following. Choose a suitable subset of frame elements, project the problem in the sense of (9) onto their span and compute the Galerkin approximation, try to estimate the current error in order to choose suitable additional frame elements and so on. But such a standard and classical approach can produce serious problems, e.g., numerical instability and difficulties to prove convergence. Indeed, since we allow redundancies within the frame, the corresponding stiffness matrices might be singular. Consequently, to handle this problem, we shall use a different strategy as outlined in detail in Section 4. Instead of using a classical Galerkin scheme, we work with an ℓ_2–problem equivalent to (1), which is treated by an approximated Richardson iteration.

3 Banach Frames

In the course of this paper, we have to consider weighted function spaces. To this end, we introduce a special class of weight functions. A *weight* w on $\mathbb{R}^d$ is a non–negative real–valued function, which we assume to be continuous without loss of generality. A weight w on $\mathbb{R}^d$ is m*–moderate* if $w(x+y) \leq m(x)w(y)$, where m is a submultiplicative weight on $\mathbb{R}^d$, i.e., $m(x+y) \leq m(x)m(y)$, and radial symmetric, i.e., $m(x) = m(\|x\|_{\mathbb{R}^d})$. A classical example of a submultiplicative and radial symmetric function is $m_s(x) = \big(1 + \|x\|_{\mathbb{R}^d}\big)^s$.

The *weighted* ℓ_p*–space* $\ell_{p,w}(\mathcal{N})$ on the countable index set $\mathcal{N} \subset \mathbb{R}^d$ with respect to the weight w is induced by the norm

$$\|c\|_{\ell_{p,w}} := \left(\sum_{n \in \mathcal{N}} |c_n|^p w(n)^p \right)^{1/p}, \tag{10}$$

with the usual modification for $p = \infty$. Throughout this paper, we will require the existence of a map $|\cdot| : \mathcal{N} \to \mathbb{Z}$, and we use the shorthand notation $\ell_{p,2^s}(\mathcal{N}) := \ell_{p,2^{s|\cdot|}}(\mathcal{N})$.

In the following, we assume that $\mathcal{H}$ is a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$. A sequence $\mathcal{F} = \{f_n\}_{n \in \mathcal{N}}$ in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if

$$\|f\|_{\mathcal{H}}^2 \sim \sum_{n \in \mathcal{N}} \big| \langle f, f_n \rangle_{\mathcal{H}} \big|^2, \quad \text{for all } f \in \mathcal{H}. \tag{11}$$

As a consequence of (11), the corresponding operators of analysis and synthesis given by

$$F : \mathcal{H} \to \ell_2(\mathcal{N}), \quad f \mapsto \big(\langle f, f_n\rangle_{\mathcal{H}}\big)_{n\in\mathcal{N}}, \tag{12}$$

$$F^* : \ell_2(\mathcal{N}) \to \mathcal{H}, \quad \mathbf{c} \mapsto \sum_{n\in\mathcal{N}} c_n f_n, \tag{13}$$

are bounded. The composition $S := F^*F$ is a boundedly invertible (positive and self–adjoint) operator called the *frame operator* and $\tilde{\mathcal{F}} := S^{-1}\mathcal{F}$ is again a frame for $\mathcal{H}$, the *canonical dual frame*, with corresponding analysis and synthesis operators

$$\tilde{F} = F(F^*F)^{-1}, \quad \tilde{F}^* = (F^*F)^{-1}F^*. \tag{14}$$

In particular, one has the following orthogonal decomposition of $\ell_2(\mathcal{N})$

$$\ell_2(\mathcal{N}) = \mathrm{ran}(F) \oplus \ker(F^*), \tag{15}$$

and

$$\mathbf{Q} := F(F^*F)^{-1}F^* : \quad \ell_2(\mathcal{N}) \to \mathrm{ran}(F), \tag{16}$$

is the orthogonal projection onto $\mathrm{ran}(F)$. The frame $\mathcal{F}$ is a Riesz basis for $\mathcal{H}$ if and only if $\ker(F^*) = \{0\}$. The importance of the canonical dual frame is its use in the reproduction of any element $f \in \mathcal{H}$. In fact, one has the following formulas:

$$f = SS^{-1}f = \sum_{n\in\mathcal{N}} \langle f, S^{-1}f_n\rangle_{\mathcal{H}} f_n = S^{-1}Sf = \sum_{n\in\mathcal{N}} \langle f, f_n\rangle_{\mathcal{H}} S^{-1}f_n. \tag{17}$$

Since a frame is typically overcomplete in the sense that the coefficient functionals $\{c_n\}_{n\in\mathcal{N}}$ in the representation

$$f = \sum_{n\in\mathcal{N}} c_n(f) f_n \tag{18}$$

are in general not unique ($\ker(F^*) \neq \{0\}$), there exist many possible non–canonical duals $\{\tilde{f}_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ for which

$$f = \sum_{n\in\mathcal{N}} \langle f, \tilde{f}_n\rangle_{\mathcal{H}} f_n. \tag{19}$$

A more general definition of frames is required for Banach spaces, cf. [44, 47], see also [35]. A *Banach frame* for a separable and reflexive Banach space $\mathcal{B}$ is a sequence $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ in $\mathcal{B}'$ with an associated sequence space $\mathcal{B}_d$ such that the following properties hold:

(B1) the coefficient operator F defined by $Ff = \big(\langle f, f_n\rangle_{\mathcal{B}\times\mathcal{B}'}\big)_{n\in\mathcal{N}}$ is bounded from $\mathcal{B}$ into $\mathcal{B}_d$;

(B2) norm equivalence:

$$\|f\|_{\mathcal{B}} \sim \left\| \left(\langle f, f_n \rangle_{\mathcal{B}\times\mathcal{B}'} \right)_{n\in\mathcal{N}} \right\|_{\mathcal{B}_d}; \tag{20}$$

(B3) there exists a bounded operator R from $\mathcal{B}_d$ onto $\mathcal{B}$, a so–called *synthesis* or *reconstruction operator*, such that

$$R\left(\left(\langle f, f_n \rangle_{\mathcal{B}\times\mathcal{B}'} \right)_{n\in\mathcal{N}} \right) = f. \tag{21}$$

Assuming that $\mathcal{B}$ is continuously and densely embedded in $\mathcal{H}$, one has

$$\mathcal{B} \subset \mathcal{H} \simeq \mathcal{H}' \subset \mathcal{B}'. \tag{22}$$

If the right inclusion is also dense, then $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ is called a *Gelfand triple.* In particular, this holds if $\mathcal{B}$ is also Hilbert space. A frame $\mathcal{F}$ (here $\tilde{\mathcal{F}}$ is the canonical dual frame) for $\mathcal{H}$ is a *Gelfand frame* for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, if $\mathcal{F} \subset \mathcal{B}$, $\tilde{\mathcal{F}} \subset \mathcal{B}'$ and there exists a Gelfand triple $\left(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\right)$ of sequence spaces such that

$$F^* : \mathcal{B}_d \to \mathcal{B},\ F^*\mathbf{c} = \sum_{n\in\mathcal{N}} c_n f_n \quad \text{and} \quad \tilde{F} : \mathcal{B} \to \mathcal{B}_d,\ \tilde{F}f = \left(\langle f, \tilde{f}_n \rangle_{\mathcal{B}\times\mathcal{B}'} \right)_{n\in\mathcal{N}} \tag{23}$$

are bounded operators.

REMARK: If $\mathcal{F}$ (again $\tilde{\mathcal{F}}$ is the canonical dual frame) is a Gelfand frame for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ with respect to the Gelfand triple of sequences $\left(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\right)$, then by duality also the operators

$$\tilde{F}^* : \mathcal{B}_d' \to \mathcal{B}',\ \tilde{F}^*\mathbf{c} = \sum_{n\in\mathcal{N}} c_n \tilde{f}_n \quad \text{and} \quad F : \mathcal{B}' \to \mathcal{B}_d',\ Ff = \left(\langle f, f_n \rangle_{\mathcal{B}'\times\mathcal{B}} \right)_{n\in\mathcal{N}} \tag{24}$$

are bounded, see, e.g., [49] for details.

The next result clarifies the relations between Gelfand and Banach frames.

Proposition 3.1. *If $\mathcal{F}$ is a Gelfand frame for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, then $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are Banach frames for $\mathcal{B}$ and $\mathcal{B}'$, respectively.*

Proof. We only show that $\tilde{\mathcal{F}}$ is a Banach frame for $\mathcal{B}$, since the second claim follows by an analogous argument. It suffices to prove (B2). $\tilde{\mathcal{F}}$ being the canonical dual of $\mathcal{F}$, we have by (17) $f = \sum_{n\in\mathcal{N}} \langle f, \tilde{f}_n \rangle_{\mathcal{H}} f_n$ for each $f \in \mathcal{H}$, with convergence in $\mathcal{H}$. But for $f \in \mathcal{B}$, we have $\langle f, \tilde{f}_n \rangle_{\mathcal{H}} = \langle f, \tilde{f}_n \rangle_{\mathcal{B}\times\mathcal{B}'}$ and $F^*\tilde{F}f \in \mathcal{B}$ by the boundedness of F^* and $\tilde{F}$, so that $\sum_{n\in\mathcal{N}} \langle f, \tilde{f}_n \rangle_{\mathcal{B}\times\mathcal{B}'} f_n = F^*\tilde{F}f = f$ also converges in $\mathcal{B}$. Hence

$$\|f\|_{\mathcal{B}} = \left\| \sum_{n\in\mathcal{N}} \langle f, \tilde{f}_n \rangle_{\mathcal{B}\times\mathcal{B}'} f_n \right\|_{\mathcal{B}} = \|F^*\tilde{F}f\|_{\mathcal{B}} \lesssim \|\tilde{F}f\|_{\mathcal{B}_d} \lesssim \|f\|_{\mathcal{B}}.$$

■

4 Adaptive Numerical Frame Schemes for Operator Equations

In this section, we want to show how the Gelfand frame setting can be used for the adaptive numerical treatment of elliptic operator equations of the form (1). Unless otherwise stated, we shall always assume that (3) and (4) hold, so that $\mathcal{L}$ is symmetric.

We want to discretize (1) by means of a suitable Gelfand frame for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, where we choose $\mathcal{B} = H$, $\mathcal{H} = L_2$ and $\mathcal{B}' = H'$. Following, e.g., [14], a natural way would be to expand the operator equation with respect to the frame and to convert the problem into an operator equation in ℓ_2. However, then the redundancy of the frame might cause problems in the sense that we might end up with a singular system matrix. Nevertheless, in Theorem 4.2 below we show that this can in fact be handled in practice and that the solution of our operator equation can in principle be computed by a version of the Richardson iteration. It is clear that the resulting scheme is not directly implementable since one has to deal with infinite matrices and vectors. Therefore, following [14, 16, 52], we also show how the scheme can be transformed into an implementable one by using approximated versions of the necessary building blocks. The result is a numerical frame algorithm which is guaranteed to converge.

As already outlined, the analysis presented in this section was inspired by the pioneering work of Stevenson [52]. Nevertheless, there is one essential difference. In [52], the frame is directly constructed for the solution Hilbert space H by identifying H with its dual via the Riesz map. This is clearly a reasonable way, however, in the Gelfand triple setting, it is then not possible to identify also $\mathcal{H} = L_2$ with its dual at the same time, see, e.g., [49] for details.

In this paper, we try to introduce the use of Gelfand frames as a more natural setting, i.e., our frames are constructed in $\mathcal{H}$ but nevertheless give rise to norm equivalences for $\mathcal{B}$ as well as for $\mathcal{B}'$. From the notational viewpoint, we deliberately denote the Hilbert space we are constructing Gelfand frames for with $\mathcal{B}$, because Gelfand frames are much more closely related to Banach than to Hilbert frames, see, e.g., Proposition 3.1.

4.1 A Series Representation of the Solution

In the following, we fix a Gelfand frame $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ with a corresponding Gelfand triple of sequence spaces $\big(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\big)$, where we will always identify $\mathcal{H}$ and $\ell_2(\mathcal{N})$ with their duals $\mathcal{H}'$ and $\ell_2(\mathcal{N})'$, respectively.

Moreover, we shall always assume that there exists an isomorphism $D_{\mathcal{B}} : \mathcal{B}_d \to \ell_2(\mathcal{N})$, so that its $\ell_2(\mathcal{N})$–adjoint $D_{\mathcal{B}}^* : \ell_2(\mathcal{N}) \to \mathcal{B}_d'$ is also an isomorphism.

Lemma 4.1. *Under the assumptions* (3), (4) *on* $\mathcal{L}$, *the operator*

$$\mathbf{G} := (D_{\mathcal{B}}^*)^{-1} F \mathcal{L} F^* D_{\mathcal{B}}^{-1} \tag{25}$$

is a bounded operator from $\ell_2(\mathcal{N})$ *to* $\ell_2(\mathcal{N})$*. Moreover* $\mathbf{G} = \mathbf{G}^*$ *and it is boundedly invertible on its range* $\mathrm{ran}(\mathbf{G}) = \mathrm{ran}((D_{\mathcal{B}}^*)^{-1}F)$.

Proof. Since $\mathbf{G}$ is a composition of bounded operators $D_{\mathcal{B}}^{-1} : \ell_2(\mathcal{N}) \to \mathcal{B}_d$, $F^* : \mathcal{B}_d \to \mathcal{B}$, $\mathcal{L} : \mathcal{B} \to \mathcal{B}'$, $F : \mathcal{B}' \to \mathcal{B}'_d$ and $(D_{\mathcal{B}}^*)^{-1} : \mathcal{B}'_d \to \ell_2(\mathcal{N})$, $\mathbf{G}$ is a bounded operator from $\ell_2(\mathcal{N})$ to $\ell_2(\mathcal{N})$. Moreover, from the decomposition (25) it is clear that

$$\ker(\mathbf{G}) = \ker(F^* D_{\mathcal{B}}^{-1}), \quad \mathrm{ran}(\mathbf{G}) = \mathrm{ran}((D_{\mathcal{B}}^*)^{-1}F). \tag{26}$$

Since $\mathcal{L}$ is symmetric, we have $\mathbf{G} = \mathbf{G}^*$ and the orthogonal decomposition

$$\ell_2(\mathcal{N}) = \ker(F^* D_{\mathcal{B}}^{-1}) \oplus \mathrm{ran}((D_{\mathcal{B}}^*)^{-1}F). \tag{27}$$

Therefore

$$\mathbf{G}_{|\,\mathrm{ran}(\mathbf{G})} : \ \mathrm{ran}(\mathbf{G}) \to \mathrm{ran}(\mathbf{G}) \tag{28}$$

is boundedly invertible. ∎

Theorem 4.2. *Let* $\mathcal{L}$ *satisfy* (3) *and* (4)*. Denote*

$$\mathbf{f} := (D_{\mathcal{B}}^*)^{-1} F f \tag{29}$$

and $\mathbf{G}$ *as in* (25)*. Then the solution* u *of* (1) *can be computed by*

$$u = F^* D_{\mathcal{B}}^{-1} \mathbf{P}\mathbf{u} \tag{30}$$

where $\mathbf{u}$ *solves*

$$\mathbf{P}\mathbf{u} = \left(\alpha \sum_{n=0}^{\infty} (\mathrm{id} - \alpha \mathbf{G})^n \right) \mathbf{f}, \tag{31}$$

with $0 < \alpha < 2/\lambda_{\max}$ *and* $\lambda_{\max} = \|\mathbf{G}\|$*. Here* $\mathbf{P} : \ell_2(\mathcal{N}) \to \mathrm{ran}(\mathbf{G})$ *is the orthogonal projection onto* $\mathrm{ran}(\mathbf{G})$.

Proof. Like in Theorem 3.1, we have $u = \sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathcal{H}} f_n$ in $\mathcal{H}$. Since $\mathcal{F}$ is a Gelfand frame, $F^*\tilde{F} : \mathcal{B} \to \mathcal{B}$ is bounded and implies $u = F^*\tilde{F}u = \sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathcal{B}\times\mathcal{B}'} f_n$ in $\mathcal{B}$. By Proposition 3.1 and using (B3) for $\mathcal{F}$, (1) is equivalent to the following system of equations

$$\sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathcal{B}\times\mathcal{B}'} \langle \mathcal{L} f_n, f_m\rangle_{\mathcal{B}'\times\mathcal{B}} = \langle f, f_m\rangle_{\mathcal{B}'\times\mathcal{B}}, \quad m \in \mathcal{N}. \tag{32}$$

Denote $\mathbf{u} := D_{\mathcal{B}}\tilde{F}u$ and $\mathbf{f}$, $\mathbf{G}$ as in (29) and (25). Then (32) can be rewritten as

$$\mathbf{G}\mathbf{u} = \mathbf{f}. \tag{33}$$

For all $\mathbf{v} \in \ell_2(\mathcal{N})$

$$\langle \mathbf{G}\mathbf{v}, \mathbf{v}\rangle_{\ell_2(\mathcal{N})} = \langle (D_{\mathcal{B}}^*)^{-1} F \mathcal{L} F^* D_{\mathcal{B}}^{-1} \mathbf{v}, \mathbf{v}\rangle_{\ell_2(\mathcal{N})} = \langle \mathcal{L} F^* D_{\mathcal{B}}^{-1} \mathbf{v}, F^* D_{\mathcal{B}}^{-1} \mathbf{v}\rangle_{\mathcal{B}'\times\mathcal{B}}.$$

Since $\mathcal{L}$ is positive, $\mathbf{G}$ is positive semi–definite. Let us denote $\lambda_{\max} := \|\mathbf{G}\|$ and $\lambda^+_{\min} := \|(\mathbf{G}_{|\operatorname{ran}(\mathbf{G})})^{-1}\|^{-1}$. For $0 < \alpha < 2/\lambda_{\max}$, one can consider the operator

$$\mathbf{B} := \alpha \sum_{n=0}^{\infty} (\operatorname{id} - \alpha \mathbf{G})^n. \tag{34}$$

Since $\rho := \|\operatorname{id} - \alpha \mathbf{G}_{|\operatorname{ran}(\mathbf{G})}\| = \max\{\alpha\lambda_{\max} - 1, 1 - \alpha\lambda^+_{\min}\} < 1$, with minimum at $\alpha^* = 2/(\lambda_{\max} + \lambda^+_{\min})$, one has that $\mathbf{B}$ is a well–defined bounded operator on $\operatorname{ran}(\mathbf{G})$. Moreover, it is also clear that

$$\mathbf{B} \circ \mathbf{G}_{|\operatorname{ran}(\mathbf{G})} = \mathbf{G} \circ \mathbf{B}_{|\operatorname{ran}(\mathbf{G})} = \operatorname{id}_{\operatorname{ran}(\mathbf{G})}. \tag{35}$$

Since $\mathbf{G}(\operatorname{id} - \mathbf{P}) = 0$,

$$\mathbf{G}\mathbf{u} = \mathbf{G}\mathbf{P}\mathbf{u} = \mathbf{f}. \tag{36}$$

Therefore $\mathbf{P}\mathbf{u} \in \operatorname{ran}(\mathbf{G})$ is the unique solution of (33) in $\operatorname{ran}(\mathbf{G})$ and by (35)

$$\mathbf{P}\mathbf{u} = \mathbf{B}\mathbf{f}. \tag{37}$$

By construction

$$\begin{aligned}
\langle f, f_m \rangle_{\mathcal{B}' \times \mathcal{B}} &= \langle \tilde{F}^* F f, f_m \rangle_{\mathcal{B}' \times \mathcal{B}} \\
&= \langle \tilde{F}^* D^*_{\mathcal{B}} \mathbf{f}, f_m \rangle_{\mathcal{B}' \times \mathcal{B}} \\
&= \langle \tilde{F}^* D^*_{\mathcal{B}} \mathbf{G}\mathbf{P}\mathbf{u}, f_m \rangle_{\mathcal{B}' \times \mathcal{B}} \\
&= \langle \mathcal{L} F^* D^{-1}_{\mathcal{B}} \mathbf{P}\mathbf{u}, f_m \rangle_{\mathcal{B}' \times \mathcal{B}}, \quad m \in \mathcal{N},
\end{aligned}$$

so that $u = F^* D^{-1}_{\mathcal{B}} \mathbf{P}\mathbf{u}$ solves (1). ■

4.2 Numerical Realization

Now we turn to the numerical treatment of (33). The computation of (31) is nothing but a damped Richardson iteration

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} - \alpha(\mathbf{G}\mathbf{u}^{(i)} - \mathbf{f}), \tag{38}$$

starting with $\mathbf{u}^{(0)} = \mathbf{0}$. Of course this iteration cannot be practically realized from infinite vectors.

In the following, we show that the approaches by fully adaptive schemes presented in [16, 52] to compute approximations of solutions of (33) can also be extended to the case where the matrix $\mathbf{G}$ is computed by a Gelfand frame discretization. Therefore, for the rest of this section, we refer to [16, 52] for details.

Assume that we have the following procedures at our disposal:

- $\mathbf{RHS}[\varepsilon, \mathbf{g}] \to \mathbf{g}_\varepsilon$: determines for $\mathbf{g} \in \ell_2(\mathcal{N})$ a finitely supported $\mathbf{g}_\varepsilon \in \ell_2(\mathcal{N})$ such that

$$\|\mathbf{g} - \mathbf{g}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon; \tag{39}$$

- $\mathbf{APPLY}[\varepsilon, \mathbf{N}, \mathbf{v}] \to \mathbf{w}_\varepsilon$: determines for $\mathbf{N} \in L(\ell_2(\mathcal{N}))$ and for a finitely supported $\mathbf{v} \in \ell_2(\mathcal{N})$ a finitely supported $\mathbf{w}_\varepsilon$ such that

$$\|\mathbf{N}\mathbf{v} - \mathbf{w}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon; \tag{40}$$

- $\mathbf{COARSE}[\varepsilon, \mathbf{v}] \to \mathbf{v}_\varepsilon$: determines for a finitely supported $\mathbf{v} \in \ell_2(\mathcal{N})$ a finitely supported $\mathbf{v}_\varepsilon \in \ell_2(\mathcal{N})$ with at most N significant coefficients, such that

$$\|\mathbf{v} - \mathbf{v}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon. \tag{41}$$

  Moreover, $N \lesssim N_{\min}$ holds, $N_{\min}$ being the minimal number of entries for which (41) is valid.

We will discuss the availability of the routines **RHS**, **APPLY** and **COARSE** later, after the proof of Theorem 4.3. Then we can define the following inexact version of the damped Richardson iteration (38):

Algorithm 1. $\mathbf{SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}] \to \mathbf{u}_\varepsilon$:

Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.
$i := 0,\ \mathbf{v}^{(0)} := 0,\ \varepsilon_0 := \|\mathbf{G}^{-1}_{|\operatorname{ran}(\mathbf{G})}\|\|\mathbf{f}\|_{\ell_2(\mathcal{N})}$
While $\varepsilon_i > \varepsilon$ do
  $i := i + 1$
  $\varepsilon_i := 3\rho^K \varepsilon_{i-1}/\theta$
  $\mathbf{f}^{(i)} := \mathbf{RHS}[\frac{\theta\varepsilon_i}{6\alpha K}, \mathbf{f}]$
  $\mathbf{v}^{(i,0)} := \mathbf{v}^{(i-1)}$
  For $j = 1, ..., K$ do
    $\mathbf{v}^{(i,j)} := \mathbf{v}^{(i,j-1)} - \alpha(\mathbf{APPLY}[\frac{\theta\varepsilon_i}{6\alpha K}, \mathbf{G}, \mathbf{v}^{(i,j-1)}] - \mathbf{f}^{(i)})$
  od
  $\mathbf{v}^{(i)} := \mathbf{COARSE}[(1-\theta)\varepsilon_i, \mathbf{v}^{(i,K)}]$
od
$\mathbf{u}_\varepsilon := \mathbf{v}^{(i)}$.

Note that here, deviating somewhat from the notation in (38), we denote by $\mathbf{v}^{(i)}$ the result after applying K approximate Richardson iterations at a time to $\mathbf{v}^{(i-1)}$.

Theorem 4.3. *In the situation of Theorem 4.2, let $\mathbf{u} \in \ell_2(\mathcal{N})$ be a solution of* (33). *Then $\mathbf{SOLVE}[\varepsilon, \mathbf{G}, \mathbf{f}]$ produces finitely supported vectors $\mathbf{v}^{(i,K)}, \mathbf{v}^{(i)}$ such that*

$$\left\|\mathbf{P}(\mathbf{u} - \mathbf{v}^{(i)})\right\|_{\ell_2(\mathcal{N})} \le \varepsilon_i, \quad i \ge 0. \tag{42}$$

In particular, one has

$$\|u - F^* D_{\mathcal{B}}^{-1}\mathbf{u}_\varepsilon\|_{\mathcal{B}} \le \|F^*\|\|D_{\mathcal{B}}^{-1}\|\varepsilon. \tag{43}$$

Moreover, it holds that

$$\left\|\mathbf{P}\mathbf{u} - (\operatorname{id} - \mathbf{P})\mathbf{v}^{(i-1)} - \mathbf{v}^{(i,K)}\right\|_{\ell_2(\mathcal{N})} \le \frac{2\theta\varepsilon_i}{3}, \quad i \ge 1. \tag{44}$$

Proof. The proof is completely analogous to [52], replacing $\mathbf{Q}$ by $\mathbf{P}$. For $i=0$, (31) and (37) yield

$$\left\|\mathbf{P}(\mathbf{u}-\mathbf{v}^{(0)})\right\|_{\ell_2(\mathcal{N})} = \|\mathbf{Pu}\|_{\ell_2(\mathcal{N})} = \|\mathbf{Bf}\|_{\ell_2(\mathcal{N})} \le \varepsilon_0.$$

Now take $i \ge 1$ and let $\left\|\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i-1)})\right\|_{\ell_2(\mathcal{N})} \le \varepsilon_{i-1}$ hold. We show (44) first. When exactly performing one damped Richardson iteration (38), from, say, some vector $\mathbf{w}^{(i)}$ to $\mathbf{w}^{(i+1)}$, equations (33) and (36) yield

$$\mathbf{Pu}-\mathbf{w}^{(i+1)} = \mathbf{Pu}-\mathbf{w}^{(i)} + \alpha(\mathbf{Gw}^{(i)}-\mathbf{f}) = (\mathrm{id}-\alpha\mathbf{G})(\mathbf{Pu}-\mathbf{w}^{(i)}). \tag{45}$$

So, by induction, the exact application of K damped Richardson iterations at a time would result in

$$\mathbf{Pu}-\mathbf{w}^{(i+K)} = (\mathrm{id}-\alpha\mathbf{G})^K(\mathbf{Pu}-\mathbf{w}^{(i)}). \tag{46}$$

But the loop of Algorithm 1 performs K *perturbed* Richardson iterations $\mathbf{v}^{(i,j)}$, starting from $\mathbf{v}^{(i,0)} = \mathbf{v}^{(i-1)}$. By construction, the stepwise error does not exceed

$$\left\|\mathbf{v}^{(i,j)}-\mathbf{v}^{(i,j-1)}+\alpha(\mathbf{Gv}^{(i,j-1)}-\mathbf{f})\right\|_{\ell_2(\mathcal{N})} \le \alpha\left(\frac{\theta\varepsilon_i}{6\alpha K}+\frac{\theta\varepsilon_i}{6\alpha K}\right) = \frac{\theta\varepsilon_i}{3K},$$

so that after K steps we end up in

$$\left\|\mathbf{Pu}-\mathbf{v}^{(i,K)}-(\mathrm{id}-\alpha\mathbf{G})^K(\mathbf{Pu}-\mathbf{v}^{(i-1)})\right\|_{\ell_2(\mathcal{N})} \le K\frac{\theta\varepsilon_i}{3K} = \frac{\theta\varepsilon_i}{3}. \tag{47}$$

It is straightforward to compute the identity

$$(\mathrm{id}-\alpha\mathbf{G})^K(\mathbf{Pu}-\mathbf{v}^{(i-1)}) = (\mathrm{id}-\alpha\mathbf{G})^K\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i-1)}) - (\mathrm{id}-\mathbf{P})\mathbf{v}^{(i-1)}. \tag{48}$$

But due to the specific choice of the relaxation parameter α in Theorem (4.2), we get

$$\left\|(\mathrm{id}-\alpha\mathbf{G})^K\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i-1)})\right\|_{\ell_2(\mathcal{N})} \le \rho^K\left\|\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i-1)})\right\|_{\ell_2(\mathcal{N})} \le \rho^K\varepsilon_{i-1} = \frac{\theta\varepsilon_i}{3}, \tag{49}$$

which, together with (48), yields (44). Now, by using (44) and the definition of **COARSE** one has

$$\begin{aligned}\|\mathbf{Pu}+(\mathrm{id}-\mathbf{P})\mathbf{v}^{(i-1)}-\mathbf{v}^{(i)}\|_{\ell_2(\mathcal{N})} \le\ & \left\|\mathbf{Pu}+(\mathrm{id}-\mathbf{P})\mathbf{v}^{(i-1)}-\mathbf{v}^{(i,K)}\right\|_{\ell_2(\mathcal{N})} \\ & +\|\mathbf{v}^{(i,K)}-\mathbf{v}^{(i)}\|_{\ell_2(\mathcal{N})} \le \frac{2\theta}{3}\varepsilon_i + (1-\theta)\varepsilon_i \le \varepsilon_i.\end{aligned}$$

Then (42) follows by

$$\begin{aligned}\|\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i)})\|^2_{\ell_2(\mathcal{N})} &\le \|\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i)})\|^2_{\ell_2(\mathcal{N})} + \|(\mathrm{id}-\mathbf{P})(\mathbf{v}^{(i-1)}-\mathbf{v}^{(i)})\|^2_{\ell_2(\mathcal{N})} \\ &= \|\mathbf{P}(\mathbf{u}-\mathbf{v}^{(i)}) + (\mathrm{id}-\mathbf{P})(\mathbf{v}^{(i-1)}-\mathbf{v}^{(i)})\|^2_{\ell_2(\mathcal{N})} \\ &= \|\mathbf{Pu}+(\mathrm{id}-\mathbf{P})\mathbf{v}^{(i-1)}-\mathbf{v}^{(i)}\|^2_{\ell_2(\mathcal{N})}.\end{aligned}$$

Since $\ker(F^* D_{\mathcal{B}}^{-1}) = \ker(\mathbf{G}) = \ker(\mathbf{P})$, we finally verify

$$\begin{aligned} \|u - F^* D_{\mathcal{B}}^{-1} \mathbf{u}_\varepsilon\|_{\mathcal{B}} &= \left\|F^*(\tilde{F}u - D_{\mathcal{B}}^{-1}\mathbf{u}_\varepsilon)\right\|_{\mathcal{B}} \\ &= \left\|F^* D_{\mathcal{B}}^{-1}(\mathbf{u} - \mathbf{u}_\varepsilon)\right\|_{\mathcal{B}} \\ &= \left\|F^* D_{\mathcal{B}}^{-1}\mathbf{P}(\mathbf{u} - \mathbf{u}_\varepsilon)\right\|_{\mathcal{B}} \\ &\leq \|F^*\|\|D_{\mathcal{B}}^{-1}\|\left\|\mathbf{P}(\mathbf{u} - \mathbf{u}_\varepsilon)\right\|_{\ell_2(\mathcal{N})} \\ &\leq \|F^*\|\|D_{\mathcal{B}}^{-1}\|\varepsilon. \end{aligned}$$

■

REMARK: Note that the single iterands $\mathbf{v}^{(i)}$ might not be elements of $\mathrm{ran}(\mathbf{G})$, which is due to the **APPLY** routine. But this has no effect on the convergence of Algorithm 1, up to the prescribed tolerance.

Of course, in concrete numerical realizations, the damped Richardson iteration might exhibit a low convergence rate when the relaxation parameter α is small. This constitutes one of key points for the efficiency of such scheme. Generalizations of Algorithm 1 towards, e.g., conjugate gradient iterations can be realized [16], even if, in some applications [19, 29], such generalizations did not give much better results.

Now clearly the question arises how the basic routines $\mathbf{RHS}[\varepsilon, \mathbf{g}]$, $\mathbf{APPLY}[\varepsilon, \mathbf{N}, \mathbf{v}]$ and $\mathbf{COARSE}[\varepsilon, \mathbf{v}]$ can be realized numerically. We refer, e.g., to [14, 16, 52] for a detailed description. However, for the reader's convenience, at least some remarks concerning $\mathbf{APPLY}[\varepsilon, \mathbf{N}, \mathbf{v}]$ are advisable. To derive a suitable version of $\mathbf{APPLY}[\varepsilon, \mathbf{N}, \mathbf{v}]$, we have to restrict ourselves to compressible matrices. As usual, for $s^* > 0$, a bounded $\mathbf{N} : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$ is called *s^*–compressible,* if for each $j \in \mathbb{N}$ there exist constants α_j and C_j and a matrix $\mathbf{N}_j$ having at most $\alpha_j 2^j$ non–zero entries per column, such that

$$\|\mathbf{N} - \mathbf{N}_j\| \leq C_j, \tag{50}$$

where $(\alpha_j)_{j\in\mathbb{N}}$ is summable, and for any $s < s^*, (C_j 2^{sj})_{j\in\mathbb{N}}$ is summable. For s^*–compressible $\mathbf{N}$, we make use of the following routine **APPLY**:

$\mathbf{APPLY}[\varepsilon, \mathbf{N}, \mathbf{v}] \to \mathbf{w}_\varepsilon$

- $q := \lceil \log((\#\,\mathrm{supp}\,\mathbf{v})^{1/2}\|\mathbf{v}\|_{\ell_2(\mathcal{N})}\|\mathbf{N}\|2/\varepsilon)\rceil$.
- *Divide the elements of* $\mathbf{v}$ *into sets* $V_0, \ldots, V_q$*, where for* $0 \leq i \leq q-1, V_i$ *contains the elements with modulus in* $(2^{-i-1}\|\mathbf{v}\|_{\ell_2(\mathcal{N})}, 2^{-i}\|\mathbf{v}\|_{\ell_2(\mathcal{N})})$ *and possible remaining elements are put into* V_q*.*
- *For* $k = 0, 1, \ldots,$ *generate vectors* $\mathbf{v}_{[k]}$ *by subsequently extracting* $2^k - \lfloor 2^{k-1} \rfloor$ *elements from* $\cup_i V_i$*, starting from* V_0 *and when it is empty continuing with* V_i

and so forth, until for some $k = l$ either $\cup_i V_i$ becomes empty or

$$\|\mathbf{N}\| \left\| \mathbf{v} - \sum_{k=0}^{l} \mathbf{v}_{[k]} \right\|_{\ell_2(\mathcal{N})} \leq \varepsilon/2. \tag{51}$$

In both cases, $\mathbf{v}_{[l]}$ may contain less than $2^l - \lfloor 2^{l-1} \rfloor$ elements.

- *Compute the smallest $j \geq l$ such that*

$$\sum_{k=0}^{l} C_{j-k} \|\mathbf{v}_{[k]}\|_{\ell_2(\mathcal{N})} \leq \varepsilon/2. \tag{52}$$

- *For $0 \leq k \leq l$, compute the non–zero entries in the matrices $\mathbf{N}_{j-k}$ which have a column index in common with one of the entries of $\mathbf{v}_{[k]}$ and compute*

$$\mathbf{w}_\varepsilon := \sum_{k=0}^{l} \mathbf{N}_{j-k} \mathbf{v}_{[k]}. \tag{53}$$

For an estimation of the computational complexity and the storage requirements of Algorithm 1, we have to introduce the *weak ℓ_τ spaces* $\ell_\tau^w(\mathcal{N})$. Given some $0 < \tau < 2$, $\ell_\tau^w(\mathcal{N})$ is defined as

$$\ell_\tau^w(\mathcal{N}) := \{\mathbf{c} \in \ell_2(\mathcal{N}) : \ |\mathbf{c}|_{\ell_\tau^w(\mathcal{N})} := \sup_{n \in \mathbb{N}} n^{1/\tau} |\gamma_n(\mathbf{c})| < \infty\}, \tag{54}$$

where $\gamma_n(\mathbf{c})$ is the nth largest coefficient in modulus of $\mathbf{c}$. We refer to [14, 31] for further details on the quasi–Banach spaces $\ell_\tau^w(\mathcal{N})$.

Theorem 4.4. *Assume that for some $s^* > 0$, $\mathbf{G}$ is s^*–compressible and that for some $s \in (0, s^*)$ and $\tau = (1/2 + s)^{-1}$, $\mathbf{G}\mathbf{u} = \mathbf{f}$ has a solution $\mathbf{u}$ in $\ell_\tau^w(\mathcal{N})$. Moreover, assume that $\mathbf{f}$ is s^*–optimal in the sense that for a suitable routine* **RHS** *for each $s \in (0, s^*)$ and all $\varepsilon > 0$ with $\mathbf{f}_\varepsilon :=$ **RHS**$[\varepsilon, \mathbf{f}]$ the following is valid:*

(I) $\# \operatorname{supp} \mathbf{f}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{f}|_{\ell_\tau^w(\mathcal{N})}^{1/s}$,

(II) the number of arithmetic operations used to compute it is at most a multiple of $\varepsilon^{-1/s} |\mathbf{f}|_{\ell_\tau^w(\mathcal{N})}^{1/s}$.

In addition, assume that there exists an $\tilde{s} \in (s, s^*)$ such that with $\tilde{\tau} = (1/2 + \tilde{s})^{-1}$, $\mathbf{P}$ is bounded on $\ell_{\tilde{\tau}}^w(\mathcal{N})$. Then, if the parameter K in **SOLVE** *is sufficiently large, for all $\varepsilon > 0$, $\mathbf{u}_\epsilon :=$* **SOLVE**$[\varepsilon, \mathbf{G}, \mathbf{f}]$ *satisfies*

(I) $\# \operatorname{supp} \mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s} |\mathbf{u}|_{\ell_\tau^w(\mathcal{N})}^{1/s}$

(II) *the number of arithmetic operations used to compute* $\mathbf{u}_\epsilon$ *is at most a multiple of* $\varepsilon^{-1/s}|\mathbf{u}|_{\ell_\tau^w(\mathcal{N})}^{1/s}$.

REMARK: In practical applications, one has to check the s^*–compressibility of $\mathbf{G}$ and especially of $\mathbf{P}$ in detail, which might be a problem, cf. [52, Section 4.3]. In Section 7, we present an application of Algorithm 1 where $\mathbf{G}$ and $\mathbf{P} = \mathbf{Q}$ are in fact s^*–compressible.

4.3 Compressible Matrices

In the previous section, it has turned out that the applicability of the fundamental Algorithm 1 essentially relies on the compressibility properties of the resulting stiffness matrices. Therefore, in this section, we want to introduce two classes of off–diagonal decaying matrices which are in fact s^*–compressible and usually appearing in applications, for instance, in signal and image processing, numerical analysis and simulation.

- The *Jaffard class* is defined as the class of matrices $\mathbf{N} = (n_{k,l})_{k,l\in\mathcal{N}}$, such that

$$|n_{k,l}| \lesssim \big(1 + \|k-l\|_{\mathbb{R}^d}\big)^{-r} \text{ for all } k,l \in \mathcal{N}, \text{ and } r > d,$$

where $\mathcal{N} \subset \mathbb{R}^d$ is assumed to be *separated*, i.e.,

$$\inf_{\substack{k\neq l,\\ k,l\in\mathcal{N}}} \|k-l\|_{\mathbb{R}^d} > \delta > 0.$$

As we will discuss in Section 7, the Jaffard class turns out to be very useful for applications of Algorithm 1 to the efficient computation of canonical dual frames and in the solution of more general operator equations. For example, suitable *brushlet systems* have been constructed by Borup and Nielsen [8] as unconditional bases for α–modulation spaces [40]. Borup showed in [7] that such systems discretize *pseudo-differential operators* in Hörmander classes into Jaffard class matrices $\mathbf{G}$ with nice polynomial off–diagonal decay, see [7, Proposition 3.2]. Alternatively, one can use α–Gabor–wavelet frames as introduced in [34, 40] for the discretization of such operators. Recently, connections between pseudodifferential operators and time–frequency analysis [45] have been recognized, with relevant applications in signal processing and transmission, radar technology, and wireless communication [37, 38].

Proposition 4.5. *Let* $\mathbf{N}$ *be a matrix in the Jaffard class and*

$$|n_{k,l}| \lesssim \big(1 + \|k-l\|_{\mathbb{R}^d}\big)^{-\eta} \textit{ for all } k,l \in \mathcal{N},$$

where $\eta/2 > r > d$. *Then the matrix* $\mathbf{N}_j = (n_{k,l}^{(j)})_{k,l\in\mathcal{N}}$ *given by*

$$n_{k,l}^{(j)} := \begin{cases} 0 & , \ \|k-l\|_{\mathbb{R}^d} > \alpha_j 2^j \\ n_{k,l} & , \ \textit{otherwise} \end{cases},$$

where $(\alpha_j)_{j\in\mathbb{N}}$ *is a positive summable sequence, is such that*

$$\|\mathbf{N}-\mathbf{N}_j\| \lesssim \alpha_j^{d-r}2^{(d-r)j} \text{ for all } j\in\mathbb{N}. \tag{55}$$

In particular, $\mathbf{N}$ *is* s^**–compressible for* $s^* = r-d$.

Proof. We want to use Schur's lemma 8.1. Let us choose as weight $w_l = \big(1+\|l\|_{\mathbb{R}^d}\big)^{-r}$, and denote $A_1 = \{l \in \mathcal{N} : \|l-k\|_{\mathbb{R}^d} \le \frac{\|k\|_{\mathbb{R}^d}}{2}\}$ and $A_2 = \mathcal{N}\backslash A_1$. If $l\in A_1$, then $\|l\|_{\mathbb{R}^d} \ge \frac{\|k\|_{\mathbb{R}^d}}{2}$ and

$$\begin{aligned}
\sum_{l\in A_1} |n_{k,l}-n_{k,l}^{(j)}|w_l &\lesssim \sum_{\substack{l\in A_1,\\ \|l-k\|_{\mathbb{R}^d}>\alpha_j 2^j}} \big(1+\|l-k\|_{\mathbb{R}^d}\big)^{-\eta}\big(1+\|l\|_{\mathbb{R}^d}\big)^{-r}\\
&\le \left(1+\frac{\|k\|_{\mathbb{R}^d}}{2}\right)^{-r} \sum_{\|l-k\|_{\mathbb{R}^d}>\alpha_j 2^j} \big(1+\|l-k\|_{\mathbb{R}^d}\big)^{-\eta}\\
&\lesssim \big(1+\|k\|_{\mathbb{R}^d}\big)^{-r}\int_{\|\xi\|_{\mathbb{R}^d}>\alpha_j 2^j}\big(1+\|\xi\|_{\mathbb{R}^d}\big)^{-\eta}d\xi\\
&\lesssim w_k\alpha_j^{d-\eta}2^{(d-\eta)j}.
\end{aligned}$$

If $l\in A_2$ then $\|l-k\|_{\mathbb{R}^d} > \frac{\|k\|_{\mathbb{R}^d}}{2}$ and

$$\begin{aligned}
\sum_{l\in A_2} |n_{k,l}-n_{k,l}^{(j)}|w_l &\lesssim \sum_{\substack{l\in A_2,\\ \|l-k\|_{\mathbb{R}^d}>\alpha_j 2^j}} \big(1+\|l-k\|_{\mathbb{R}^d}\big)^{-\eta}\big(1+\|l\|_{\mathbb{R}^d}\big)^{-r}\\
&\le \left(1+\frac{\|k\|_{\mathbb{R}^d}}{2}\right)^{-\eta} \sum_{\|l-k\|_{\mathbb{R}^d}>\alpha_j 2^j} \big(1+\|l\|_{\mathbb{R}^d}\big)^{-r}.
\end{aligned}$$

The assumption $\eta > 2r$ implies

$$\begin{aligned}
\sum_{l\in A_1} |n_{k,l}-n_{k,l}^{(j)}|w_l &\lesssim \big(1+\|k\|_{\mathbb{R}^d}\big)^{r-\eta} \sum_{\|l-k\|_{\mathbb{R}^d}>\alpha_j 2^j} \big(1+\|l-k\|_{\mathbb{R}^d}\big)^{-r}\\
&\lesssim \big(1+\|k\|_{\mathbb{R}^d}\big)^{-r}\int_{\|\xi\|_{\mathbb{R}^d}>\alpha_j 2^j}\big(1+\|\xi\|_{\mathbb{R}^d}\big)^{-r}d\xi\\
&\lesssim w_k\alpha_j^{d-r}2^{(d-r)j}.
\end{aligned}$$

Since $\eta > r$, we have $\alpha_j^{d-\eta}2^{(d-\eta)j} \le \alpha_j^{d-r}2^{(d-r)j}$ and

$$\sum_l |n_{k,l}-n_{k,l}^{(j)}|w_l \lesssim w_k\alpha_j^{d-r}2^{(d-r)j}, \text{ for all } k\in\mathcal{N}, j\in\mathbb{N}.$$

In the same way one can show that

$$\sum_k |n_{k,l}-n_{k,l}^{(j)}|w_k \lesssim w_l\alpha_j^{d-r}2^{(d-r)j}, \text{ for all } l\in\mathcal{N}, j\in\mathbb{N}.$$

By Schur's lemma 8.1, one has $\|\mathbf{N}-\mathbf{N}_j\| \lesssim \alpha_j^{d-r} 2^{(d-r)j}$. The s^*–compressibility of $\mathbf{N}$ is obvious. ■

Another important class of off–diagonal decaying matrices is the Lemarié class. This kind of matrices typically arises in the discretization of non–local operators with Schwartz kernels satisfying (8) by using wavelet frames.

- The *Lemarié class* is defined as the class of matrices $\mathbf{N} = (n_{\lambda,\lambda'})_{\lambda,\lambda'\in\mathcal{J}}$, such that
$$|n_{\lambda,\lambda'}| \lesssim 2^{-s||\lambda|-|\lambda'||}\big(1+\delta(\lambda,\lambda')\big)^{-r} \text{ for all } \lambda,\lambda' \in \mathcal{J}. \tag{56}$$
Here we require $r > d$ and $s > d/2$, d is the *spatial dimension* and the index $\lambda \in \mathcal{J}$ typically encodes several types of information simultaneously, namely the *scale* often denoted by $|\lambda| \in \mathbb{Z}$ and the *spatial localization*, see [14] for more details on the notation. Furthermore, we assume that $\delta : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ fulfills the following properties:

  (a) $\sum_{|\lambda'|=j'} (1+\delta(\lambda,\lambda'))^{-r} \lesssim 2^{d\max\{0,j'-|\lambda|\}}$, for all $\lambda \in \mathcal{J}$,

  (b) $\sum_{\{\lambda':\delta(\lambda,\lambda')>R\}} (1+\delta(\lambda,\lambda'))^{-r} \lesssim R^{-r+d} 2^{d\max\{0,|\lambda'|-|\lambda|\}}$, for all $\lambda \in \mathcal{J}$ and $R > 0$,

  (c) $\#\{\lambda' \in \mathcal{J} : \delta(\lambda,\lambda') \le R\} \lesssim R^d 2^{d||\lambda|-|\lambda'||}$, for all $\lambda \in \mathcal{J}$ and any $R > 0$.

A typical example of a function δ fulfilling (a)–(c) is given by

$$\delta(\lambda,\lambda') = 2^{\min\{|\lambda|,|\lambda'|\}} \operatorname{dist}(\operatorname{supp}\psi_\lambda, \operatorname{supp}\psi_{\lambda'}),$$

where $\{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ is a wavelet system.

In Section 6, we construct suitable Gelfand wavelet frames on domains which ensure that the stiffness matrices corresponding to elliptic differential operators are contained in the Lemarié class. We also refer to [14, 16, 50] for the relevant and related literature. The s^*–compressibility of Lemarié class matrices, depending on the parameters s, r, is discussed in [14, Section 2.4, Proposition 3.4, Corollary 3.7]. In particular, one has the following:

Proposition 4.6. *Suppose that $s > d/2$, $r > d$ and $\mathbf{N} = (n_{\lambda,\lambda'})_{\lambda,\lambda'\in\mathcal{J}}$ satisfies* (56). *Then $\mathbf{N}$ is s^*–compressible, where*

$$s^* := \min\left\{\frac{s}{d}-\frac{1}{2}, \frac{r}{d}-1\right\}. \tag{57}$$

5 Localization of Frames and Gelfand Frames

As we have discussed in the previous sections, given an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}'$ which satisfies our basic ellipticity assumptions, one should choose a suitable Gelfand frame for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ such that the corresponding matrix $\mathbf{G}$ can exhibit compressibility properties, maybe belonging to one of the off–diagonal decay classes illustrated in Section 4. This ensures that Algorithm 1 can work properly and maybe converges with optimal complexity.

The point is that already the extension of a Hilbert frame for $\mathcal{H}$ to a Gelfand frame for $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ is not a trivial problem, especially when one is dealing with highly unstructured situations or complex geometries. For this it is important to have a quite flexible and general machinery to ensure the Gelfand property also for these cases. For example, we will discuss in Section 6 the construction of Gelfand wavelet type frames for domains. In order to preserve all the freedom that frames can ensure, we will not exploit any additional structure, e.g., an underlying multiresolution analysis.

In this section, we illustrate a very general method for ensuring that suitable frames for the Hilbert space $\mathcal{H}$ indeed extend to Gelfand frames for the class of associated Banach spaces $\mathcal{B} = \mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}})$ of the functionals that admit Banach frame expansions with canonical dual coefficients in $\ell_{p,w}(\mathcal{N})$. Such abstract machinery can be concretely applied on a very large class of relevant frames appearing, for instance, in several problems of signal processing, for example, Gabor frames [37, 38, 45] and more general α–Gabor–wavelet frames [34, 39, 40]. In the next section, we will modify the approach to treat also the more complicated case of wavelets on domains and manifolds. Especially, our goal is to prove the following theorem:

Theorem 5.1. *Let $\mathcal{F}$ be an $\mathcal{A}$–self–localized frame for $\mathcal{H}$, i.e., a frame for which the corresponding Gramian matrix belongs to a suitable Banach algebra of matrices $\mathcal{A}$. Assume that w is an m–moderate weight (for a suitable choice of m as we will discuss in the following), and that $\big(\ell_{p,w}(\mathcal{N}), \ell_2(\mathcal{N}), \ell_{p',1/w}(\mathcal{N})\big)$, $1/p + 1/p' = 1$ for some $1 \leq p \leq \infty$, is a Gelfand triple. Then $\mathcal{F}$ is a Gelfand frame for the Gelfand triple $(\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}}), \mathcal{H}, \mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}})')$.*

Some preparations are necessary. In Subsection 5.1, we discuss the localization properties of frames. Especially, the concept of $\mathcal{A}$–self–localized frames is introduced. Then, in Subsection 5.2, we show that $\mathcal{A}$–self–localized frames give rise to characterizations of associated Banach spaces in a very natural way. In particular, the spaces $\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}})$ and $\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}})'$ are introduced and discussed.

5.1 Localization Properties

In the following $\mathcal{N}$ and $\mathcal{X}$ denote index sets taken in $\mathbb{R}^d$. We assume that all index sets are *relatively separated*, this means that, for all $k \in \mathbb{Z}^d$,

$$\sup_{k \in \mathbb{Z}^d} \operatorname{card}\big(\mathcal{N} \cap (k + [0,1]^d)\big) := \nu < \infty\,.$$

We want to recall here a concept of mutual localization of two frames measured by their (cross–)Gramian matrix belonging to a Banach $*$–algebra $\mathcal{A}$ of matrices which is inverse–closed in $L(\ell_2(\mathcal{N}))$. The theory of localized frames with respect to an algebra has been introduced in [48] and further developed in [42]. We also refer to [17, 44, 46, 47] for relevant and related papers on localization of frames. In particular, in [42], it has been shown that a localized frame can extend to a Banach frame in a natural way for a large family of Banach spaces together with its canonical dual. As we shall see, this will be useful as a tool for constructing Gelfand frames. In the literature there have already been developed other tools to extend Riesz bases to unconditional bases of Banach spaces. These techniques were based, for example, on multilevel nested sequences of spaces satisfying Jackson and Bernstein inequalities, see, e.g., [22, 23]. The localization of frames theory proves to be another very useful and flexible tool to extend frames to Gelfand frames.

In the following, $\mathcal{A}$ is a Banach $*$–algebra of infinite matrices indexed by $\mathcal{N} \times \mathcal{N}$ with the following properties:

(A0) $\mathcal{A} \subseteq L(\ell_2(\mathcal{N}))$, i.e., each $\mathbf{A} \in \mathcal{A}$ defines a bounded operator on $\ell_2(\mathcal{N})$;

(A1) if $\mathbf{A} \in \mathcal{A}$ is invertible on $\ell_2(\mathcal{N})$, then $\mathbf{A}^{-1} \in \mathcal{A}$ as well. In the language of Banach algebras, $\mathcal{A}$ is called *inverse–closed* in $L(\ell_2(\mathcal{N}))$;

(A2) $\mathcal{A}$ is *solid*: i.e., if $\mathbf{A} \in \mathcal{A}$ and $|b_{kl}| \leq |a_{kl}|$ for all $k, l \in \mathcal{N}$, then $\mathbf{B} \in \mathcal{A}$ as well, and $\|\mathbf{B}\| \leq \|\mathbf{A}\|$.

In the sequel we will call a Banach $*$–algebra $\mathcal{A}$ satisfying properties (A0)–(A2) a *solid spectral matrix algebra*, or, for brevity, simply a *spectral algebra*.

There are many examples of spectral algebras such as the Jaffard class introduced in Section 4.3. We refer to [48] for further information where a characterization of a large class of spectral algebras is presented. The results have been proved there for infinite matrices indexed by $\mathbb{Z}^d \times \mathbb{Z}^d$, but they can be generalized to separated set of indices. A submultiplicative and radial symmetric weight m is called $\mathcal{A}$*–admissible* if

(W) any matrix $\mathbf{A} \in \mathcal{A}$ extends to a bounded operator $\mathbf{A} : \ell_{p,w}(\mathcal{N}) \to \ell_{p,w}(\mathcal{N})$ for all $1 \leq p \leq \infty$ and for all m–moderate weights w.

We assume in the following that m is an $\mathcal{A}$–admissible weight and $\mathcal{A}$ is a solid spectral matrix algebra.

By means of the algebra $\mathcal{A}$, we can now introduce a general localization principle, i.e., we can consider $\mathcal{A}$–localized frames. Given two frames $\mathcal{F} = \{f_x\}_{x \in \mathcal{N}}$ and $\mathcal{G} = \{g_y\}_{y \in \mathcal{N}}$ for the Hilbert space $\mathcal{H}$, the (cross–) Gramian matrix $G = G(\mathcal{G}, \mathcal{F})$ of $\mathcal{G}$ with respect to $\mathcal{F}$ is the $\mathcal{N} \times \mathcal{N}$-matrix with entries

$$g_{x,y} = \langle g_x, f_y \rangle_{\mathcal{H}}.$$

A frame $\mathcal{G}$ for $\mathcal{H}$ is called $\mathcal{A}$*–localized* with respect to another frame $\mathcal{F}$ if $G(\mathcal{G}, \mathcal{F}) \in \mathcal{A}$. In this case we write $\mathcal{G} \sim_{\mathcal{A}} \mathcal{F}$. If $\mathcal{G} \sim_{\mathcal{A}} \mathcal{G}$, then $\mathcal{G}$ is called $\mathcal{A}$*–self–localized* or *intrinsically* $\mathcal{A}$*–localized*. In particular, the following theorem has been proved in [42]:

Theorem 5.2. *Each* $\mathcal{A}$*–self–localized frame* $\mathcal{G}$ *has an* $\mathcal{A}$*–self–localized canonical dual.*

5.2 Associated Banach Spaces

In this subsection, we want to show that $\mathcal{A}$–self–localized frames can characterize suitable families of Banach spaces in a natural way. Let $\mathcal{F}$, $\tilde{\mathcal{F}}$ be two mutually dual $\mathcal{A}$–self–localized frames for $\mathcal{H}$ and assume that $\ell_{p,w}(\mathcal{N}) \subset \ell_2(\mathcal{N})$. Then the Banach space $\mathcal{H}_p^w$ is defined to be

$$\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}}) := \left\{ f \in \mathcal{H} : \ f = \sum_{n \in \mathcal{N}} \langle f, \tilde{f}_n \rangle_{\mathcal{H}} f_n, \ \left(\langle f, \tilde{f}_n \rangle_{\mathcal{H}} \right)_{n \in \mathcal{N}} \in \ell_{p,w}(\mathcal{N}) \right\} \tag{58}$$

with the norm $\|f\|_{\mathcal{H}_p^w} = \left\| \left(\langle f, \tilde{f}_n \rangle_{\mathcal{H}} \right)_{n \in \mathcal{N}} \right\|_{\ell_{p,w}(\mathcal{N})}$ and $1 \leq p \leq \infty$. Since $\ell_{p,w}(\mathcal{N}) \subset \ell_2(\mathcal{N})$, $\mathcal{H}_p^w$ is a dense subspace of $\mathcal{H}$. If $\ell_{p,w}(\mathcal{N})$ is not included in $\ell_2(\mathcal{N})$ and $1 \leq p < \infty$, then we define $\mathcal{H}_p^w$ to be the completion of the subspace $\mathcal{H}_0$ of all finite linear combinations in $\mathcal{F}$ with respect to the norm $\|f\|_{\mathcal{H}_p^w} = \left\| \left(\langle f, \tilde{f}_n \rangle_{\mathcal{H}} \right)_{n \in \mathcal{N}} \right\|_{\ell_{p,w}(\mathcal{N})}$. If $p = \infty$, then we take the weak*–completion of $\mathcal{H}_0$ to define $\mathcal{H}_\infty^w$. The definition of $\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}})$ does not depend on the particular $\mathcal{A}$–self localized dual chosen, and any other $\mathcal{A}$–self–localized frame $\mathcal{G}$ which is $\mathcal{A}$-localized to $\mathcal{F}$ generates in fact the same spaces, cf. [42].

The next step is to show that $\mathcal{A}$–self–localized frames $\mathcal{F}$ extend to Banach frames. To do that, according to the definition of a Banach frame, we have to embed $\mathcal{F}$ and its canonical dual $\tilde{\mathcal{F}}$ into a suitable space of continuous functionals. Since $\mathcal{H}_1^m$ is continuously and densely embedded into $\mathcal{H}$, one has the following continuous inclusions

$$\mathcal{H}_1^m \subset \mathcal{H} \simeq \mathcal{H}' \subset (\mathcal{H}_1^m)'. \tag{59}$$

The following characterization of the spaces $\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}})$ has been proved in [42]:

Theorem 5.3. *Let $\mathcal{F}$ be an $\mathcal{A}$–self–localized frame for $\mathcal{H}$. Then the abstract Banach space $\mathcal{H}_p^w$ from (58) can be described as*

$$\mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}}) \simeq \left\{ f \in (\mathcal{H}_1^m)' : \ f = \sum_{n \in \mathcal{N}} \langle f, \tilde{f}_n \rangle_{(\mathcal{H}_1^m)' \times \mathcal{H}_1^m} f_n, \ \left(\langle f, \tilde{f}_n \rangle_{(\mathcal{H}_1^m)' \times \mathcal{H}_1^m} \right)_{n \in \mathcal{N}} \in \ell_{p,w}(\mathcal{N}) \right\} \tag{60}$$

with the norm $\|f\|_{\mathcal{H}_p^w} = \left\| \left(\langle f, \tilde{f}_n \rangle_{(\mathcal{H}_1^m)' \times \mathcal{H}_1^m} \right)_{n \in \mathcal{N}} \right\|_{\ell_{p,w}}$. The convergence of the series in (60) is unconditional for $1 \leq p < \infty$ and the series are convergent in the sense of the norm of $(\mathcal{H}_1^w)'$ for $p = \infty$. In particular, the linear operators

$$\begin{aligned} F^* &: \ell_{p,w}(\mathcal{N}) \to \mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}}), \quad F^* \mathbf{c} = \sum_{n \in \mathcal{N}} c_n f_n, \\ \tilde{F} &: \mathcal{H}_p^w(\mathcal{F}, \tilde{\mathcal{F}}) \to \ell_{p,w}(\mathcal{N}), \quad \tilde{F} f = \left(\langle f, \tilde{f}_n \rangle_{(\mathcal{H}_1^m)' \times \mathcal{H}_1^m} \right)_{n \in \mathcal{N}} \end{aligned} \tag{61}$$

and

$$\tilde{F}^* : \ell_{p,w}(\mathcal{N}) \to \mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}}), \quad \tilde{F}^*\mathbf{c} = \sum_{n\in\mathcal{N}} c_n \tilde{f}_n,$$
$$F : \mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}}) \to \ell_{p,w}(\mathcal{N}), \quad Ff = \big(\langle f, f_n\rangle_{(\mathcal{H}_1^m)'\times\mathcal{H}_1^m}\big)_{n\in\mathcal{N}} \tag{62}$$

are bounded. $\mathcal{F}$ *and* $\tilde{\mathcal{F}}$ *are Banach frames for* $\mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}})$*.*

5.3 Proof of Theorem 5.1

We start showing that if $\big(\ell_{p,w}(\mathcal{N}), \ell_2(\mathcal{N}), \ell_{p',1/w}(\mathcal{N})\big)$, $1/p + 1/p' = 1$, is a Gelfand triple, then $\big(\mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}}), \mathcal{H}, \mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}})'\big)$ is also a Gelfand triple. To this end, we have in particular to prove that $\mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}})' = \mathcal{H}_{p'}^{1/w}(\mathcal{F},\tilde{\mathcal{F}})$.

Let us show first $(\mathcal{H}_p^w)' \subset \mathcal{H}_{p'}^{1/w}$. Since $\mathcal{H}_1^m \subset \mathcal{H}_p^w$ densely, then $(\mathcal{H}_p^w)' \subset (\mathcal{H}_1^m)'$ and $\theta(\varphi) = \langle \theta, \varphi\rangle_{(\mathcal{H}_p^w)'\times\mathcal{H}_p^w}$ for all $\theta \in (\mathcal{H}_p^w)'$ and $\varphi \in \mathcal{H}_1^m$. By (62) $f = \sum_{n\in\mathcal{N}} c_n \tilde{f}_n$, $\mathbf{c} \in \ell_{p,w}(\mathcal{N})$ implies $f \in \mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}})$ and

$$\infty > \big|\theta(f)\big| = \left|\sum_{n\in\mathcal{N}} c_n \langle \theta, \tilde{f}_n\rangle_{(\mathcal{H}_p^w)'\times\mathcal{H}_p^w}\right|.$$

The dual of $\ell_{p,w}(\mathcal{N})$ coincides with its *Köthe dual*

$$\big(\ell_{p,w}(\mathcal{N})\big)^\alpha := \{\mathbf{a} \in (\ell_1(\mathcal{N}))_{\mathrm{loc}} : \mathbf{ac} \in \ell_1(\mathcal{N}),\ \text{for all } \mathbf{c} \in \ell_{p,w}(\mathcal{N})\}$$

for all $1 < p < \infty$, and $\big(\ell_{p,w}(\mathcal{N})\big)^\alpha = \ell_{p',w}(\mathcal{N})$ for $p = 1$ or $p = \infty$, see, e.g., [54]. Therefore one has $\big(\langle \theta, \tilde{f}_n\rangle_{(\mathcal{H}_p^w)'\times\mathcal{H}_p^w}\big)_{n\in\mathcal{N}} \in \ell_{p',1/w}(\mathcal{N})$, where

$$\|\theta\|_{(\mathcal{H}_p^w)'} \sim \left\|\big(\langle \theta, \tilde{f}_n\rangle_{(\mathcal{H}_p^w)'\times\mathcal{H}_p^w}\big)_{n\in\mathcal{N}}\right\|_{\ell_{p',1/w}(\mathcal{N})},$$

and, since $\mathcal{H}_0$ is dense in $(\mathcal{H}_p^w)'$, one has $\theta = \sum_{n\in\mathcal{N}} \langle \theta, \tilde{f}_n\rangle_{(\mathcal{H}_p^w)'\times\mathcal{H}_p^w} f_n$. This implies that $(\mathcal{H}_p^w)' \subset \mathcal{H}_{p'}^{1/w}$. Conversely, if $\theta \in \mathcal{H}_{p'}^{1/w}$, one defines the action of θ on $\mathcal{H}_1^m \subset \mathcal{H}_p^w$ by $\theta(\varphi) = \sum_n \langle \theta, \tilde{f}_n\rangle_{(\mathcal{H}_1^m)'\times\mathcal{H}_1^m} \langle f_n, \varphi\rangle_{\mathcal{H}}$, $\varphi \in \mathcal{H}_1^m$ and, observing that θ in fact extends to a unique element in $(\mathcal{H}_p^w)'$ by density, one has $\mathcal{H}_{p'}^{1/w} \subseteq (\mathcal{H}_p^w)'$.

By Theorem 5.3 one immediately has that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are Gelfand frames for $\big(\mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}}), \mathcal{H}, \mathcal{H}_p^w(\mathcal{F},\tilde{\mathcal{F}})'\big)$. This concludes the proof. □

Example 1. For any function f on $\mathbb{R}^d$ write

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i\omega t} f(t), \tag{63}$$

the *translation* and *modulation operators*. Their combination

$$\pi(\xi) = M_\omega T_x \quad \text{for} \quad \xi = (x,\omega) \in \mathbb{R}^{2d} \tag{64}$$

is called a *time–frequency shift*. Let $\mathcal{X}$ be a relatively separated set in the time–frequency plane $\mathbb{R}^{2d}$ and let $g \in L_2(\mathbb{R}^d)$ be a fixed analyzing function. If the sequence $\mathcal{F} = \mathcal{G}(g, \mathcal{X}) = \{\pi(\xi)g\}_{\xi\in\mathcal{X}}$ is a frame for $L_2(\mathbb{R}^d)$, then it is called a *Gabor frame* if $\mathcal{X}$ is a regular lattice, a *non–uniform* or *irregular Gabor frame* otherwise.

Consider in the following $\mathcal{A} = \mathcal{A}_r$, $r > d$, the Jaffard inverse–closed Banach $*$–algebra and an r–moderate weight w. In this situation, an $\mathcal{A}_r$–localized frame $\mathcal{F}$ is called *r–localized*. Let us fix $w_t(\omega) = \big(1 + \|\omega\|_{\mathbb{R}^d}\big)^t$ for any $|t| \leq r$. If $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$ generates a Gabor frame $\mathcal{F} = \mathcal{G}(g, \mathcal{X})$, then, for any $r > d$, $\mathcal{F}$ is intrinsically r-localized and has an intrinsically r-localized canonical dual $\tilde{\mathcal{F}} = \{\tilde{e}_\xi\}_{\xi\in\mathcal{X}}$. Moreover, it has been shown in [42] that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are Banach frames for suitable classes of *modulation spaces* and in particular for any L_2-Sobolev space $H^t(\mathbb{R}^d)$, $t \in \mathbb{R}$. This means that

- the frame expansions

$$f = \sum_{\xi\in\mathcal{X}} \langle f, \tilde{e}_\xi\rangle \pi(\xi)g = \sum_{\xi\in\mathcal{X}} \langle f, \pi(\xi)g\rangle \tilde{e}_\xi, \tag{65}$$

  converge unconditionally in $H^t(\mathbb{R}^d)$;

- $H^t(\mathbb{R}^d)$ can be characterized by the frame coefficients as follows:

$$\|f\|_{H^t} \sim \left\| \big(\langle f, \tilde{e}_\xi\rangle\big)_{\xi\in\mathcal{X}} \right\|_{\ell_{2,w_t}(\mathcal{X})} \sim \left\| \big(\langle f, \pi(\xi)g\rangle\big)_{\xi\in\mathcal{X}} \right\|_{\ell_{2,w_t}(\mathcal{X})}. \tag{66}$$

Therefore the spaces $\mathcal{H}_2^{w_t}(\mathcal{F}, \tilde{\mathcal{F}})$ and $H^t(\mathbb{R}^d)$ in fact coincide with equivalent norms. This implies also that $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are Gelfand frames for (H^t, L_2, H^{-t}), $t \geq 0$.

6 Wavelet Gelfand Frames on Domains

In this section, we want to come back to the original motivation of this paper, namely the adaptive treatment of elliptic operator equations by means of frame algorithms. To this end, it is clearly necessary to construct Gelfand frames of wavelet type on a bounded domain $\Omega \subset \mathbb{R}^d$. It turns out that for the construction of such frames, we have to generalize the Lemarié localization concept from Section 4, and to modify the strategy illustrated in Section 5, mainly because the corresponding matrix class with exponential off–diagonal decay is not inverse–closed and thus not a spectral algebra. We introduce and analyze the new localization concept in Subsection 6.1. In Subsection 6.2, we present an explicit construction of wavelet Gelfand frames on Ω which fit into the new localization setting. The major tool is the specific localization to a smooth $L_2(\Omega)$ auxiliary reference Riesz basis.

6.1 ϱ–exponential Localization of Frames

The construction of Gelfand frames of wavelet type for L_2–Sobolev (and Besov) spaces requires localization properties of wavelet frames of exponential type, see, e.g., [17]. However, in the case of bounded domains, the setting of the previous sections has to be slightly generalized. It is mainly the definition of exponential localization that has to be modified, because the index sets of different frames over Ω will no longer be mutually isomorphic in general, cf. [24, 26, 27]. To cope with this difficulty, let us consider in the following three countable sets of indices $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, a triple $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ of functions $\varrho_i : \mathcal{N}_j \times \mathcal{N}_k \to \mathbb{R}$ and projections

$$\pi_i : \mathcal{N}_1 \times \mathcal{N}_2 \times \mathcal{N}_3 \to \mathcal{N}_j \times \mathcal{N}_k, \qquad \pi_i(\mathbf{x}) = (x_j, x_k), \tag{67}$$

where $i \in \{1, 2, 3\}$, $j, k \in \{1, 2, 3\} \setminus \{i\}$ and $j < k$. We assume that the following three generalized triangle inequalities hold for some fixed $w_0 > 0$:

$$\varrho_i\big(\pi_i(\mathbf{x})\big) \le \varrho_j\big(\pi_j(\mathbf{x})\big) + w_0 \varrho_k\big(\pi_k(\mathbf{x})\big) \quad \text{for all } \mathbf{x} \in \mathcal{N}_1 \times \mathcal{N}_2 \times \mathcal{N}_3. \tag{68}$$

Given two frames $\mathcal{F} = \{f_x\}_{x\in\mathcal{M}}$ and $\mathcal{G} = \{g_y\}_{y\in\mathcal{N}}$ for the Hilbert space $\mathcal{H}$, we say that $\mathcal{F}$ is *ϱ–exponentially localized* with respect to $\mathcal{G}$ (or simply *exponentially localized* once ϱ is fixed) if there exists a choice $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \in \{\mathcal{M}, \mathcal{N}\}$ and a function triple $\varrho = (\varrho_1, \varrho_2, \varrho_3)$ satisfying (68) as above, such that for some $s > 0$ and some $i \in \{1, 2, 3\}$

$$\big|\langle f_x, g_y\rangle_{\mathcal{H}}\big| \lesssim e^{-s\varrho_i(x,y)} \quad \text{for all } x \in \mathcal{N}_j,\ y \in \mathcal{N}_k. \tag{69}$$

In such a case we write $\mathcal{F} \sim_{\exp} \mathcal{G}$. A frame $\mathcal{F}$ such that $\mathcal{F} \sim_{\exp} \mathcal{F}$, is called *intrinsically ϱ–exponentially localized*. Let us denote here $\mathcal{A}_{\exp}$ the class of matrices which have *ϱ–exponential off–diagonal decay*, i.e, $\mathbf{A} \in \mathcal{A}_{\exp}$ whenever $|a_{x,y}| \lesssim e^{-s\varrho_i(x,y)}$ for some $s > 0$, and $\mathcal{B}_{\exp} \subset \mathcal{A}_{\exp}$ is the class of matrices in $\mathcal{A}_{\exp}$ such that there exists some $s' \in (0, s/w_0)$ so that, for $s'' := s - w_0 s'$,

$$\sup_{y\in\mathcal{N}_k} \sum_{x\in\mathcal{N}_j} e^{-s''\varrho_i(x,y)} < \infty, \tag{70}$$

where $s > 0$ is as in formula (69). Note that $\mathcal{A}_{\exp}$ is neither closed under multiplication nor inversion and thus not a spectral algebra. Therefore the theory developed in the previous section cannot be applied. However, by Lemma 8.4 in the appendix, one has that $\mathcal{B}_{\exp}\mathcal{A}_{\exp} \subset \mathcal{A}_{\exp}$ and Theorem 8.5 shows that for invertible matrices $\mathbf{M} \in \mathcal{B}_{\exp}$, we still have $\mathbf{M}^{-1} \in \mathcal{A}_{\exp}$.

The following result concerning exponential localization is a slight generalization of those presented in [17, 47] and it will turn out to be a helpful alternative technical tool to show localization properties of canonical dual wavelet frames on domains.

Theorem 6.1. *Let $\mathcal{F} = \{f_x\}_{x\in\mathcal{M}}$ be a frame for $\mathcal{H}$ and $\mathcal{G} = \{g_y\}_{y\in\mathcal{N}}$ a Riesz basis for $\mathcal{H}$ with dual basis $\tilde{\mathcal{G}} = \{\tilde{g}_y\}_{y\in\mathcal{N}}$ such that $\mathcal{F} \sim_{\exp} \tilde{\mathcal{G}}$. Moreover, we assume that $G(\mathcal{F}, \tilde{\mathcal{G}}) \in \mathcal{B}_{\exp}$ and $G(\tilde{\mathcal{G}}, \mathcal{F})G(\mathcal{F}, \tilde{\mathcal{G}}) \in \mathcal{B}_{\exp}$. Then $\tilde{\mathcal{F}} \sim_{\exp} \mathcal{G}$, where $\tilde{\mathcal{F}}$ is the canonical dual frame of $\mathcal{F}$.*

Proof. Consider the map

$$\Gamma : \mathcal{H} \to \ell_2(\mathcal{N}), \quad \Gamma f := \left(\langle f, \tilde{g}_y \rangle_{\mathcal{H}}\right)_{y \in \mathcal{N}},$$

its adjoint

$$\Gamma^* : \ell_2(\mathcal{N}) \to \mathcal{H}, \quad \Gamma^* \mathbf{c} := \sum_{y \in \mathcal{N}} c_y \tilde{g}_y$$

and the operator $\mathbf{T} := \Gamma S \Gamma^* : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$, where

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{x \in \mathcal{M}} \langle f, f_x \rangle_{\mathcal{H}} f_x$$

is the frame operator associated to $\mathcal{F}$. $\mathbf{T}$ is an automorphism of $\ell_2(\mathcal{N})$ with

$$\mathbf{T}_{x,y} = \langle \mathbf{e}_x, \mathbf{T}\mathbf{e}_y \rangle_{\ell_2(\mathcal{N})} = \sum_{z \in \mathcal{M}} \langle \tilde{g}_y, f_z \rangle_{\mathcal{H}} \langle f_z, \tilde{g}_x \rangle_{\mathcal{H}},$$

and therefore $\mathbf{T} \in \mathcal{B}_{\text{exp}}$. A straightforward computation yields

$$\begin{aligned}
\langle f_x, \tilde{g}_y \rangle_{\mathcal{H}} &= \langle S S^{-1} f_x, \tilde{g}_y \rangle_{\mathcal{H}} \\
&= \left\langle \sum_{z \in \mathcal{M}} \langle S^{-1} f_x, f_z \rangle_{\mathcal{H}} f_z, \tilde{g}_y \right\rangle_{\mathcal{H}} \\
&= \sum_{z \in \mathcal{M}} \langle S^{-1} f_x, f_z \rangle_{\mathcal{H}} \langle f_z, \tilde{g}_y \rangle_{\mathcal{H}} \\
&= \sum_{z \in \mathcal{M}} \langle \tilde{f}_x, f_z \rangle_{\mathcal{H}} \langle f_z, \tilde{g}_y \rangle_{\mathcal{H}} \\
&= \sum_{z \in \mathcal{M}} \left(\sum_{\xi \in \mathcal{N}} \langle \tilde{f}_x, g_\xi \rangle_{\mathcal{H}} \langle \tilde{g}_\xi, f_z \rangle_{\mathcal{H}} \right) \langle f_z, \tilde{g}_y \rangle_{\mathcal{H}} \\
&= (\mathbf{A}\mathbf{T})_{x,y},
\end{aligned}$$

where

$$\mathbf{A} := \left(\langle \tilde{f}_x, g_\xi \rangle_{\mathcal{H}}\right)_{x \in \mathcal{M}, \xi \in \mathcal{N}}.$$

By Theorem 8.5, we have $\mathbf{T}^{-1} \in \mathcal{A}_{\text{exp}}$, and the claim immediately follows by $\mathbf{A}_{x,y} = \sum_{\xi \in \mathcal{N}} \langle f_x, \tilde{g}_\xi \rangle_{\mathcal{H}} (\mathbf{T}^{-1})_{\xi,y}$, and by $\mathcal{B}_{\text{exp}} \mathcal{A}_{\text{exp}} \subset \mathcal{A}_{\text{exp}}$. ∎

6.2 Aggregated Wavelet Frames

In this subsection, we want to establish a straightforward construction of Gelfand wavelet frames on a bounded open domain $\Omega \subset \mathbb{R}^d$. The very natural key idea is to lift an appropriate template (Gelfand) wavelet frame $\Psi^\square = \{\psi_x^\square\}_{x \in \mathcal{N}^\square}$ on the d–dimensional unit cube $\square := (0,1)^d$ to Ω, using only a sufficiently smooth parametrization of Ω by local charts, and then just to merge all local basis functions into a global system Ψ, cf. [41, 52]. Under some conditions on $\Psi^\square$ and the other ingredients of the construction, Ψ is again a (Gelfand) frame. According to the nature of its construction, let us call such a system Ψ an *aggregated wavelet frame*.

In the following, we fix $\mathcal{B} := H_0^t(\Omega)$, the L_2–Sobolev space of Sobolev smoothness $t \geq 0$ on Ω of functions vanishing on the boundary $\partial\Omega$. The continuous and dense inclusions

$$H_0^t \subset L_2 \simeq (L_2)' \subset H^{-t}$$

ensure that (H_0^t, L_2, H^{-t}) is a Gelfand triple. Assume then that $\mathcal{C} := \{\Omega_i\}_{i=1}^n$ is an overlapping, relatively compact covering of Ω, such that

(C1) there exist C^m–diffeomorphisms $\kappa_i : \square \to \Omega_i$ of Ω_i, $m \geq t$, for all $i = 1, \ldots, n$;

(C2) there exists a C^m–partition of unity $\Sigma := \{\sigma_i\}_{i=1,\ldots,n}$ subordinate to $\mathcal{C}$.

Clearly, the set of admissible domains Ω is restricted by raising the conditions (C1) and (C2), e.g., the boundary of Ω has to be piecewise smooth enough. But since the particularly attractive case of polyhedral domains is still covered, these assumptions on the parametrizations κ_i are no principal limitations. The partition of unity Σ affects the construction of Ψ only in so far as we will use it as a tool for proving the Gelfand frame properties. In particular, we will exploit that the operators $\mathcal{P}$: $H_0^t(\Omega) \to H_0^t(\Omega_i)$, $\mathcal{P}_i(f) = f \cdot \sigma_i$ are bounded and constitute a *Bessel resolution of the identity* for $H_0^t(\Omega)$, i.e.,

(P1) $\sum_{i=1}^n \mathcal{P}_i = \mathrm{id}$, in the strong operator topology;

(P2) $\sum_{i=1}^n \|\mathcal{P}_i f\|^2_{H^t(\Omega)} \sim \|f\|^2_{H^t(\Omega)}$, for all $f \in H_0^t(\Omega)$.

As already mentioned, let us consider a template wavelet frame $\Psi^\square = \{\psi_x^\square\}_{x \in \mathcal{N}^\square}$ in $L_2(\square)$, with canonical dual $\tilde{\Psi}^\square = \{\tilde{\psi}_x^\square\}_{x \in \mathcal{N}^\square}$. We assume that $\Psi^\square \subset H_0^\gamma(\square)$ for some $\gamma > 0$. In practice, since we will have to raise some vanishing moment conditions on $\Psi^\square$ to ensure the Gelfand frame properties of Ψ, we will choose $\Psi^\square$ to be a Riesz basis, constructed as a tensor product of biorthogonal wavelet bases on the unit interval with complementary boundary conditions [24, 26]. Those bases are particularly attractive since they can be designed to exhibit any given Sobolev smoothness and any given number of vanishing moments of the *primal* wavelets. Moreover, one has that $\Psi^\square$ is a Gelfand wavelet frame in $H_0^t(\square)$ for some $t > 0$. Another possibility would be to consider genuine wavelet frames on the interval from the very start of the construction [13], but in the following, we confine the discussion to the Riesz basis case.

Concerning the frame indices $\mathcal{N}^\square$, we will use the same notation as in [24, 26, 27]. Let $j_0 \in \mathbb{Z}$ be a fixed coarsest level. Each wavelet frame on $\square$ consists of a set of scaling functions (or generators) on the level j_0 and of the wavelets for all levels $j \geq j_0$. Let $\Delta_{j_0}^\square$, $\nabla_j^\square \subset \mathbb{Z}^d$ be fixed index sets for $j \geq j_0$. To simplify the notation, basis elements of the form $\psi_{j_0-1,\mathbf{k}}^\square$, for $\mathbf{k} \in \Delta_{j_0}^\square$, correspond to the scaling functions on the level j_0 (with an index shift by 1 in j), whereas $\psi^{\square}_{j,\mathbf{k}}$, for $j \geq j_0$ and $\mathbf{k} \in \nabla^{\square}_j$, are wavelets on the level j. We will use the index sets

$$\mathcal{N}^{(i)} := \left\{(j_0 - 1, \mathbf{k}) : \ \mathbf{k} \in \Delta^{\square}_{j_0}\right\} \cup \left\{(j, \mathbf{k}) : \ j \geq j_0, \mathbf{k} \in \nabla^{\square}_j\right\}, \quad \text{for } 1 \leq i \leq n, \tag{71}$$

and

$$\mathcal{N} := \bigcup_{i=1}^{n} \{i\} \times \mathcal{N}^{(i)}, \tag{72}$$

where we define the mapping $|\cdot| : \mathcal{N}^{(i)} \to \mathbb{Z}$ by

$$\big|(j, \mathbf{k})\big| := j, \quad \text{for all } (j, \mathbf{k}) \in \mathcal{N}^{(i)}. \tag{73}$$

We will have to assume that the geometrical and the dyadic physical grid are compatible in the sense that for $(i, j, \mathbf{k}) \in \mathcal{N}$, we always have $\mathbf{k} \in \{0, \ldots, 2^j\}^d$, so that $2^{-j}\mathbf{k} \in \square$. But this condition indeed holds when using an appropriate biorthogonal Riesz basis on $\square$ [24, 26].

Our aim is now to show that the system

$$\Psi := (\psi_{i,j,\mathbf{k}})_{(i,j,\mathbf{k}) \in \mathcal{N}}, \tag{74}$$

where

$$\psi_{i,j,\mathbf{k}}(x) := \frac{\psi^{\square}_{j,\mathbf{k}}\big(\kappa_i^{-1}(x)\big)}{\big|\det D\kappa_i\big(\kappa_i^{-1}(x)\big)\big|^{1/2}}, \quad \text{for all } (i, j, \mathbf{k}) \in \mathcal{N}, \ x \in \Omega_i, \tag{75}$$

and $\psi_{i,j,\mathbf{k}}(x) = 0$ for $x \in \Omega \setminus \Omega_i$, is a Gelfand frame for $(H^t_0(\Omega), L_2(\Omega), H^{-t}(\Omega))$ with (global) canonical dual

$$\tilde{\Psi} := (\widetilde{\psi_{i,j,\mathbf{k}}})_{(i,j,\mathbf{k}) \in \mathcal{N}}. \tag{76}$$

By (75), the (local) duals of $\psi_{i,j,\mathbf{k}}|_{\Omega_i} \in H^t_0(\Omega_i)$ can be written as

$$\tilde{\psi}_{i,j,\mathbf{k}} = \frac{\tilde{\psi}^{\square}_{j,\mathbf{k}}}{|\det D\kappa_i|^{1/2}} \circ \kappa_i^{-1}.$$

In the following, let $\varrho_1 : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ be given by

$$\begin{aligned} \varrho_{\Psi,\Psi}\big((i, j, \mathbf{k}), (i', j', \mathbf{k}')\big) &:= \varrho_1\big((i, j, \mathbf{k}), (i', j', \mathbf{k}')\big) \\ &:= \frac{r}{s} \log\left(1 + 2^{\min(j,j')}\big\|\kappa_i(2^{-j}\mathbf{k}) - \kappa_{i'}(2^{-j'}\mathbf{k}')\big\|_{\mathbb{R}^d}\right) \\ &\quad + |j - j'| \log 2 + \frac{9r}{2s} \log 2, \end{aligned} \tag{77}$$

for $(i, j, \mathbf{k}), (i', j', \mathbf{k}') \in \mathcal{N}$ and $r, s > 0$, i.e.,

$$e^{-s\varrho_1((i,j,\mathbf{k}),(i',j',\mathbf{k}'))} = 2^{-9r/2}\left(1 + 2^{\min(j,j')}\big\|\kappa_i(2^{-j}\mathbf{k}) - \kappa_{i'}(2^{-j'}\mathbf{k}')\big\|_{\mathbb{R}^d}\right)^{-r} 2^{-s|j-j'|}. \tag{78}$$

By Lemma 8.6, ϱ_1 fulfills the triangle inequality (68) for $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}$, $\varrho_2 = \varrho_3 = \varrho_1$ and $w_0 = 1$.

The frame and Gelfand frame properties of Ψ can be ensured by raising some conditions on $\Psi_i := (\psi_{i,j,\mathbf{k}})_{(j,\mathbf{k}) \in \mathcal{N}^{(i)}} \subset H^t_0(\Omega_i)$:

Theorem 6.2. *For each $1 \le i \le n$, Ψ_i is a frame for $L_2(\Omega_i)$ and Ψ is a frame for $L_2(\Omega)$ with canonical dual $\tilde{\Psi}$. Moreover, if*

$$\left|\langle \psi_{i,j,\mathbf{k}}, \widetilde{\psi_{i',j',\mathbf{k}'}}\rangle\right| \lesssim e^{-s\varrho_1((i,j,\mathbf{k}),(i',j',\mathbf{k}'))}, \tag{79}$$

for some $r > d$ and $s > d$, where ϱ_1 is defined by (77), and, for each $1 \le i \le n$, Ψ_i is a Gelfand frame for $\left(H_0^t(\Omega_i), L_2(\Omega_i), H^{-t}(\Omega_i)\right)$ with respect to the Gelfand triple of sequence spaces $\left(\ell_{2,2^t}(\mathcal{N}^{(i)}), \ell_2(\mathcal{N}^{(i)}), \ell_{2,2^{-t}}(\mathcal{N}^{(i)})\right)$ for some $t \in (0, s-d)$, then Ψ is a Gelfand aggregated wavelet frame for $\left(H_0^t(\Omega), L_2(\Omega), H^{-t}(\Omega)\right)$ with respect to the Gelfand triple of sequence spaces $\left(\ell^n_{2,2^t}, \ell^n_2, \ell^n_{2,2^{-t}}\right)$, where

$$\ell^n_{2,2^t} := \{\mathbf{c} = (\mathbf{c}_1, ..., \mathbf{c}_n) : \mathbf{c}_i \in \ell_{2,2^t}(\mathcal{N}^{(i)})\}, \quad \|\mathbf{c}\|_{\ell^n_{2,2^t}} := \left(\sum_{i=1}^n \|\mathbf{c}_i\|^2_{\ell_{2,2^t}(\mathcal{N}^{(i)})}\right)^{1/2}.$$

Proof. Since Ψ_i is a (Hilbert) frame for $L_2(\Omega_i)$, [52, Theorem 4.1] implies that Ψ is a frame for $L_2(\Omega)$. We have to show that the operators

$$F^* : \ell^n_{2,2^t} \to H_0^t(\Omega), \quad F^*\mathbf{c} := \sum_{(i,j,\mathbf{k})\in\mathcal{N}} c_{i,j,\mathbf{k}}\psi_{i,j,\mathbf{k}} \tag{80}$$

and

$$\tilde{F} : H_0^t(\Omega) \to \ell^n_{2,2^t}, \quad \tilde{F}f := \left(\langle f, \tilde{\psi}_{i,j,\mathbf{k}}\rangle\right)_{(i,j,\mathbf{k})\in\mathcal{N}} \tag{81}$$

are bounded. For $\mathbf{c} \in \ell^n_{2,2^t}$, we have $\mathbf{c}_i \in \ell_{2,2^t}(\mathcal{N}^{(i)})$. The corresponding operators $F_i^* : \ell_{2,2^t}(\mathcal{N}^{(i)}) \to H_0^t(\Omega_i)$, $F_i^*\mathbf{d} = \sum_{(j,\mathbf{k})\in\mathcal{N}^{(i)}} d_{j,\mathbf{k}}\psi_{i,j,\mathbf{k}}$ for the Gelfand frames Ψ_i are bounded, so that $F_i^*\mathbf{c}_i \in H_0^t(\Omega_i)$ for $1 \le i \le n$. Hence $F^*\mathbf{c} = \sum_{i=1}^n F_i^*\mathbf{c}_i \in H_0^t(\Omega)$ using the trivial embedding $H_0^t(\Omega_i) \subset H_0^t(\Omega)$, and thus

$$\|F^*\mathbf{c}\|_{H^t(\Omega)} \le \sum_{i=1}^n \|F_i^*\mathbf{c}_i\|_{H^t(\Omega_i)} \lesssim \sum_{i=1}^n \left(\sum_{(j,\mathbf{k})\in\mathcal{N}^{(i)}} 2^{2tj}|c_{i,j,\mathbf{k}}|^2\right)^{1/2} \lesssim \|\mathbf{c}\|_{\ell^n_{2,2^t}}.$$

To show that (81) is bounded, take an arbitrary $f \in H_0^t(\Omega)$. Then $f = \sum_{i=1}^n \sigma_i f$, where $\sigma_i f \in H_0^t(\Omega_i)$. Since $\tilde{F}_i : H_0^t(\Omega_i) \to \ell_{2,2^t}(\mathcal{N}^{(i)})$, $\tilde{F}_i g = \left(\langle g, \tilde{\psi}_{i,j,\mathbf{k}}\rangle\right)_{(j,\mathbf{k})\in\mathcal{N}^{(i)}}$ are bounded operators, we get

$$\|f\|^2_{H^t(\Omega)} \sim \sum_{i=1}^n \|\sigma_i f\|^2_{H^t(\Omega_i)} \sim \sum_{i=1}^n \sum_{(j,\mathbf{k})\in\mathcal{N}^{(i)}} 2^{2tj}\left|\langle \sigma_i f, \tilde{\psi}_{i,j,\mathbf{k}}\rangle_{L_2(\Omega_i)}\right|^2.$$

By (P1) one concludes

$$
\begin{aligned}
\langle f, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega)} &= \sum_{i'=1}^{n} \langle \sigma_{i'} f, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega)} \\
&= \sum_{i'=1}^{n} \langle \sigma_{i'} f, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega_{i'})} \\
&= \sum_{i'=1}^{n} \sum_{(j',\mathbf{k}') \in \mathcal{N}^{(i')}} \langle f, \sigma_{i'} \tilde{\psi}_{i',j',\mathbf{k}'} \rangle_{L_2(\Omega_{i'})} \langle \psi_{i',j',\mathbf{k}'}, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega_{i'})} \\
&= \sum_{i'=1}^{n} \sum_{(j',\mathbf{k}') \in \mathcal{N}^{(i')}} \langle f, \sigma_{i'} \tilde{\psi}_{i',j',\mathbf{k}'} \rangle_{L_2(\Omega)} \langle \psi_{i',j',\mathbf{k}'}, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega)}.
\end{aligned}
$$

Since $\big(\langle f, \sigma_{i'} \tilde{\psi}_{i',j',\mathbf{k}'} \rangle_{L_2(\Omega)}\big)_{(i',j',\mathbf{k}') \in \mathcal{N}} \in \ell^n_{2,2^t}$ and (79), then, by Proposition 8.3 it follows that also $\big(\langle f, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega)}\big)_{(i,j,\mathbf{k}) \in \mathcal{N}} \in \ell^n_{2,2^t}$ and

$$
\sum_{i=1}^{n} \sum_{(j,\mathbf{k}) \in \mathcal{N}^{(i)}} 2^{2tj} \big| \langle f, \widetilde{\psi_{i,j,\mathbf{k}}} \rangle_{L_2(\Omega)} \big|^2 \lesssim \sum_{i'=1}^{n} \sum_{(j',\mathbf{k}') \in \mathcal{N}^{(i')}} 2^{2tj'} \big| \langle f, \sigma_{i'} \tilde{\psi}_{i',j',\mathbf{k}'} \rangle_{L_2(\Omega_{i'})} \big|^2 \lesssim \|f\|^2_{H^t(\Omega)}.
$$

This ensures that $\tilde{F}$ in (81) is bounded. ■

To ensure (79), we utilize another template Riesz basis $\Psi^{\square,\circ} := \{\psi^{\square,\circ}_{j,\mathbf{k}}\}_{(j,\mathbf{k}) \in \mathcal{N}^{\square,\circ}}$ in $L_2(\square)$ with $\Psi^{\square,\circ} \subset H^{\gamma}_0(\square)$. We may choose $\Psi^{\square,\circ} = \Psi^{\square}$, but since we want to leave open the possibility to choose a genuine wavelet frame $\Psi^{\square}$, let us distinguish between the two template bases in the following. Given the covering $\mathcal{C} = \{\Omega_i\}_{i=1}^{n}$, it is possible to construct a non–overlapping, C^m auxiliary covering $\mathcal{C}^\circ = \{\Omega^\circ_i\}_{i=1}^{n'}$ with diffeomorphisms $\kappa^\circ_i : \square \to \Omega^\circ_i$. Then we can define an associated aggregated system $\Psi^\circ := \{\psi^\circ_{i,j,\mathbf{k}}\}_{(i,j,\mathbf{k}) \in \mathcal{N}^\circ}$, where $\mathcal{N}^\circ$ is constructed in the same way as $\mathcal{N}$ and

$$
\psi^\circ_{i,j,\mathbf{k}}(x) := \frac{\psi^{\square,\circ}_{j,\mathbf{k}}\big((\kappa^\circ_i)^{-1}(x)\big)}{\big|\det D\kappa^\circ_i\big((\kappa^\circ_i)^{-1}(x)\big)\big|^{1/2}}, \quad \text{for all } (i,j,\mathbf{k}) \in \mathcal{N}^\circ. \tag{82}
$$

By construction, Ψ° is a Riesz basis in $L_2(\Omega)$ with the same Sobolev regularity as $\Psi^{\square,\circ}$. It turns out that the localization property (79) is in fact fulfilled by the canonical dual of Ψ for any aggregated wavelet frame constructed in this way, as long as $s, r > 0$ are appropriately chosen and Ψ as well as the system $\{\psi_{i,j,\mathbf{k}}\}_{(i,j,\mathbf{k}) \in \mathcal{N}}$ are localized with respect to Ψ°:

Proposition 6.3. *Let Ψ and Ψ° be constructed as above. If*

$$
\big|\langle \psi_{i,j,\mathbf{k}}, \psi^\circ_{i',j',\mathbf{k}'} \rangle\big| \lesssim e^{-s\varrho_2((i,j,\mathbf{k}),(i',j',\mathbf{k}'))}, \tag{83}
$$

where $\varrho_2 : \mathcal{N} \times \mathcal{N}^\circ \to \mathbb{R}$ *is defined by*

$$\begin{aligned}\varrho_2\big((i,j,\mathbf{k}),(i',j',\mathbf{k}')\big) &:= \varrho_{\Psi,\Psi^\circ}\big((i,j,\mathbf{k}),(i',j',\mathbf{k}')\big) \\ &:= \frac{r}{s}\log\Big(1+2^{\min(j,j')}\big\|\kappa_i(2^{-j}\mathbf{k})-\kappa^\circ_{i'}(2^{-j'}\mathbf{k}')\big\|_{\mathbb{R}^d}\Big) \\ &\quad + |j-j'|\log 2 + \frac{9r}{2s}\log 2,\end{aligned} \tag{84}$$

completely analogous to (77)*, and* $r, s > 2d$*, then there exist* $s' \in (0, s)$ *and* $r' \in (0, r)$ *such that*

$$\big|\langle \psi_{i',j',\mathbf{k}'}, \widetilde{\psi_{i,j,\mathbf{k}}}\rangle\big| \lesssim e^{-s'\varrho_1((i',j',\mathbf{k}'),(i,j,\mathbf{k}))}, \tag{85}$$

where (77) *is valid for* r'*.*

Proof. By Lemma 8.7, ϱ_2 fulfills all the necessary triangle inequalities (68). Since Ψ° is a Riesz basis for $L_2(\Omega)$, we have

$$\langle \psi_{i',j',\mathbf{k}'}, \widetilde{\psi_{i,j,\mathbf{k}}}\rangle = \sum_{(i'',j'',\mathbf{k}'')\in\mathcal{N}^\circ} \langle \psi_{i',j',\mathbf{k}'}, \psi^\circ_{i'',j'',\mathbf{k}''}\rangle \langle \widetilde{\psi^\circ_{i'',j'',\mathbf{k}''}}, \widetilde{\psi_{i,j,\mathbf{k}}}\rangle. \tag{86}$$

By hypothesis and the proof of Proposition 8.3, one has $G(\Psi, \Psi^\circ) \in \mathcal{B}_{\exp}$, and by Lemma 8.4, it is not difficult to show that $G(\Psi^\circ, \Psi)G(\Psi, \Psi^\circ) \in \mathcal{B}_{\exp}$. Theorem 6.1 yields $\tilde{\Psi} \sim_{\exp} \tilde{\Psi}^\circ$, so that (85) follows by (86) and by $\mathcal{B}_{\exp}\mathcal{A}_{\exp} \subset \mathcal{A}_{\exp}$. ∎

The main results of this section can be summarized in the following Theorem:

Theorem 6.4. *Assume that* Ψ *is an aggregated wavelet frame for* $L_2(\Omega)$ *generated by local Gelfand frames* Ψ_i *for* $\big(H^t_0(\Omega_i), L_2(\Omega_i), H^{-t}(\Omega_i)\big)$*, where* $t \in (0, s-d)$ *and* $s > 2d$*. If* Ψ *has the localization property* (83)*, and* r', s' *in* (85) *are such that* $r', s' > d$*, then* Ψ *is a Gelfand aggregated wavelet frame for* $(H^t_0(\Omega), L_2(\Omega), H^{-t}(\Omega))$*.*

It remains to show how (83) can be realized in practice. To this end, we will exploit the fact that the supports of $\psi^\square_{j,\mathbf{k}}$ and $\psi^{\square,\circ}_{j',\mathbf{k}'}$ are essentially localized at $2^{-j}\mathbf{k}$ and $2^{-j'}\mathbf{k}'$, respectively:

$$\sup_{x\in\operatorname{supp}(\psi^\square_{j,\mathbf{k}})} \|x - 2^{-j}\mathbf{k}\|_{\mathbb{R}^d} \lesssim 2^{-j}, \quad \text{for all } (j,\mathbf{k}) \in \mathcal{N}^\square, \tag{87}$$

$$\sup_{x\in\operatorname{supp}(\psi^{\square,\circ}_{j',\mathbf{k}'})} \|x - 2^{-j'}\mathbf{k}'\|_{\mathbb{R}^d} \lesssim 2^{-j'}, \quad \text{for all } (j',\mathbf{k}') \in \mathcal{N}^{\square,\circ}. \tag{88}$$

(87) and (88) indeed hold for the constructions from [24, 26]. Since the local parametrizations κ_i and $\kappa^\circ_{i'}$ are sufficiently smooth, it immediately follows that also

$$\sup_{x\in\operatorname{supp}\psi_{i,j,\mathbf{k}}} \big\|x - \kappa_i(2^{-j}\mathbf{k})\big\|_{\mathbb{R}^d} \lesssim 2^{-j} \tag{89}$$

and

$$\sup_{x\in\operatorname{supp}\psi^{\circ}_{i',j',\mathbf{k}'}} \left\|x-\kappa^{\circ}_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d} \lesssim 2^{-j'}, \tag{90}$$

for $(i,j,\mathbf{k})\in\mathcal{N}$ and $(i',j',\mathbf{k}')\in\mathcal{N}^{\circ}$, respectively. Then, raising some vanishing moment conditions on $\Psi^{\square}$ and $\Psi^{\square,\circ}$ is sufficient to guarantee (83):

Theorem 6.5. *Assume that, for $N\in\mathbb{N}$ with $N\geq\max\{\gamma,t\}$, $\Psi^{\square}$ and $\Psi^{\square,\circ}$ fulfill the following moment conditions:*

$$\begin{aligned}\int_{\square} x^{\beta}\psi^{\square}_{j,\mathbf{k}}(x)\,dx &= 0, \quad \text{for all } |\beta|\leq N,\ j\geq j_0,\ \mathbf{k}\in\nabla^{\square}_{j}, &(91)\\ \int_{\square} x^{\beta}\psi^{\square,\circ}_{j',\mathbf{k}'}(x)\,dx &= 0, \quad \text{for all } |\beta|\leq N,\ j'\geq j_0^{\circ},\ \mathbf{k}'\in\nabla^{\square,\circ}_{j'}. &(92)\end{aligned}$$

Then, lifting $\Psi^{\square}$ and $\Psi^{\square,\circ}$ as in (75) *and* (82)*, Ψ is exponentially ϱ–localized to Ψ°, i.e., there exists a constant $C>0$, only depending on global parameters, such that*

$$\left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\Omega)}\right| \leq Ce^{-\gamma\varrho_2((i,j,\mathbf{k}),(i',j',\mathbf{k}'))}, \tag{93}$$

where ϱ_2 is given by (84)*.*

Proof. First of all, assume that $j'\geq j$. Using (92) and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\Omega)}\right| &= \left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\operatorname{supp}\psi^{\circ}_{i',j',\mathbf{k}'})}\right|\\ &= \left|\int_{\operatorname{supp}\psi^{\square,\circ}_{j',\mathbf{k}'}} \psi_{i,j,\mathbf{k}}(\kappa^{\circ}_{i'}(x))\psi^{\square,\circ}_{j',\mathbf{k}'}(x)|\det D\kappa^{\circ}_{i'}(x)|^{1/2}\,dx\right|\\ &= \left|\int_{\operatorname{supp}\psi^{\square,\circ}_{j',\mathbf{k}'}} \big(\psi_{i,j,\mathbf{k}}(\kappa^{\circ}_{i'}(x))|\det D\kappa^{\circ}_{i'}(x)|^{1/2}-P(x)\big)\psi^{\square,\circ}_{j',\mathbf{k}'}(x)\,dx\right|\\ &\lesssim \left\|(\psi_{i,j,\mathbf{k}}\circ\kappa^{\circ}_{i'})|\det D\kappa^{\circ}_{i'}|^{1/2}-P\right\|_{L_2(\operatorname{supp}\psi^{\square,\circ}_{j',\mathbf{k}'})}\end{aligned}$$

where P is an arbitrary polynomial of degree at most N. Then a Whitney–type estimate yields

$$\begin{aligned}\left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\Omega)}\right| &\lesssim 2^{-\gamma j'}\left|(\psi_{i,j,\mathbf{k}}\circ\kappa^{\circ}_{i'})|\det D\kappa^{\circ}_{i'}|^{1/2}\right|_{H^{\gamma}(\operatorname{supp}\psi^{\square,\circ}_{j',\mathbf{k}'})}\\ &\lesssim 2^{-\gamma j'}\left\|\psi_{i,j,\mathbf{k}}\circ\kappa^{\circ}_{i'}\right\|_{H^{\gamma}(\operatorname{supp}\psi^{\square,\circ}_{j',\mathbf{k}'})}\\ &\lesssim 2^{-\gamma j'}\|\psi_{i,j,\mathbf{k}}\|_{H^{\gamma}(\Omega_i)}\\ &\lesssim 2^{-\gamma(j'-j)}.\end{aligned}$$

In the case $j' \leq j$, one can show in a completely analogous way

$$\begin{aligned}
\left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\Omega)}\right| &= \left|\langle\psi^{\circ}_{i',j',\mathbf{k}'},\psi_{i,j,\mathbf{k}}\rangle_{L_2(\operatorname{supp}\psi_{i,j,\mathbf{k}})}\right| \\
&= \left|\int_{\operatorname{supp}\psi^{\square}_{j,\mathbf{k}}} \psi^{\circ}_{i',j',\mathbf{k}'}(\kappa_i(x))\psi^{\square}_{j,\mathbf{k}}(x)|\det D\kappa_i(x)|^{1/2}\,dx\right| \\
&= \left|\int_{\operatorname{supp}\psi^{\square}_{j,\mathbf{k}}} \big(\psi^{\circ}_{i',j',\mathbf{k}'}(\kappa_i(x))|\det D\kappa_i(x)|^{1/2} - P(x)\big)\psi^{\square}_{j,\mathbf{k}}(x)\,dx\right| \\
&\lesssim \left\|(\psi^{\circ}_{i',j',\mathbf{k}'}\circ\kappa_i)|\det D\kappa_i|^{1/2} - P\right\|_{L_2(\operatorname{supp}\psi^{\square}_{j,\mathbf{k}})} \\
&\lesssim 2^{-\gamma j}\|\psi^{\circ}_{i',j',\mathbf{k}'}\|_{H^{\gamma}(\Omega^{\circ}_{i'})} \\
&\lesssim 2^{-\gamma(j-j')},
\end{aligned}$$

so that

$$\left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\Omega)}\right| \lesssim 2^{-\gamma|j-j'|}.$$

Now let us analyze the situations where the integrals $\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle$ can be nontrivial at all. By (89) and (90), a necessary condition for $\operatorname{supp}\psi_{i,j,\mathbf{k}}\cap\operatorname{supp}\psi^{\circ}_{i',j',\mathbf{k}'}$ having nontrivial measure is

$$\left\|\kappa_i(2^{-j}\mathbf{k}) - \kappa^{\circ}_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d} \lesssim 2^{-\min(j,j')},$$

i.e.,

$$\Big(1 + 2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k}) - \kappa^{\circ}_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\Big)^{-r} \gtrsim 2^{-r} \tag{94}$$

for any $r > 0$. Hence, if (94) is fulfilled, we obtain the estimate

$$\begin{aligned}
\left|\langle\psi_{i,j,\mathbf{k}},\psi^{\circ}_{i',j',\mathbf{k}'}\rangle_{L_2(\Omega)}\right| &\lesssim \Big(1 + 2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k}) - \kappa^{\circ}_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\Big)^{-r} 2^{-\gamma|j-j'|} \\
&\sim e^{-\gamma\varrho_2((i,j,\mathbf{k}),(i',j',\mathbf{k}'))}.
\end{aligned}$$

■

REMARKS:

1. The theoretical estimation of the localization properties and exponents (85) via (115) is sub–optimal, and we conjecture that in practice it can be improved. In principle, the strict requirements of Theorems 6.2 and 6.4 can be met just by choosing appropriate template bases $\Psi^{\square}$, $\Psi^{\square,\circ}$.

2. Note that in Theorem 6.5, we did not use any information about the dual bases $\tilde{\Psi}^{\square}$, $\tilde{\Psi}^{\square,\circ}$. In fact, there are principal limitations on (wavelet) Riesz bases in $H^s_0(\square)$ as to the number of vanishing moments of the dual wavelets. One can easily show that for a given Riesz basis $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ in $H^s_0(\square)$, an infinite subset of the dual Riesz basis does *not* have a vanishing first moment. This can be seen as a consequence of the necessary boundary modifications, cf. [24, 26].

3. The reader might wonder why the additional wavelet basis Ψ° is of any use at all since, e.g., it does usually not give rise to norm equivalences for the solution space H_0^t. However, in our applications, only a localization property is needed, i.e., a specific cross–Gramian matrix has to exhibit a certain off–diagonal decay. To establish these decay properties, in the wavelet setting only two features are needed: smoothness and cancellation properties, and both of them hold for our additional wavelet basis.

4. Let us conclude this section by the observation that under smoothness, locality, and cancellation conditions as illustrated in Theorem 6.5, one can ensure compressibility properties of the system matrices arising by the discretization of suitable differential and integral operators using aggregate wavelet frames. Following the lines of Lemma 5.1 in [25], for any (non–local) operator $\mathcal{L}$ satisfying (8), and for any (local) operator $\mathcal{L}$ such that $\langle \mathcal{L}u, v\rangle = 0$ whenever $\operatorname{supp} v \cap \operatorname{supp} u = \emptyset$, the corresponding stiffness matrix

$$\mathbf{G} = \left(2^{-(j+j')}\langle \mathcal{L}\psi_{i,j,\mathbf{k}}, \psi_{i',j',\mathbf{k}'}\rangle\right)_{(i,j,\mathbf{k}),(i',j',\mathbf{k}')\in\mathcal{N}} \tag{95}$$

is s^*–compressible for a suitable $s^* > 0$. We refer the reader to [25, 52] for major details.

7 Adaptive Computation of Localized Canonical Duals

As we have discussed in Sections 5 and 6, localization properties of the canonical dual are in fact relevant for the characterization of Banach spaces. Unfortunately, the canonical dual frame $\tilde{\mathcal{F}}$ of the frame $\mathcal{F}$ is only implicitly defined by the equation

$$S\tilde{\mathcal{F}} = \mathcal{F}, \tag{96}$$

where S is the associated frame operator, see Section 3. Usually no explicit formulas are available to describe $\tilde{\mathcal{F}}$. Therefore, any property of the canonical dual is very difficult to be checked. For this reason, efficient numerical methods to approximate it will be a very helpful tool of investigation. In this section, we want to present an application of Algorithm 1 for the computation of canonical dual frames. Methods for computing canonical duals for general frames are still a matter of investigation and no implemented solutions are presently available for infinite frames. Some techniques have been suggested for infinite Gabor and wavelet frames in [9, 10], but it seems that they do not have computationally efficient realizations. In [12] the use of the *finite section method* combined with localization properties of the frame is shown to be a very useful tool for an accurate approximation of inverse frame operators. Unfortunately, the finite section method appears again to be computationally expensive, since it works by means of the inversion of matrices with potentially high dimension. Moreover, the method assumes that the frame should be localized with respect to a given orthonormal basis and the approximation of the canonical dual is in fact given as a linear combination of elements of the orthonormal basis. It is known that orthonormal bases usually cannot have good time–frequency localization and, by consequence, nor does the approximation of the dual either in general. In this section, we want to show how, under intrinsic localization properties (no auxiliary (bi)orthogonal basis is required), it is possible to compute the canonical dual efficiently and with optimal computational complexity by means of the adaptive Algorithm 1. The approximation will be given as a linear combination of the original frame elements, therefore inheriting their nice properties, for example, regularity, compact support and vanishing moment properties, in the case of wavelet frames.

We assume that $\mathcal{B} = \mathcal{H} \simeq \mathcal{H}' = \mathcal{B}'$, $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ is a frame for $\mathcal{H}$, and $D_{\mathcal{B}} = \mathrm{id} : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$. Because of (11), the frame operator S is in fact elliptic on $\mathcal{H}$ in the sense of (2). Clearly the solution of the equation

$$Su = f_n, \tag{97}$$

is the canonical dual element $u = \tilde{f}_n = S^{-1} f_n$. Since $u = \sum_{l\in\mathcal{N}} \langle u, \tilde{f}_l\rangle_{\mathcal{H}} f_l$, (97) is equivalent to the discrete equations

$$\sum_{l\in\mathcal{N}} \langle u, \tilde{f}_l\rangle_{\mathcal{H}} \langle S f_l, f_m\rangle_{\mathcal{H}} = \langle f_n, f_m\rangle_{\mathcal{H}}, \quad \text{for all } m \in \mathcal{N}. \tag{98}$$

Denote $\mathbf{u} := \tilde{F} u = \left(\langle u, \tilde{f}_m\rangle_{\mathcal{H}}\right)_{m\in\mathcal{N}}$, $\mathbf{f}_n := F f_n = \left(\langle f_n, f_m\rangle_{\mathcal{H}}\right)_{m\in\mathcal{N}}$ and, finally, $\mathbf{G} := \left(\langle S f_n, f_m\rangle_{\mathcal{H}}\right)_{n,m\in\mathcal{N}}$. Then equations (98) can be rewritten as

$$\mathbf{G}\mathbf{u} = \mathbf{f}_n. \tag{99}$$

It is not difficult to show that $\mathbf{G} = F S F^*$. Moreover,

$$\langle S f_n, f_m\rangle_{\mathcal{H}} = \sum_{l\in\mathcal{N}} \langle f_n, f_l\rangle_{\mathcal{H}} \langle f_l, f_m\rangle_{\mathcal{H}}. \tag{100}$$

In the situation at hand, we have $\operatorname{ran} \mathbf{G} = \operatorname{ran} F$, and thus the orthogonal projection $\mathbf{Q} = F(F^*F)^{-1}F^*$ from (16) coincides with the orthogonal projection $\mathbf{P} : \ell_2(\mathcal{N}) \to \operatorname{ran} \mathbf{G}$. Then one can apply Algorithm 1 for the adaptive computation of the canonical dual $\tilde{\mathcal{F}}$:

Theorem 7.1. *If $\mathcal{F}$ is a frame, then $\tilde{\mathcal{F}}$ can be computed by*

$$\tilde{f}_n = F^* \mathbf{P} \tilde{\mathbf{f}}_n = \sum_{m\in\mathcal{N}} (\tilde{\mathbf{f}}_n)_m f_m, \quad \mathbf{P}\tilde{\mathbf{f}}_n = \left(\alpha \sum_{n=0}^{\infty} (\mathrm{id} - \alpha \mathbf{G})^n \right) \mathbf{f}_n, \tag{101}$$

for $0 < \alpha < 2/B_{\mathcal{F}}$. Moreover, if $\mathcal{F}$ is intrinsically $\mathcal{A}_\eta$–localized for $\eta > 2r$, $r > d$, then for any $\varepsilon > 0$ there exists a finite vector $\tilde{\mathbf{f}}_n^\varepsilon$ such that $\left\|\mathbf{P}(\tilde{\mathbf{f}}_n - \tilde{\mathbf{f}}_n^\varepsilon)\right\|_{\ell_2(\mathcal{N})} \leq \varepsilon$. This vector is the result of the application of Algorithm 1,

$$\mathbf{SOLVE}[\varepsilon, \mathbf{G}, f_n] \to \tilde{\mathbf{f}}_n^\varepsilon, \tag{102}$$

and $\tilde{\mathbf{f}}_n^\varepsilon$ has the following properties:

(I) $\#\operatorname{supp}\tilde{\mathbf{f}}_n^\varepsilon \lesssim \varepsilon^{-1/s}|\tilde{\mathbf{f}}_n|_{\ell_\tau^w(\mathcal{N})}^{1/s}$;

(II) the number of arithmetic operations used to compute it is at most a multiple of $\varepsilon^{-1/s}|\tilde{\mathbf{f}}_n|_{\ell_\tau^w(\mathcal{N})}^{1/s}$.

Here $\tau = (1/2 + s)^{-1}$ and $s \in (0, r - d)$. Therefore, by (43), one has the following approximation of the canonical dual:

$$\left\|\tilde{f}_n - \sum_{m \in \mathcal{N}} (\tilde{\mathbf{f}}_n^\varepsilon)_m f_m\right\|_{\mathcal{H}} \leq B_{\mathcal{F}}^{1/2}\varepsilon. \tag{103}$$

Proof. Let us denote $s^* = r - d$ and $\tau = (1/2 + s)^{-1}$, for any $s \in (0, s^*)$. The frame $\mathcal{F}$ is intrinsically $\mathcal{A}_\eta$–localized, i.e., it is localized in the sense of the Jaffard's algebra. This implies that

(i) by formula (100) and [47, Lemma 2.2]

$$\left|\langle Sf_n, f_m\rangle_{\mathcal{H}}\right| \lesssim \left(1 + \|n - m\|_{\mathbb{R}^d}\right)^{-\eta},$$

and by Proposition 4.5, one has that $\mathbf{G}$ is s^*–compressible;

(ii) by Theorem 5.2, the canonical dual is also intrinsically $\mathcal{A}_\eta$–localized, and $\tilde{\mathcal{F}}$ is $\mathcal{A}_\eta$–localized with respect to $\mathcal{F}$. In fact, one has

$$\langle f_n, \tilde{f}_m\rangle_{\mathcal{H}} = \sum_{l \in \mathcal{N}} \langle f_n, f_l\rangle_{\mathcal{H}} \langle \tilde{f}_l, \tilde{f}_m\rangle_{\mathcal{H}}, \quad n, m \in \mathcal{N} \tag{104}$$

and, therefore $\mathbf{P} = G(\mathcal{F}, \tilde{\mathcal{F}}) \in \mathcal{A}_\eta$;

(iii) by Proposition 4.5 and (ii), the orthogonal projection $\mathbf{P} = \left(\langle f_n, \tilde{f}_m\rangle_{\mathcal{H}}\right)_{n,m \in \mathcal{N}}$ is s^*–compressible, and by [14, Proposition 3.8], $\mathbf{P}$ is a bounded operator from $\ell_\tau^w(\mathcal{N})$ to $\ell_\tau^w(\mathcal{N})$;

(iv) by Proposition 4.5 and [14, Proposition 3.8], $\tilde{\mathbf{f}}_n = \tilde{F}\tilde{f}_n$ and $\mathbf{f}_n = Ff_n \in \ell_\tau^w(\mathcal{N})$ and therefore $\mathbf{f}_n$ is s^*–optimal and $\mathbf{G}\mathbf{u} = \mathbf{f}_n$ has a solution $\mathbf{u} = \tilde{\mathbf{f}}_n \in \ell_\tau^w(\mathcal{N})$.

Combining (i)-(iv) and Theorem 4.4, one concludes the proof. ∎

Examples 1. (a) Gabor frames. If $0 \neq g \in \mathcal{S}(\mathbb{R}^d)$ (or g time–frequency localized enough) generates an irregular Gabor frame $\mathcal{F} = \mathcal{G}(g, \mathcal{X})$, then, for any $\eta > d$, the frame $\mathcal{F}$ is intrinsically η–localized and has an intrinsically η–localized canonical dual $\tilde{\mathcal{F}} = \{\tilde{e}_\xi\}_{\xi \in \mathcal{X}}$. Therefore, an approximation of the canonical dual frame can be computed by using Theorem 7.1.

(b) Wavelet frames. The Lemarié class of matrices is not a spectral algebra and therefore the previous theorem cannot be modified to work with (wavelet) ρ–exponentially localized frames. Nevertheless, for simple cases, for instance wavelet frames on the Euclidean spaces, it is possible to estimate the localization in time and scale separately as follows. If $\Psi = \{\psi_{j,\mathbf{k}} := 2^{d/2}\psi(2^j \cdot -\mathbf{k})\}_{j\in\mathbb{Z},\mathbf{k}\in\mathbb{Z}^d}$ is a wavelet frame for $L_2(\mathbb{R}^d)$ with enough regularity and vanishing moments, then

$$\left|\langle \psi_{j,\mathbf{k}}, \psi_{j',\mathbf{k}'}\rangle\right| \lesssim \left(1 + \|\mathbf{k} - \mathbf{k}'\|_{\mathbb{R}^d}\right)^{-\eta} 2^{-s(|j-j'|)}, \text{ for } \eta > 2r, r > d, s > 0. \quad (105)$$

Then Ψ is also intrinsically $\mathcal{A}_\eta$–localized:

$$\left|\langle \psi_{j,\mathbf{k}}, \psi_{j',\mathbf{k}'}\rangle\right| \lesssim \left(1 + \|\mathbf{k} - \mathbf{k}'\|_{\mathbb{R}^d} + |j - j'|\right)^{-\eta}, \text{ for } \eta > 2r, r > d, \quad (106)$$

Therefore, an approximation of the canonical dual frame can be computed by using Theorem 7.1.

REMARK: Formula (101) is nothing but a discrete version of the well–known *frame algorithm* to compute canonical dual frames, see for example [17] for a recent discussion of its convergence in Banach spaces. Neumann series inversion of synthesis operators and Richardson iterations were also the key idea for relevant iterative methods in irregular sampling problems, see for example [36] and the related literature.

8 Appendix

We collect in this Appendix some relevant auxiliary lemmata.

Lemma 8.1 (Schur). *If for an infinite matrix* $\mathbf{N} = (n_{k,l})_{k,l\in\mathcal{N}}$ *there exists weights* $w_l > 0$, $l \in \mathcal{N}$ *such that*

$$\sum_{l\in\mathcal{N}} |n_{k,l}| w_l \leq C w_k, \text{ for all } k \in \mathcal{N},$$

and

$$\sum_{k\in\mathcal{N}} |n_{k,l}| w_k \leq C w_l, \text{ for all } l \in \mathcal{N},$$

then $\mathbf{N}$ *is a bounded operator from* $\ell_2(\mathcal{N})$ *to* $\ell_2(\mathcal{N})$ *and* $\|\mathbf{N}\| \leq C$.

Proposition 8.2. *Let $\alpha > 0$. If $g_{x,y} := 2^{-\alpha\|x-y\|_{\mathbb{R}^d}}$ for $x, y \in \mathcal{N}$, then $\mathbf{G} := (g_{x,y})_{x,y\in\mathcal{N}}$ is a bounded operator from $\ell_{p,w}(\mathcal{N})$ to $\ell_{p,w}(\mathcal{N})$ for all $p \in [1, \infty]$ and for any β–exponential moderate weight w, i.e., $w(x + y) \le 2^{\beta\|x\|_{\mathbb{R}^d}} w(y)$, for all $\beta \in (0, \alpha)$.*

Proof. Boundedness on $\ell_{1,w}(\mathcal{N})$:

$$\begin{aligned}\|\mathbf{Gc}\|_{\ell_{1,w}(\mathcal{N})} &\le \sum_{x\in\mathcal{N}}\sum_{y\in\mathcal{N}} 2^{-\alpha\|x-y\|_{\mathbb{R}^d}}|c_y|w(x)\\ &\le \sup_{y\in\mathcal{N}}\left(\sum_{x\in\mathcal{N}} 2^{-(\alpha-\beta)\|x-y\|_{\mathbb{R}^d}}\right)\left(\sum_{y\in\mathcal{N}}|c_y|w(y)\right)\\ &= C\|\mathbf{c}\|_{\ell_{1,w}(\mathcal{N})},\end{aligned}$$

since $w(x - y + y) \le 2^{\beta\|x-y\|_{\mathbb{R}^d}} w(y)$.
Boundedness on $\ell_{\infty,w}(\mathcal{N})$: it can be shown with a similar argument. One concludes by interpolation of weighted $\ell_p(\mathcal{N})$ spaces, see, e.g., [5] and [21], Appendix B, for details. ■

An analogous mapping property holds for matrices with more general ϱ–exponential off–diagonal decay:

Proposition 8.3. *Let $r, s > d$ and $\varrho_1 : \mathcal{N} \times \mathcal{N} \to \mathbb{R}$ be given by (77). Then $\mathbf{G} := (e^{-s\varrho_1((i,j,\mathbf{k}),(i',j',\mathbf{k}'))})_{(i,j,\mathbf{k}),(i',j',\mathbf{k}')\in\mathcal{N}\times\mathcal{N}}$ is a bounded operator from $\ell^n_{p,2^t}$ to $\ell^n_{p,2^t}$ for all $p \in [1, \infty]$ and for any $t \in (0, s - d)$.*

Proof. The boundedness of $\mathbf{G}$ on $\ell^n_{1,2^t}$ can be shown as follows:

$$\begin{aligned}&\|\mathbf{Gc}\|_{\ell^n_{1,2^t}}\\ &\lesssim \sum_{(i,j,\mathbf{k})\in\mathcal{N}}\sum_{(i',j',\mathbf{k}')\in\mathcal{N}} e^{-s\varrho_1((i,j,\mathbf{k}),(i',j',\mathbf{k}'))}|c_{i',j',\mathbf{k}'}|2^{tj}\\ &\lesssim \sum_{(i,j,\mathbf{k})\in\mathcal{N}}\sum_{(i',j',\mathbf{k}')\in\mathcal{N}} 2^{-s|j-j'|}2^{tj}\left(1+2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\right)^{-r}|c_{i',j',\mathbf{k}'}|\\ &= \sum_{i=1}^{n}\sum_{j\ge j_0-1}\sum_{i'=1}^{n}\sum_{j'\ge j_0-1} 2^{-s|j-j'|}2^{tj}\\ &\quad\cdot\left(\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}\sum_{\mathbf{k}\in\nabla^{\square}_{j}}\left(1+2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\right)^{-r}|c_{i',j',\mathbf{k}'}|\right).\end{aligned}$$

By the properties of the diffeomorphisms κ_i, one can further estimate

$$\begin{aligned}\|\mathbf{G}\mathbf{c}\|_{\ell^n_{1,2^t}} &\lesssim \sum_{i=1}^{n}\sum_{j\geq j_0-1}\sum_{i'=1}^{n}\sum_{j'\geq j_0-1} 2^{-s|j-j'|}2^{tj} \\ &\quad\cdot\left(\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}\left(\int_{\mathbb{R}^d}\left(1+2^{\min(j,j')}\left\|2^{-j}x\right\|_{\mathbb{R}^d}\right)^{-r}dx\right)|c_{i',j',\mathbf{k}'}|\right).\end{aligned}$$

These computations imply

$$\begin{aligned}\|\mathbf{G}\mathbf{c}\|_{\ell^n_{1,2^t}} &\lesssim \sum_{i'=1}^{n}\sum_{j,j'\geq j_0-1} 2^{-s|j-j'|}2^{\max\{0,d(j-j')\}}2^{tj}\left(\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}|c_{i',j',\mathbf{k}'}|\right) \\ &\lesssim \sum_{i'=1}^{n}\sum_{j,j'\geq j_0-1} 2^{-(s-d)|j-j'|}2^{tj}\left(\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}|c_{i',j',\mathbf{k}'}|\right).\end{aligned}$$

Let us denote $d^{i'}_{j'} := \left(\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}|c_{i',j',\mathbf{k}'}|\right)$. By an application of Proposition 8.2 for $\mathcal{N}=\mathbb{Z}_{\geq j_0}$, one has

$$\|\mathbf{G}\mathbf{c}\|_{\ell^n_{1,2^t}} \lesssim \sum_{i'=1}^{n}\sum_{j\geq j_0-1} d^{i'}_{j}2^{tj} = \|\mathbf{c}\|_{\ell^n_{1,2^t}}.$$

Let us now show the boundedness of $\mathbf{G}$ on $\ell^n_{\infty,2^t}$ and conclude by interpolation of weighted ℓ_p–spaces.

$$\begin{aligned}&\|\mathbf{G}\mathbf{c}\|_{\ell^n_{\infty,2^t}} \\ &\lesssim \sup_{(i,j,\mathbf{k})\in\mathcal{N}}\sum_{(i',j',\mathbf{k}')\in\mathcal{N}} e^{-s\varrho_1((i,j,\mathbf{k}),(i',j',\mathbf{k}'))}|c_{i',j',\mathbf{k}'}|2^{tj} \\ &= \sup_{(i,j,\mathbf{k})\in\mathcal{N}}\sum_{(i',j',\mathbf{k}')\in\mathcal{N}} 2^{-s|j-j'|}2^{tj}\left(1+2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\right)^{-r}|c_{i',j',\mathbf{k}'}| \\ &= \sup_{\substack{1\leq i\leq n\\ j\geq j_0-1}}\sum_{i'=1}^{n}\sum_{j'\geq j_0-1} 2^{-s|j-j'|}2^{tj} \\ &\qquad\cdot\left(\sup_{\mathbf{k}\in\nabla^{\square}_j}\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}\left(1+2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\right)^{-r}|c_{i',j',\mathbf{k}'}|\right).\end{aligned}$$

Since, as computed above, one has

$$\sum_{\mathbf{k}'\in\nabla^{\square}_{j'}}\left(1+2^{\min(j,j')}\left\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\right\|_{\mathbb{R}^d}\right)^{-r} \lesssim 2^{\max\{0,d(j'-j)\}},$$

then

$$\|\mathbf{G}\mathbf{c}\|_{\ell^n_{\infty,2^t}} \lesssim \sup_{j\geq j_0-1} \sum_{i'=1}^{n} \sum_{j'\geq j_0-1} 2^{-(s-d)|j-j'|} 2^{tj} \left(\sup_{\mathbf{k}'\in\nabla^{\square}_{j'}} |c_{i',j',\mathbf{k}'}| \right).$$

Again by an application of Proposition 8.2 one finally has

$$\|\mathbf{G}\mathbf{c}\|_{\ell^n_{\infty,2^t}} \lesssim \|\mathbf{c}\|_{\ell^n_{\infty,2^t}}.$$

This concludes the proof. ■

The following lemma is a slight generalization of a lemma from [50, section 5]:

Lemma 8.4. *Let $D, w_0 > 0$, $E \in (0, D/w_0)$ and $i \in \{1,2,3\}$, $j,k \in \{1,2,3\} \setminus \{i\}$ with $j < k$ be given. Furthermore, assume that (68) holds and*

$$S_F := \sup_{y\in\mathcal{N}_j} \sum_{x\in\mathcal{N}_i} e^{-F\varrho_k(x,y)} < \infty \tag{107}$$

for $F := D - Ew_0$. Then the matrix product $\mathbf{A}_1 := \mathbf{A}_2\mathbf{A}_3 =: (a_{z,y})_{y\in\mathcal{N}_j, z\in\mathcal{N}_k}$ of $\mathbf{A}_2 := (e^{-E\varrho_j(x,z)})_{z\in\mathcal{N}_k, x\in\mathcal{N}_i}$ and $\mathbf{A}_3 := (e^{-D\varrho_k(x,y)})_{x\in\mathcal{N}_i, y\in\mathcal{N}_j}$ fulfills

$$|a_{z,y}| \leq S_F e^{-E\varrho_i(y,z)} \quad \text{for all } y \in \mathcal{N}_j,\ z \in \mathcal{N}_k. \tag{108}$$

If, moreover,

$$S'_F := \sup_{z\in\mathcal{N}_k} \sum_{x\in\mathcal{N}_i} e^{-F\varrho_j(x,z)} < \infty \tag{109}$$

then an analogous estimate holds for reversed roles of E and D. If $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3$ and $\varrho_1 = \varrho_2 = \varrho_3$ is symmetric, powers $\mathbf{A}_3^n =: (a^{(n)}_{y,z})$ have the decay

$$|a^{(n)}_{z,y}| \leq S_F^{n-1} e^{-E\varrho_i(y,z)}. \tag{110}$$

Proof. In [50, Section 5], only the special case $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3$ and $\varrho_1 = \varrho_2 = \varrho_3$ was considered, but the proof also works in the general setting. Using (68) and (107), a direct calculation yields (108):

$$\begin{aligned} |a_{z,y}| &= \sum_{x\in\mathcal{N}_i} e^{-E\varrho_j(x,z)} e^{-D\varrho_k(x,y)} \\ &\leq \sum_{x\in\mathcal{N}_i} e^{-E(\varrho_i(y,z) - w_0\varrho_k(x,y))} e^{-D\varrho_k(x,y)} \\ &= e^{-E\varrho_i(y,z)} \sum_{x\in\mathcal{N}_i} e^{-(D-Ew_0)\varrho_k(x,y)} \\ &\leq S_F e^{-E\varrho_i(y,z)}. \end{aligned}$$

For interchanged roles of D and E, one calculates analogously

$$\begin{aligned}
|a_{z,y}| &= \sum_{x\in\mathcal{N}_i} e^{-D\varrho_j(x,z)}e^{-E\varrho_k(x,y)} \\
&\le \sum_{x\in\mathcal{N}_i} e^{-D\varrho_j(x,z)}e^{-E(\varrho_i(y,z)-w_0\varrho_j(x,z))} \\
&= e^{-E\varrho_i(y,z)}\sum_{x\in\mathcal{N}_i} e^{-(D-Ew_0)\varrho_j(x,z)} \\
&\le S'_F e^{-E\varrho_i(y,z)}.
\end{aligned}$$

(110) follows by induction. ■

We will also use the following generalization of [50, Théorème 5]:

Theorem 8.5. *Assume that $\mathcal{N} := \mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3$, $\varrho_1 = \varrho_2 = \varrho_3$ is symmetric,* (68) *and* (107) *hold for $F > 0$ and*

$$\varrho_1(x,x) = \varrho_0 \quad \textit{for all } x \in \mathcal{N} \tag{111}$$

where $\varrho_0 \ge 0$ is some constant. Let $\mathbf{M} = (m_{x,y})_{x,y\in\mathcal{N}}$ be an automorphism of $\ell_2(\mathcal{N})$ with

$$A\|\mathbf{c}\|_{\ell_2(\mathcal{N})} \le \|\mathbf{M}\mathbf{c}\|_{\ell_2(\mathcal{N})} \le B\|\mathbf{c}\|_{\ell_2(\mathcal{N})} \tag{112}$$

and the off–diagonal decay estimate

$$|m_{x,y}| \le Ce^{-D\varrho_1(x,y)} \tag{113}$$

for some constants $A, B, C, D > 0$. Then the inverse $\mathbf{M}^{-1} =: (p_{x,y})_{x,y\in\mathcal{N}}$ has exponential off–diagonal decay as well:

$$|p_{x,y}| \le C_1 e^{-D_1\varrho_1(x,y)} \tag{114}$$

for some $C_1 > 0$ and

$$D_1 = \min\left\{\frac{E}{2}, -\left\lfloor\frac{E}{2\log\left(\left(e^{D\varrho_0}+\frac{C}{B}\right)S_F\right)}\right\rfloor \log\left(1-\frac{A}{B}\right)\right\}. \tag{115}$$

Proof. In [50, Théorème 5], only the case $\varrho_0 = 0$ was considered, but the proof for $\varrho_0 > 0$ is completely analogous. Without loss of generality, assume that $\mathbf{M}$ is positive self–adjoint, otherwise use $\mathbf{M}^{-1} = \mathbf{M}^*(\mathbf{M}\mathbf{M}^*)^{-1}$ and Lemma 8.4. By (112), the spectrum $\sigma(\mathbf{M})$ is contained in $[A, B]$, i.e., $\sigma(\mathbf{S}) \subset [0, 1-\frac{A}{B}]$ for $\mathbf{S} := \mathrm{id} - \frac{1}{B}\mathbf{M}$. Moreover, $\|\mathbf{S}\| \le 1 - \frac{A}{B} =: q < 1$, so that the Neumann series $\mathbf{M}^{-1} = \frac{1}{B}\sum_{n=0}^{\infty}\mathbf{S}^n$ can be used to estimate $|p_{x,y}|$ by the entries of $\mathbf{S}^n =: (s^{(n)}_{x,y})_{x,y\in\mathcal{N}}$. For large n, we use

$|s_{x,y}^{(n)}| \leq \|\mathbf{S}^n\| \leq q^n$. For small n, we choose a number $E \in (0, D/w_0)$. By (113), we have

$$|s_{x,y}^{(1)}| \leq \delta_{x,y} + \frac{C}{B} e^{-D\varrho_1(x,y)} \leq \left(e^{D\varrho_0} + \frac{C}{B}\right) e^{-D\varrho_1(x,y)},$$

so that by Lemma 8.4, for $F := D - Ew_0$,

$$|s_{x,y}^{(n)}| \leq \left(e^{D\varrho_0} + \frac{C}{B}\right)^n \left((e^{-D\varrho_1(\zeta,\xi)})_{\zeta,\xi\in\mathcal{N}}\right)_{x,y}^n \leq \left(e^{D\varrho_0} + \frac{C}{B}\right)^n S_F^{n-1} e^{-E\varrho_1(x,y)}.$$

Hence for any $n_0 \in \mathbb{N}$ it follows that

$$\begin{aligned}
|p_{x,y}| &\leq \frac{1}{B}\left(\delta_{x,y} + \left(\sum_{n=1}^{n_0}\left(e^{D\varrho_0} + \frac{C}{B}\right)^n S_F^{n-1}\right) e^{-E\varrho_1(x,y)}\right) + \frac{1}{B}\sum_{n=n_0+1}^{\infty} q^n \\
&\leq \frac{1}{B}\left(e^{E\varrho_0} + \sum_{n=1}^{n_0}\left(e^{D\varrho_0} + \frac{C}{B}\right)^n S_F^{n-1}\right) e^{-E\varrho_1(x,y)} + \frac{q^{n_0+1}}{B(1-q)} \\
&\leq \frac{e^{E\varrho_0}}{B}\left(1 + \sum_{n=1}^{n_0}\left(e^{D\varrho_0} + \frac{C}{B}\right)^n S_F^{n-1}\right) e^{-E\varrho_1(x,y)} + \left(\frac{1}{A} - \frac{1}{B}\right)\left(1 - \frac{A}{B}\right)^{n_0}.
\end{aligned}$$

Since

$$1 = e^{D\varrho_0} e^{-D\varrho_0} \leq e^{D\varrho_0} e^{-F\varrho_0} \leq e^{D\varrho_0} S_F,$$

we can estimate

$$\begin{aligned}
|p_{x,y}| &\leq \frac{e^{(D+E)\varrho_0}}{B}\left(\sum_{n=0}^{n_0}\left(e^{D\varrho_0} + \frac{C}{B}\right)^n S_F^n\right) e^{-E\varrho_1(x,y)} + \left(\frac{1}{A} - \frac{1}{B}\right)\left(1 - \frac{A}{B}\right)^{n_0} \\
&= \frac{e^{(D+E)\varrho_0}\left(\left(e^{D\varrho_0} + \frac{C}{B}\right)^{n_0+1} S_F^{n_0+1} - 1\right)}{B\left(\left(e^{D\varrho_0} + \frac{C}{B}\right) S_F - 1\right)} e^{-E\varrho_1(x,y)} + \left(\frac{1}{A} - \frac{1}{B}\right)\left(1 - \frac{A}{B}\right)^{n_0} \\
&\leq C_1\left(\left(e^{D\varrho_0} + \frac{C}{B}\right)^{n_0} S_F^{n_0} e^{-E\varrho_1(x,y)} + \left(1 - \frac{A}{B}\right)^{n_0}\right),
\end{aligned}$$

where

$$C_1 := \max\left\{\frac{e^{(D+E)\varrho_0}\left(e^{D\varrho_0} + \frac{C}{B}\right) S_F}{B\left(\left(e^{D\varrho_0} + \frac{C}{B}\right) S_F - 1\right)}, \frac{1}{A} - \frac{1}{B}\right\}.$$

Now we choose

$$n_0 := \left\lfloor \frac{E}{2\log\left(\left(e^{D\varrho_0} + \frac{C}{B}\right) S_F\right)} \rho_1(x,y) \right\rfloor \geq 0,$$

so that (115) follows by

$$\begin{aligned}
\left(e^{D\varrho_0} + \frac{C}{B}\right)^{n_0} S_F^{n_0} e^{-E\varrho_1(x,y)} + \left(1 - \frac{A}{B}\right)^{n_0} &= e^{n_0 \log\left(\left(e^{D\varrho_0} + \frac{C}{B}\right) S_F\right) - E\varrho_1(x,y)} + e^{n_0 \log\left(1 - \frac{A}{B}\right)} \\
&\leq e^{-D_1\varrho_1(x,y)}.
\end{aligned}$$

∎

A crucial ingredient of the localization arguments is the validity of various triangle inequalities:

Lemma 8.6. *The function* ϱ_1 *from* (77) *fulfills* (68) *for* $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}_3 = \mathcal{N}$, $\varrho_2 = \varrho_3 = \varrho_1$ *and* $w_0 = 1$.

Proof. As in [43], consider an analogy ϱ_P to the Poincaré metric on the upper half plane $\mathbb{R}^d \times \mathbb{R}_+$

$$\varrho_P\big((\mathbf{x},t),(\mathbf{x}',t')\big) := \operatorname{Artanh}\vartheta = \log\left(\frac{1+\vartheta}{1-\vartheta}\right)^{1/2}, \tag{116}$$

where

$$\vartheta := \vartheta\big((\mathbf{x},t),(\mathbf{x}',t')\big) := \left(\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}^2 + |t'-t|^2}{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}^2 + |t'+t|^2}\right)^{1/2} \in [0,1). \tag{117}$$

ϱ_P is indeed a metric. Like in [43], one observes that

$$\left(\frac{1+\vartheta}{1-\vartheta}\right)^{1/2} = \frac{1+\vartheta}{2}\left(\frac{|t'+t|^2}{t't}\right)^{1/2}\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}^2}{|t'+t|^2}\right)^{1/2}. \tag{118}$$

We have the equivalence

$$\frac{1}{2}\left(\frac{1+\vartheta}{1-\vartheta}\right)^{1/2} \le \max\left(\sqrt{\frac{t'}{t}},\sqrt{\frac{t}{t'}}\right)\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}}{\max\{t,t'\}}\right) \le \sqrt{32}\left(\frac{1+\vartheta}{1-\vartheta}\right)^{1/2}, \tag{119}$$

since by (117) and (118)

$$\begin{aligned}
\frac{1+\vartheta}{1-\vartheta} &= \frac{(1+\vartheta)^2}{4}\left(\frac{|t'+t|^2}{t't}\right)\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}^2}{|t'+t|^2}\right) \\
&\le \left(\frac{t'}{t}+2+\frac{t}{t'}\right)\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}}{|t'+t|}\right)^2 \\
&\le 4\max\left\{\frac{t'}{t},\frac{t}{t'}\right\}\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}}{\max\{t,t'\}}\right)^2 \\
&= 4\left(\max\left\{\sqrt{\frac{t'}{t}},\sqrt{\frac{t}{t'}}\right\}\right)^2\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}}{\max\{t,t'\}}\right)^2 \\
&\le 16(1+\vartheta)^2\left(\sqrt{\frac{t'}{t}}+\sqrt{\frac{t}{t'}}\right)^2\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}}{|t'+t|}\right)^2 \\
&\le 32(1+\vartheta)^2\left(\frac{t'}{t}+2+\frac{t}{t'}\right)\left(1+\frac{\|\mathbf{x}'-\mathbf{x}\|_{\mathbb{R}^d}^2}{|t'+t|^2}\right) \\
&= 128\frac{1+\vartheta}{1-\vartheta}.
\end{aligned}$$

In the following, we use (119) at points of the form $(\mathbf{x},t) = \big(\kappa_i(2^{-j}\mathbf{k}), 2^{-j}\big)$. Since (77) transforms into

$$\begin{aligned}
&\varrho_1\big((i,j,\mathbf{k}),(i',j',\mathbf{k}')\big)\\
&= \frac{r}{s}\log\Big(1+2^{\min(j,j')}\big\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\big\|_{\mathbb{R}^d}\Big) + |j-j'|\log 2 + \frac{9r}{2s}\log 2\\
&= \frac{r}{s}\log\left(2^{|j-j'|(s/r-1/2)}2^{|j-j'|/2}\left(1+\frac{\big\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\big\|_{\mathbb{R}^d}}{\max\{2^{-j},2^{-j'}\}}\right)\right) + \frac{9r}{2s}\log 2,
\end{aligned}$$

(119) and the metric properties of ϱ_P yield for $r < 2s$

$$\begin{aligned}
&\varrho_1\big((i,j,\mathbf{k}),(i'',j'',\mathbf{k}'')\big)\\
&= \frac{r}{s}\log\left(2^{|j-j''|(s/r-1/2)}2^{|j-j''|/2}\left(1+\frac{\big\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i''}(2^{-j''}\mathbf{k}'')\big\|_{\mathbb{R}^d}}{\max\{2^{-j},2^{-j''}\}}\right)\right) + \frac{9r}{2s}\log 2\\
&\le \frac{r}{s}\log\left(\sqrt{32}\left(\frac{1+\vartheta\big((\mathbf{x},t),(\mathbf{x}'',t'')\big)}{1-\vartheta\big((\mathbf{x},t),(\mathbf{x}'',t'')\big)}\right)^{1/2}\right) + |j-j''|\left(1-\frac{r}{2s}\right)\log 2 + \frac{9r}{2s}\log 2\\
&\le \frac{r}{s}\varrho_P\big((\kappa_i(2^{-j}\mathbf{k}),2^{-j}),(\kappa_{i'}(2^{-j'}\mathbf{k}'),2^{-j'})\big) + |j-j'|\left(1-\frac{r}{2s}\right)\log 2\\
&\quad +\frac{r}{s}\varrho_P\big((\kappa_{i'}(2^{-j'}\mathbf{k}'),2^{-j'}),(\kappa_{i''}(2^{-j''}\mathbf{k}''),2^{-j''})\big) + |j'-j''|\left(1-\frac{r}{2s}\right)\log 2 + \frac{7r}{s}\log 2\\
&\le \frac{r}{s}\log\left(2^{|j-j'|/2}\left(1+\frac{\big\|\kappa_i(2^{-j}\mathbf{k})-\kappa_{i'}(2^{-j'}\mathbf{k}')\big\|_{\mathbb{R}^d}}{\max\{2^{-j},2^{-j'}\}}\right)\right) + |j-j'|\left(1-\frac{r}{2s}\right)\log 2\\
&\quad +\frac{r}{s}\log\left(2^{|j'-j''|/2}\left(1+\frac{\big\|\kappa_{i'}(2^{-j'}\mathbf{k}')-\kappa_{i''}(2^{-j''}\mathbf{k}'')\big\|_{\mathbb{R}^d}}{\max\{2^{-j'},2^{-j''}\}}\right)\right) + |j'-j''|\left(1-\frac{r}{2s}\right)\log 2\\
&\quad +\frac{9r}{s}\log 2\\
&= \varrho_1\big((i,j,\mathbf{k}),(i',j',\mathbf{k}')\big) + \varrho_1\big((i',j',\mathbf{k}'),(i'',j'',\mathbf{k}'')\big).
\end{aligned}$$

■

Since the proof of Lemma 8.6 completely relies on the metric properties of ϱ_P and since the structure of the index sets $\mathcal{N}$ and $\mathcal{N}^\circ$ from (72) is identical, one can immediately prove the following:

Lemma 8.7. *For any choice of $\Psi_1, \Psi_2, \Psi_3 \in \{\Psi, \Psi^\circ\}$ and appropriate definitions of $\varrho_i = \varrho_{\Psi_j,\Psi_k}$, where $i \in \{1,2,3\}$, $j,k \in \{1,2,3\}$ and $j<k$, (68) is fulfilled.*

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