ON THE EXISTENCE OF QUASI-SELF-SIMILAR SOLUTIONS OF THE WEAKLY SHEAR-THINNING EQUATION

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ABSTRACT. We consider the spreading of a thin droplet of viscous liquid on a plane surface driven by capillarity alone in the complete wetting regime. In the case of constant viscosity, the no-slip condition leads to a force singularity at advancing contact lines. It is well known nowadays that the introduction of appropriate slip conditions removes this paradox. Here, we investigate a different approach, which consists in keeping the no-slip condition and assuming instead a shear-thinning rheology. This relaxation leads, in lubrication approximation, to fourth order degenerate parabolic equations of quasilinear type. We obtain results on the existence of quasi-self-similar solutions to these equations in the limit of Newtonian rheology.

1. Introduction and results

We study the spreading of a thin droplet of viscous liquid on a plane surface driven by capillarity in the complete wetting regime. In the case of constant viscosity, the no-slip condition leads to a force singularity at advancing contact lines. Different possibilities have been proposed to remove the contact-line paradox; all of them introduce an additional "microscopic" scale in the model, which is expected to influence only weakly the effective description of the macroscopic dynamics. For instance, it is nowadays well known that the introduction of appropriate slip conditions removes the force singularity at advancing contact lines and thus the paradox. Moreover, it has been shown (first by asymptotic methods [16], [9], [3] and then rigorously [13]) that the macroscopic behaviour of solutions is only logarithmically affected by the ("microscopic") slip parameter at intermediate time scales. The results described in [2] demostrate rigorously that shear-thinning liquids are not affected by the contact-line paradox. This suggests the possibility of adopting weakly shear-thinning rheology in order to describe the macroscopic dynamics of liquid films. The present study is devoted to the investigation of such possibility. A way to encode weakly shear-thinning rheology, first proposed by Weidner and Schwartz [25], is to consider Ellis rheology of the form:

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left(1 + \left|\frac{\tau}{\tilde{\tau}}\right|^{p-2}\right), \tag{1.1}$$

where η is the viscosity, τ denotes the shear stress, η_0 is the viscosity at zero shear stress and $\tilde{\tau} > 0$ is the shear stress at which viscosity is reduced by a factor 1/2. If $p = 2$ or $1/\tilde{\tau} = 0$ the liquid is Newtonian, whereas it is "Ellis" shear–thinning for $p > 2$ and $\tilde{\tau} \in (0, \infty)$. The difference with respect to similar nonlinear relations between the viscosity and the shear stress, such as "power-law" rheology, is that (1.1) does not have a singularity at zero shear stress for $p > 2$, and therefore allows to recover the Newtonian case:

$$\frac{1}{\eta} = \frac{1}{\eta_0}\left(1 + \left|\frac{\tau}{\tilde{\tau}}\right|^{p-2}\right) \longrightarrow \frac{1}{\eta_0} \quad \forall \tau \in \mathbb{R} \quad \text{whenever } \tilde{\tau}^{p-2} \to \infty. \tag{1.2}$$

In lubrication approximation, this relaxation of the pair shear-dependent rheology / no-slip condition yields the following partial differential equation for the rescaled height $h(t, x)$ on its positivity set:

$$h_t + \kappa \left[h^3 \left(1 + |b\, h\, h_{xxx}|^{p-2}\right) h_{xxx}\right]_x = 0, \tag{1.3}$$

where

$$b = \left(\frac{3}{p+1}\right)^{\frac{1}{p-2}} \frac{1}{\tilde{\tau}} \tag{1.4}$$

and $h(t, x) = \gamma^{-1}\overline{h}(\gamma t, \gamma x)$, where $\overline{h}$ is the dimensional height, t is the time, x is the spatial coordinate and γ is the surface tension. The equation is coupled to conditions of vanishing flux and zero contact angle at triple junctions:

$$h_x\Big|_{\partial\{h>0\}} = 0, \qquad \lim_{x \to \partial\{h>0\}} h^3(1 + |b\, h\, h_{xxx}|^{p-2})h_{xxx} = 0. \tag{1.5}$$

We want to give a qualitative and quantitative description of solutions of (1.3)-(1.5). In this sense, our point of view will be twofold: Firstly, we intend to analyze the scaling law for macroscopic quantities for an almost Newtonian rheology, which in view of (1.2) and (1.4) corresponds to the smallness of the parameter b (the 'microscopic' parameter):

$$\tilde{\tau}^{p-2} \gg 1 \quad \Longrightarrow \quad b^{p-2} \ll 1. \tag{1.6}$$

Secondly, we are interested in describing — if possible — the large time behaviour of solutions for a given Ellis law (not necessarily satisfying $b^{p-2} \ll 1$):

$$t \gg 1 \qquad \text{for fixed } p > 2,\ b \in (0, \infty). \tag{1.7}$$

Both regimes will be investigated through formal asymptotic expansions using a method introduced in [3], based on the analysis of a class of quasi self–similar solutions of (1.3)–(1.5). Let

$$h(t,x) = (7\kappa t)^{-\frac{1}{7}} u(t,y), \quad y = x(7\kappa t)^{-\frac{1}{7}}.$$

Then (1.3) can be rewritten as

$$(yu)_y - 7tu_t = \left[u^3 u_{yyy} \left(1 + \left|b\,(7\kappa t)^{-\frac{5}{7}}\, u\, u_{yyy}\right|^{p-2}\right)\right]_y.$$

We expect that, after an initial time layer during which the evolution is governed by the specific form of the initial datum, solutions "forget" the initial droplet shape and relax to a profile which depends on the initial condition only through its mass (which is a conserved quantity). This is the basis of our first main assumption:

($H1$) *The term tu_t is negligible after a transient time T_0 which depends on the specific initial datum.*

Remark 1.1. The link between ($H1$) and the aforementioned expectation becomes transparent when considering an operator with self-similar structure: In that case ($H1$) would imply a small discrepancy between a rescaled mass-conserving solution u and the corresponding self-similar profile, which indeed depends only on initial mass. For other degenerate parabolic equations with self-similar structure, such as the porous medium equation, this discrepancy is known to decay to zero, and the rate of decay (hence T_0) has been recently quantified in terms of the initial datum using gradient-flow based approaches (cf. [7]–[23]).

In view of ($H1$) we have

$$(yu)_y \sim \left[u^3 u_{yyy} \left(1 + \left|b\,(7\kappa t)^{-\frac{5}{7}}\, u\, u_{yyy}\right|^{p-2}\right)\right]_y,$$

which using (1.5) can be integrated once with respect to y, obtaining

$$y \sim u^2 u_{yyy} \left(1 + \left|b\,(7\kappa t)^{-\frac{5}{7}}\, u\, u_{yyy}\right|^{p-2}\right). \tag{1.8}$$

Our second main assumption too concerns the time scale:

($H2$) *The time scale is such that* $\left(b\,t^{-\frac{5}{7}}\right)^{p-2} \ll 1.$

This assumption comprises both the relevant regimes (1.6) and (1.7) introduced before. If $b = 0$ or $p = 2$ — that is, in case of exact self–similarity — solutions of (1.8) would correspond to steady–states in (t, y) variables. In our case solutions of (1.8) correspond to quasi–steady states in the (t, y) variables provided we look at regions where time increments of lower order $\Delta t \ll t$ are negligible:

$$\left| b\,(7\kappa t)^{-\frac{5}{7}}\, u\, u_{yyy} \right|^{p-2} \le 1 \quad \overset{(1.8),(H2)}{\Longleftrightarrow} \quad \frac{|y|}{u} \lesssim 2\,|u\, u_{yyy}| \le 2\,(7\kappa t)^{\frac{5}{7}}\, b^{-1}$$

$$\Longleftrightarrow \quad h(t,x) \gtrsim \frac{b\,|x|}{14\,\kappa\, t}. \tag{1.9}$$

In view of $(H2)$, in order to analyze (1.8) we replace $b\,(7\kappa t)^{-\frac{5}{7}}$ by a constant ϵ:

$$\epsilon := b\,(7\kappa t)^{-\frac{5}{7}}, \quad \epsilon^{p-2} \ll 1. \tag{1.10}$$

Thus, we will consider the following boundary value problem:

$$(I) \quad \begin{cases} y &= u^2 u''' \left(1 + |\epsilon\, u\, u'''|^{p-2}\right), \quad u > 0 \quad y \in (0,a) \\ u'(0) &= 0 \\ u(a) &= 0, \quad u'(a) \;=\; 0 \\ M &= \int_0^a u(y)\, dy \end{cases}$$

Since the contact point a is itself an unknown of the problem, by a solution of (I) we mean a pair (a, u), with $a > 0$ and $u \in C^3([0,a)) \cap C^1([0,a])$. Let us state a well–posedness result for problem (I), which will be proved in Section 4:

Theorem (Existence of quasi-self-similar solutions). *For any $M > 0$, $p > 2$ and $\epsilon > 0$, problem (I) admits a solution (a,u).*

Since this problem is not invariant under rescaling, we will first consider $a > 0$ as fixed and prove existence and uniqueness for the following problem

$$(P_a) \begin{cases} u''' &= F(y,u) \quad \text{in} \quad (0,a) \\ u'(0) &= 0, \\ u(a) &= 0, \quad u'(a) = 0. \end{cases}$$

This will be achieved by an argument used by Ferreira and Bernis [12] in a similar context, based on estimates of the Green's function and on a fixed point argument. Then we will prove that there exists a positive number a such that $\int_0^a u_a(y)\, dy = M$, where u_a is the solution to (P_a).

2. Preliminaries

Introducing the function

$$W(y,u,\xi) := u^2\xi\left[1+(\epsilon\, u\, \xi)^{p-2}\right]-y,$$

the equation of (I) can be rewritten as

$$W(y,u,\xi)=0 \tag{2.1}$$

with $u'''=\xi$. Since (2.1) implies

$$\epsilon^{p-2}\, u^p\, \xi^{p-1} = y - u^2\,\xi,$$

for any fixed $(y,u)\in(0,\infty)\times(0,\infty)$ there exists a unique value $\xi\in(0,\infty)$ such that $W(y,u,\xi)=0$. This allows to define the function $\xi = F(y,u)$:

$$\begin{aligned}\{(y,u,\xi)\in(0,\infty)\times(0,\infty)\times(0,\infty) : W(y,u,\xi)=0\} &= \\ = \{(y,u,F(y,u)) : (y,u)\in(0,\infty)\times(0,\infty)\}\,. \end{aligned} \tag{2.2}$$

Hence we obtain the esplicit form:

$$u''' = F(y,u). \tag{2.3}$$

Since W is continuous, differentiable and strictly increasing with respect to ξ, we see that $F\in C^1((0,\infty)\times(0,\infty))$. Moreover $F\in C([0,\infty)\times(0,\infty))$ and

$$F(y,u)\sim\begin{cases}\frac{y}{u^2} & (\epsilon u u''')^{p-2}\ll 1\\ \left(\frac{y}{\epsilon^{p-2}u^p}\right)^{\frac{1}{p-1}} & (\epsilon u u''')^{p-2}\gg 1,\end{cases}$$

that is

$$F(y,u)\sim\begin{cases}\frac{y}{u^2} & \epsilon y\ll u\\ \left(\frac{y}{\epsilon^{p-2}u^p}\right)^{\frac{1}{p-1}} & \epsilon y\gg u.\end{cases} \tag{2.4}$$

This expansion already shows that the macroscopic behaviour of the solution is governed by the limit equation, whereas the shear–thinning rheology takes over for small values of u. Due to the nonlinearity in the third derivative, such phenomenon is not transparent from the PDE itself. In addition, simple computations show that

$$F(0,u)=0,\quad \frac{\partial F}{\partial y}>0,\quad \text{and}\quad \frac{\partial F}{\partial u}<0\quad \text{in}\ \ (0,\infty)\times(0,\infty) \tag{2.5}$$

and

$$\lim_{u\to 0^+} F(y,u) = +\infty \quad \forall\, y>0. \tag{2.6}$$

3. Green's function and properties

We consider the following problem:

$$(P_\psi) \begin{cases} u''' &= \psi(y) \quad \text{in} \quad (0,a) \\ u'(0) &= 0, \quad u(a)=0, \quad u'(a)=0. \end{cases} \tag{3.1}$$

For $t \in (0,a)$, we introduce the parabolas $P_-(y,t)$ defined in $y \in [0,t]$ and $P_+(y,t)$ defined in $y \in [t,a]$ such that

$$P_-'(0,t) = P_+(a,t) = P_+'(a,t) = 0 \tag{3.2}$$

and

$$P_-(t,t) = P_+(t,t), \quad P_-'(t,t) = P_+'(t,t), \quad P_+''(t,t) - P_-''(t,t) = 1 \tag{3.3}$$

where here and throughout the section, $'$ denotes differentiation w.r.t. y. Condition (3.2) and (3.3) give

$$P_-(y,t) = -\frac{(a-t)}{2a}\, y^2 + \frac{t}{2}\,(a-t), \quad P_+(y,t) = \frac{t}{2a}\,(a-y)^2.$$

Then the Green's function associated to the linear problem (3.1) is defined by the formula

$$G(y,t) = \begin{cases} \frac{t}{2}\,(a-t) - \frac{(a-t)}{2a}\, y^2 & \text{if } y \le t \\ \frac{t}{2a}\,(a-y)^2 & \text{if } y \ge t. \end{cases} \tag{3.4}$$

Note that $G(\cdot,t) \in C^1([0,a])$, and we have

$$G'(y,t) = \begin{cases} -\frac{(a-t)}{a}\, y & \text{if } y \le t \\ -\frac{t}{a}\,(a-y) & \text{if } y \ge t \end{cases} \tag{3.5}$$

$$G''(y,t) = \begin{cases} -\frac{(a-t)}{a} & \text{if } y \le t \\ \frac{t}{a} & \text{if } y \ge t \end{cases} \tag{3.6}$$

$$G'''(y,t) = \delta(y-t), \quad 0<y<a, \quad 0<t<a,$$
$$G'(0,t) = G(a,t) = G'(a,t) = 0, \quad 0<t<a. \tag{3.7}$$

We collect some properties of the Green's function in the following Lemma.

Lemma 3.1. *The function defined by (3.4) satisfies the following properties, where C_1 and C_2 are positive constants:*

1. $G(y,t) > 0$ *if* $0 \le y \le a$ *and* $0 < t < a$*;*
2. $G'(y,t) < 0$ *if* $y,t \in (0,a)$*;*
3. $G(y,t) \le C_1(a-t)$ *and* $|G'(y,t)| < C_1(a-t)$ *for all* $y,t \in [0,a]$*;*
4. $\int_y^a G(y,t)\,dt \ge C_2\,(a-y)^3$ *for all* $y \in [0,a]$.

Proof. Property (2) is evident from (3.5), while (1) follows from (2) and $G(a,t)=0$. The assertion (3) for G'' and G follows respectively from (3.5) and by integration in y. Since $G(y,t) \geq G(t,t)$ when $y \leq t$, and $G(t,t)$ can be rewritten as

$$G(t,t) = \frac{t}{2a}\,(a-t)^2 = \frac{(a-t)^2}{2} - \frac{(a-t)^3}{2a},$$

we have

$$\begin{aligned} \int_y^a G(y,t)\,dt &\geq \int_y^a G(t,t)\,dt \\ &= \int_y^a \frac{(a-t)^2}{2}\,dt - \int_y^a \frac{(a-t)^3}{2a}\,dt \\ &= \frac{(a-y)^3\,(a+3y)}{24\,a} \geq C_2\,(a-y)^3 \end{aligned}$$

which is assertion (4). □

The solution of (P_ψ) can of course be obtained through the Green's function G, as stated in the following Lemma:

Lemma 3.2. *For any $\psi \in C([0,a])$ there exists a unique solution $u \in C^3([0,a])$ of problem (P_ψ). Furthermore, u satisfies*

$$u^{(j)}(y) = \int_0^a G^{(j)}(y,t)\psi(t)\,dt, \quad j = 0, 1. \tag{3.8}$$

Proof. Let $u(y) = \int_0^a G(y,t)\psi(t)\,dt$. Since $G(\cdot,t) \in C^1([0,a])$, by (3) of Lemma 3.1 and (3.7) we obtain

$$u'(y) = \int_0^a G'(y,t)\psi(t)\,dt,$$

and $u'(0) = u(a) = u'(a) = 0$. Given a test function φ such that $\mathrm{supp}(\varphi) \subset (0,a)$, integrating by parts we obtain

$$\int_0^a u(y)\,\varphi'''(y)\,dy = -\int_0^a u'''(y)\,\varphi(y)\,dy \overset{(3.1)}{=} -\int_0^a \psi(y)\,\varphi(y)\,dt.$$

This means that $u''' = \psi$ in the sense of distributions. Hence u is a solution of (3.1). Since uniqueness is elementary, the proof is complete. □

4. Existence proof

The proof of the Theorem proceeds along several steps. We first consider $a > 0$ as fixed and prove the following result.

Proposition 4.1. *Let $p > 2$ and F defined by (2.2). For any $a > 0$ there exists $u \in C^3([0,a)) \cap C^1([0,a])$, $u > 0$ in $[0,a)$ which solves the following problem:*

$$(P_a) \begin{cases} u''' &= F(y,u) \quad in \quad (0,a) \\ u'(0) &= 0, \\ u(a) &= 0, \quad u'(a) = 0. \end{cases} \tag{4.1}$$

Furthermore,

$$u^{(j)}(y) = \int_0^a G^{(j)}(y,t)F(t,u(t))\,dt, \quad j = 0,1. \tag{4.2}$$

To this aim, we consider the approximating problem

$$(P_\delta) \begin{cases} u''' &= F(y,u) \quad \text{in} \quad (0,a) \\ u'(0) &= 0, \quad u(a) = \delta, \quad u'(a) = 0, \end{cases} \tag{4.3}$$

where δ is a positive number.

Remark 4.2. *By (2.4), it follows that*

$$\frac{y}{2u^2} \le F(y,u) \le \frac{y}{u^2} \quad for \quad u \ge \epsilon y \tag{4.4}$$

$$\left(\frac{y}{2\epsilon^{p-2}\,u^p}\right)^{1/p-1} \le F(y,u) \le \left(\frac{y}{\epsilon^{p-2}\,u^p}\right)^{1/p-1} \quad for \quad u \le \epsilon y. \tag{4.5}$$

Lemma 4.3. *For every $p > 2$ problem (P_δ) has at least a positive solution $u_\delta \in C^3([0,a])$, which satisfies*

$$u_\delta(y) = \delta + \int_0^a G(y,t)\,F(t,u_\delta(t))\,dt, \quad u_\delta'(y) = \int_0^a G'(y,t)\,F(t,u_\delta(t))\,dt. \tag{4.6}$$

Proof. We proceed to apply Schauder's fixed point theorem. Let S be the closed convex set of the Banach space $C([0,a])$ defined by

$$S = \{v \in C([0,a]) : \delta \le v \le A \text{ in } [0,a]\},$$

where A is a constant to be chosen later. We introduce a (nonlinear) operator T by setting $T(v) = u$ for each $v \in S$, where u is the unique solution (cf. Lemma 3.2) of the problem

$$\begin{cases} u''' &= F(y,v) \quad \text{in} \quad (0,a) \\ u'(0) &= 0, \quad u(a) = \delta, \quad u'(a) = 0. \end{cases}$$

By (3.8),

$$u(y) = \delta + \int_0^a G(y,t)\, F(t, v(t))\, dt, \tag{4.7}$$

$$u'(y) = \int_0^a G'(y,t)\, F(t, v(t))\, dt. \tag{4.8}$$

We claim that $T(S) \subset S$ for A sufficiently large. Indeed, by (2.5), $u''' > 0$ in $(0,a)$ implies that u' is a convex function with $u'(0) = u'(a) = 0$. Therefore $u' < 0$ in $(0,a)$, which means that $u(y) \geq u(a) = \delta$. By (4.7),(2.5) and (3) of Lemma 3.1, for $y \in [0,a]$ and $\delta \leq v \leq A$ we obtain $u(y) \leq \delta + \frac{1}{2} F(a,\delta)\, C_1\, a^2 := A$. This proves the claim. Again by (4.7), since $F(t,\cdot)$ is uniformly continuous on $[\delta, A]$, T is continuous. By (4.8) and (3) of Lemma 3.1, $|u'(y)| \leq A - \delta$; therefore $T(S)$ is bounded in $C^1([0,a])$ and hence relatively compact in $C([0,a])$. By Schauder's fixed point theorem there exists $u_\delta \in S$ such that $T(u_\delta) = u_\delta$, which is the desired solution. Finally, (4.6) follows from (4.7) and (4.8). □

For $y \in (0,a]$, we consider

$$\bar{H}_y(\xi) := H(y,\xi) = \frac{\xi}{F(y,\xi)}. \tag{4.9}$$

In view of (2.5), $\frac{d\bar{H}_y}{d\xi} > 0$ in $(0,\infty)$. Hence its inverse $\xi = \bar{H}_y^{-1}(r)$ is well-defined and increasing in $(0,\infty)$ for any $y \in (0,a]$.

Lemma 4.4. *The solution $u_\delta(y)$ of problem (P_δ) satisfies for all $y \in (0,a]$:*

(1) *$u_\delta(y) \geq \bar{H}_y^{-1}(C_2(a-y)^3)$ where $\bar{H}_y(\xi)$ is defined by (4.9);*
(2) *$u_\delta(y) \leq C$ and $|u'_\delta(y)| \leq C$ independently by δ.*

Proof. By (4.6), (2.5) and (4) of Lemma 3.1, denoting with C a generic positive constant independently by δ, we have

$$u_\delta(y) \geq F(y, u_\delta(y)) \int_y^a G(y,t)\, dt \geq C\, (a-y)^3\, F(y, u_\delta(y)). \tag{4.10}$$

Hence

$$\bar{H}_y(u_\delta(y)) = H(y, u_\delta(y)) = \frac{u_\delta(y)}{F(y, u_\delta(y))} \geq C(a-y)^3. \tag{4.11}$$

Since $\bar{H}_y^{-1}$ is increasing, (4.11) means that

$$u_\delta(y) = \bar{H}_y^{-1}\left(\bar{H}_y(u_\delta(y))\right) \geq \bar{H}_y^{-1}(C(a-y)^3). \tag{4.12}$$

By Remark 4.2, the following inequalities hold:

$$\frac{\xi^3}{y} \leq \bar{H}_y(\xi) \leq \frac{2\,\xi^3}{y} \quad \text{for} \quad \xi \geq \epsilon y, \tag{4.13}$$

$$\left(\frac{\epsilon^{p-2}\,\xi^{2p-1}}{y}\right)^{1/p-1} \leq \bar{H}_y(\xi) \leq \left(\frac{2\,\epsilon^{p-2}\,\xi^{2p-1}}{y}\right)^{1/p-1} \quad \text{for} \quad \xi \leq \epsilon y. \tag{4.14}$$

In turn, (4.13) and (4.14) imply that

$$\left(\frac{1}{2}\,y\,r\right)^{1/3} \leq \bar{H}_y^{-1}(r) \leq (y\,r)^{1/3} \quad \text{for} \quad r \geq \bar{H}_y(\epsilon\,y), \tag{4.15}$$

$$\left(\frac{1}{2}\,\epsilon^{2-p}\,y\,r^{p-1}\right)^{1/2p-1} \leq \bar{H}_y^{-1}(r) \leq \left(\epsilon^{2-p}\,y\,r^{p-1}\right)^{1/2p-1} \quad \text{for} \quad r \leq \bar{H}_y(\epsilon\,y). \tag{4.16}$$

Using also the monotonicity of F, if $u_\delta(y) \leq \epsilon\,y$ we see that

$$\begin{aligned} F(y, u_\delta(y)) &\overset{(4.12)}{\leq} F(y, \bar{H}_y^{-1}(C(a-y)^3)) \\ &\overset{(4.16)}{\leq} F(y, C\,y^{\frac{1}{2p-1}}\,(a-y)^{\frac{3(p-1)}{2p-1}}) \\ &\overset{(4.5)}{\leq} C\,y^{\frac{1}{2p-1}}\,(a-y)^{\frac{-3p}{2p-1}}. \end{aligned} \tag{4.17}$$

Let $y^* \in (0,a)$ such that $\epsilon y^* = u_\delta(y^*)$. This point y^* exists and is unique for δ sufficiently small since $u_\delta' < 0$ in $(0,a)$ and as it has been proved in Lemma 4.3, $u_\delta \in S$. Moreover since u_δ is decreasing we observe that $u_\delta(y) \geq u_\delta(y^*) = \epsilon\,y^* \geq \epsilon\,y$ for $0 < y \leq y^*$ and $u_\delta(y) \leq u_\delta(y^*) = \epsilon\,y^* \leq \epsilon\,y$ for $y^* \leq y \leq a$. By (4.6), (3) of Lemma 3.1 , (4.4) and (4.17), we obtain

$$\begin{aligned} u_\delta(y) &\leq 1 + C\int_0^{y^*} (a-t)\,F(y^*, u_\delta(y^*))\,dt \\ &\quad + C\int_{y^*}^a t^{\frac{1}{2p-1}}\,(a-t)^{-\frac{p+1}{2p-1}}\,dt \\ &\leq 1 + C\,\frac{a\,{y^*}^2}{u(y^*)^2} + C\,a^{\frac{3p-2}{2p-1}} \\ &= 1 + C\,a + C\,a^{\frac{p-1}{2p-1}}. \end{aligned} \tag{4.18}$$

Hence $u_\delta(y) \leq C$ independently by δ. In the same way one proves that $|u_\delta'(y)| \leq C$. $\square$

Proof of Proposition 4.1. We wish to pass to the limit as $\delta \downarrow 0$ in the approximating problems. By (2) of Lemma 4.4, there exists a subsequence (still labelled by δ) such that

$$u_\delta \to u \quad \text{uniformly in} \quad [0,a] \quad \text{as} \quad \delta \downarrow 0.$$

Since $u > 0$ in $[0,a)$ by (1) of Lemma 4.4, then

$$u_\delta''' = F(y,u_\delta) \to F(y,u) \quad \text{uniformly in compact subsets of} \quad [0,a).$$

On the other hand, $u_\delta''' \to u'''$ in the sense of distributions and hence u satisfies the differential equation of problem (4.1). By (3) of Lemma 3.1 and (4.17), we have

$$|G^{(j)}(y,t)|\, F(t,u_\delta(t)) \le C\, t^{\frac{1}{2p-1}}\,(a-t)^{-\frac{p+1}{2p-1}}, \quad y^* \le t \le a \quad j = 0,1.$$

Since $-\frac{p+1}{2p-1} + 1 = \frac{p-2}{2p-1} > 0$, it follows from (4.6) and Lebesgue's dominated convergence theorem that u_δ converges in $C^1([0,a])$ and hence u' satisfies the boundary conditions of problem (4.1). This argument also proves (4.2) and completes the proof of Proposition 4.1. □

In the next result we show that the solution u of problem (P_a) obtained in Proposition (4.1) is in fact unique.

Proposition 4.5. *The solution of problem (P_a) is unique.*

Proof. Let u and v be two solutions of problem (4.1) and let $w = u - v$; then

$$w'(0) = 0, \quad w(a) = 0, \quad w'(a) = 0.$$

Since $w\,w''' = (u-v)(u'''-v''') = (u-v)(F(y,u) - F(y,v))$ and the function $u \to F(y,u)$ is decreasing, it follows that

$$w\,w''' \le 0. \tag{4.19}$$

On the other hand, the following identity holds:

$$y\,w\,w''' = (y\,w\,w'')' - (w\,w')' - \frac{1}{2}(y(w')^2)' + \frac{3}{2}(w')^2. \tag{4.20}$$

Therefore the function

$$g(y) = y\,w\,w'' - w\,w' - \frac{1}{2}y(w')^2 \tag{4.21}$$

is non-increasing. Clearly $g(0) = 0$. Since g is non-increasing the following limits exists:

$$\lim_{y\to a} g(y) = \lim_{y\to a} w(y)\,w''(y) = L.$$

Since u' and v' are bounded, and zero in $y = a$, we have that $|w(y)| \le C(a-y)$. If $L \ne 0$ then $|w''(y)| \ge |L|/C(a-y)$ near $y = a$, which contradicts the continuity of w'. Hence

$L = 0$. Since $g(0) = 0$ and g is non-increasing, we conclude that $g \equiv 0$. Then by (4.20) and (4.21)

$$g' = y\,w\,w''' - \frac{3}{2}\,(w')^2 \equiv 0,$$

and it follows from (4.19) that $w' \equiv 0$. Therefore $w \equiv 0$ and the proof is complete. $\square$

Now we are ready to prove the Theorem.

Proof of the Theorem. Let $M_a = \int_0^a u_a(y)\,dy$. In view of Propositions 4.1 and 4.5, it suffices to prove that

$$\lim_{a\to\infty} M_a = \infty \quad \text{and} \quad \lim_{a\to 0} M_a = 0.$$

Let $\overline{y}_a \in (0,a)$ such that $u_a(\overline{y}_a) = {\overline{y}_a}^{\beta}$, $\beta > 0$. If $\overline{y}_a \geq \frac{a}{4}$, we have

$$M_a \geq \int_0^{\overline{y}_a} u_a(y)\,dy \geq u_a(\overline{y}_a)\,\overline{y}_a \geq C\,a^{\beta+1} \to \infty \quad \text{as} \quad a \uparrow \infty.$$

If $\overline{y}_a < \frac{a}{4}$ and $t \leq 2\overline{y}_a \leq y$, since $a - 2\overline{y}_a > \frac{a}{2}$, we have

$$M_a \geq \int_{2\overline{y}_a}^{a} dy \int_{\overline{y}_a}^{2\overline{y}_a} G(y,t)\,F(t,u_a(t))\,dt > C\,F(\overline{y}_a, u_a(\overline{y}_a))\,{\overline{y}_a}^2\,a^2. \tag{4.22}$$

From Remark 4.2 and (4.22), it follows that

$$M_a > C\,{\overline{y}_a}^{3-2\beta}\,a^2 \quad \text{if} \quad u_a(\overline{y}_a) \geq \epsilon\,\overline{y}_a,$$

and

$$M_a > C\,{\overline{y}_a}^{\frac{2p-1-\beta p}{p-1}}\,a^2 \quad \text{if} \quad u_a(\overline{y}_a) \leq \epsilon\,\overline{y}_a.$$

Then

$$M_a > C\,a^2 \min\left\{{\overline{y}_a}^{3-2\beta}, {\overline{y}_a}^{\frac{2p-1-\beta p}{p-1}}\right\}.$$

Choosing $\beta = 2$ we obtain (since $\overline{y}_a < a/4$)

$$\begin{aligned} M_a &> C\,a^2 \min\left\{{\overline{y}_a}^{-1}, {\overline{y}_a}^{-\frac{1}{p-1}}\right\} \\ &> C\,a^2 \min\left\{a^{-1}, a^{-\frac{1}{p-1}}\right\} \\ &> C \min\left\{a, a^{\frac{2p-3}{p-1}}\right\}, \end{aligned}$$

and therefore M_a tends to infinity as $a \to \infty$. In the limit $a \downarrow 0$, we consider

$$\begin{aligned} M_a &= \int_0^a dy \int_0^{y^*} G(y,t)\,F(t,u_a(t))\,dt + \int_0^a dy \int_{y^*}^a G(y,t)\,F(t,u_a(t))\,dt \\ &= I_1 + I_2. \end{aligned}$$

As observed before, since $u_a(y) \geq u_a(y^*) = \epsilon\, y^* \geq \epsilon\, y$ for $0 < y \leq y^*$ and $u_a(y) \leq u_a(y^*) = \epsilon\, y^* \leq \epsilon\, y$ for $y^* \leq y \leq a$, by (3) of Lemma 3.1 and (4.4),

$$I_1 \leq C \int_0^a dy \int_0^{y^*} (a-t)\, F(y^*, u_a(y^*))\, dt \leq C\, a^2 \left(\frac{y^*}{u(y^*)}\right)^2 = \frac{C\, a^2}{\epsilon^2}.$$

By (3) of Lemma 3.1 and passing to the limit $\delta \downarrow 0$ in (4.17),

$$I_2 \leq C \int_0^a dy \int_{y^*}^a t^{\frac{1}{2p-1}}\, (a-t)^{-\frac{p+1}{2p-1}}\, dt \leq C\, a^{\frac{3p-2}{2p-1}},$$

and the proof is complete. □

Remark 4.6. It's also interesting to consider solutions of (I) with non-zero contact angle, more precisely, with $u'(a) = 0$ replaced by $u'(a) = -\theta$, where $\theta > 0$ is prescribed. For any $M > 0$, $p > 2$, $\epsilon > 0$ and $\theta > 0$, problem (I) admits a solution. Since its proof is identical to the previous case, we omit it.

5. Conclusions and discussion

In view of (1.10), we introduce a generic curve Γ in the (p, ϵ)–plane

$$s \mapsto \Gamma_s = (p_s, \epsilon_s) \in (2, \overline{p}] \times (0, \infty), \quad 2 < \overline{p} < \infty$$

such that

$$\lim_{s\to\infty} \epsilon_s^{p_s - 2} = 0 \qquad \Big(\Rightarrow \lim_{s\to\infty} \epsilon_s = 0\Big).$$

The behaviour of the corresponding solutions (a_s, u_s) in the limit $s \to \infty$ has been studied in [1]. It turns out that u_s converge to a Dirac mass concentrated at the origin, whose rescaled profile is a symmetric concave parabola determined by M:

$$a_s \to 0, \;\; a_s u_s(0) \to \frac{3M}{2}, \;\; \frac{1}{u_s(0)} u_s(a_s \xi) \to 1 - \xi^2 \quad \text{as} \;\; s \to \infty. \tag{5.1}$$

The limiting profile is a parabola, but does not give any information on its rate of convergence with respect to $\Gamma_s = (p_s, \epsilon_s)$. The next result concerns the rate of convergence of u_s:

$$a s^7 \log\left(\frac{1}{\epsilon_s}\right) \to 9M^3 \quad \text{as} \;\; s \to \infty. \tag{5.2}$$

For $\epsilon^{p-2} \ll 1$ solutions of (I) are expected to describe the profile of exact solutions of (1.3)-(1.5) on their macroscopic support, whose definition is motivated by (1.9):

$$S(t) = \left\{ x \in \mathbb{R} : h(t,x) \gtrsim \frac{b\,|x|}{t} \right\}.$$

After a transient time T_0 (during which the evolution is governed by the specific shape of the initial datum), and provided

$$\left(b\,t^{-\frac{5}{7}}\right)^{p-2} \ll 1, \quad \mathrm{diam}\{h(t)>0\} \ll t^{\frac{1}{7}}\sqrt{b^{-1}t^{\frac{5}{7}}}, \tag{5.3}$$

(5.1) and (5.2) yield the following formal asymptotic expansion for the macroscopic profile of h: At leading order in $\left(b\,t^{-\frac{5}{7}}\right)^{p-2} \ll 1$

$$h(t,x) \sim \frac{3M}{2[x_c(t)]^3}\left([x_c(t)]^2 - x^2\right)_+ \quad \text{for } x \in S(t),$$

where $2M$ is the mass of the droplet and

$$[x_c(t)]^7 \sim 9(7\kappa)M^3 \frac{t}{\log\left(b^{-1}t^{\frac{5}{7}}\right)}.$$

Here $x_c(t)$ indicates the boundary of the macroscopic support, i.e. of $S(t)$. In addition, a simple computation shows that

$$(\theta^e)^3 \sim \frac{3}{\kappa}\,\dot{x}_c(t)\,\log\left(b^{-1}t^{\frac{5}{7}}\right),$$

which gives the appropriate logarithmic correction to Tanner's law [24] (the contact angle is proportional to the cubic root of the velocity). This shows that the scaling law in time for macroscopic quantities (length of support, effective contact angle) is only logarithmically affected by the shear-thinning parameter b. It is remarkable, in contrast with slippage models, that if the second constraint in (5.3) holds true for large times, then the scaling law will be valid without any upper bound on the time scale. This fact would represent a big difference with respect to the slip regularization, and might make the shear–thinning thin-film equation (1.3) more reliable in the effective modelling of spreading droplets.

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