A relaxation result in BV for integral functionals with discontinuous integrands

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Abstract - We prove a relaxation theorem in BV for a non coercive functional with linear growth. No continuity of the integrand with respect to the spatial variable is assumed.

1 Introduction

In this paper we study the relaxation in BV(Ω) of an integral functional of the type

$$F(u) = \int_\Omega f(x, u(x), \nabla u(x))\, dx\,,$$

where u is a scalar function from $W^{1,1}(\Omega)$.

In recent years there has been a renewed interest in the L^1-lower semicontinuity of such an integral functional and of its BV counterpart

$$\mathcal{F}(u) = \int_\Omega f(x, u, \nabla u)dx + \int_\Omega f^\infty\left(x, \widetilde{u}, \frac{D^c u}{|D^c u|}\right) d|D^c u| + \int_{J_u \cap \Omega} \left(\int_{u^-(x)}^{u^+(x)} f^\infty(x, s, \nu_u)\, ds \right) d\mathcal{H}^{N-1}(x)$$

with the aim of lessening the regularity assumptions on the integrand f with respect to the spatial variable x (see [20], [21], [9], [7], [8], [18]). Roughly speaking, one can show that the L^1-lower semicontinuity still holds if one replaces the classical continuity and coerciveness assumptions with the weak differentiability of f with respect to x. Therefore the results proved in the above papers suggest that a similar assumption should be also enough to prove that the relaxation in BV of the functional F is represented by $\mathcal{F}$.

In this paper we prove that this representation formula actually holds (see Theorem 6.1) under the assumption that for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ the function $f(\cdot, s, \xi)$ is weakly differentiable and coincides $\mathcal{H}^{N-1}$-a.e. with its precise representative, i.e., it is $\mathcal{H}^{N-1}$-a.e. approximately continuous in Ω. Notice that though the functional F does not change its values if we modify f in a subset of zero Lebesgue measure of Ω, this modification may affect the values of $\mathcal{F}$ on $\mathrm{BV}(\Omega)$. Therefore, if f is not assumed to be $\mathcal{H}^{N-1}$-a.e. approximately continuous with respect to x, then it is not true in general that the relaxation of F is represented by $\mathcal{F}$. On the other hand the approximate continuity alone is not enough to assure the relaxation result, as shown in a counterexample given in [1]. In that paper the representation formula proved here has been obtained in the particular case where the integrand admits a separate dependence on the spatial and the gradient variables. In that case the authors prove the relaxation result by suitably refining the techniques introduced in [6] through the use of a new Reshetnyak-type theorem which applies only to measures which are gradients of BV-function, but does not require the continuity of the integrand. Unfortunately, this technique does not work for a general integrand like the ones considered here.

Thus, we have to follow another approach to relaxation by using the blow-up technique introduced by Fonseca and Müller in [16] and [17] (see also [3] and [15]). However, in all these papers the use of this technique relies strongly on the continuity of the integrand with respect to x, an assumption that here is replaced by the $\mathcal{H}^{N-1}$-a.e. approximate continuity.

This fact introduces some relevant difficulties and requires a delicate study of the approximate continuity of $(N-1)$-dimensional restrictions of BV-functions. Differently from the usual continuity, the approximate continuity is not inherited by the sections of measurable functions. However, in the first part of this paper we prove that given a $\mathcal{H}^{N-1}$-almost everywhere approximately continuous BV-function, its sections keep the same property, as long as we restrict them to a countably $\mathcal{H}^{N-1}$-rectifiable set whose normal is "never" orthogonal to the hyperplane with respect to whom the sections are taken (see Theorem 4.6). This theorem is the main tool needed for dealing with the jump part of the functional via the blow-up technique. More precisely, given a jump point x_0 of a BV-function u, we study the behaviour of the integrand on the tangent hyperplane Π to the jump set at x_0. If the restriction of the integrand to Π is not approximately continuous, we have to approximate Π with a sequence of "good" hyperplanes (where the restriction of f is approximately continuous). In fact, in Proposition 6.5 we prove that this property holds at $\mathcal{H}^{N-1}$-point x_0 of the jump set of u.

The paper is organized as follows: Section 2 is devoted to notations; in Section 3 we recall some properties of BV-functions and some results of geometric measure theory needed for the sequel. In Section 4 we carry on a thorough analysis of the fine properties of the $(N-1)$-dimensional sections of BV-functions. In Section 5 we set the problem and state some technical lemmas; moreover, we discuss some properties of the recession function that do not follow from the corresponding ones of integrand (see Example 5.3). Finally, in Section 6 we state and prove our main result, i.e. the relaxation theorem.

2 Notation

Throughout the paper, $N \geq 2$ is a fixed integer and the letter c denotes a strictly positive constant, whose value may vary from line to line. Given $x_0 \in \mathbb{R}^N$ and $\rho > 0$, $B_\rho(x_0)$ denotes the ball in $\mathbb{R}^N$ centered in x_0 with radius ρ, while $\mathbb{S}^{N-1}$ is the unit sphere of $\mathbb{R}^N$.
Let Ω be a bounded open set in $\mathbb{R}^N$. We denote by $\mathcal{A}(\Omega)$ the family of all bounded open subsets A of Ω and by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets B of Ω. Moreover, $\mathcal{M}(\Omega;\mathbb{R}^N)$ is the space of the $\mathbb{R}^N$-valued Radon measures on Ω; in particular, $\mathcal{M}(\Omega) := \mathcal{M}(\Omega;\mathbb{R})$.

As usual, $\mathcal{L}^N$ stands for the Lebesgue measure on $\mathbb{R}^N$ and $\mathcal{H}^k$ for the k-dimensional Hausdorff measure on $\mathbb{R}^N$. The Lebesgue measure of the unit ball in $\mathbb{R}^N$ is denoted by ω_N, hence $\mathcal{L}^N(B_\rho(x_0)) = \omega_N \rho^N$.

Given a direction $\nu \in \mathbb{S}^{N-1}$, every point $x \in \mathbb{R}^N$ can be decomposed as $x = (x_\nu^\perp, x_\nu)$, with $x_\nu = \langle x, \nu\rangle \nu$ and $x_\nu^\perp = x - x_\nu$. By $\pi_{\nu^\perp}$ we denote the projection of $\mathbb{R}^N$ onto the plane through the origin orthogonal to ν and π_ν denotes the projection of $\mathbb{R}^N$ over the line through the origin in the direction ν. We shall often identify $\pi_{\nu^\perp}(\mathbb{R}^N)$ with $\mathbb{R}^{N-1}$ and $\pi_\nu(\mathbb{R}^N)$ with $\mathbb{R}$, so that, for instance, the $\mathcal{H}^{N-1}$-measure on $\pi_{\nu^\perp}(\mathbb{R}^N)$ will be identified with $\mathcal{L}^{N-1}$. Finally, if E is a given subset of $\mathbb{R}^N$, we set

$$E_{x_\nu^\perp} = \{x_\nu \in \pi_\nu(E) : (x_\nu^\perp, x_\nu) \in E\} \qquad \text{and} \qquad E_{x_\nu} = \{x_\nu^\perp \in \pi_{\nu^\perp}(E) : (x_\nu^\perp, x_\nu) \in E\}\,.$$

Similarly, if $g : \mathbb{R}^N \to \mathbb{R}$ is a given function, for every $x_\nu^\perp \in \mathbb{R}^{N-1}$, we denote by $g_{x_\nu^\perp}$ the restriction of the function g to $\mathbb{R}$; i.e., the function $x_\nu \in \mathbb{R} \mapsto g(x_\nu^\perp, x_\nu)$; for every $x_\nu \in \mathbb{R}$, the restriction g_{x_ν} is defined analogously.
When $\nu = \mathrm{e}_N$, we simply write π_{N-1}, π_1, (x', y), $E_{x'}$, E_y, $g_{x'}$, g_y, instead of $\pi_{\mathrm{e}_N^\perp}$, π_{e_N}, $(x_{\mathrm{e}_N}^\perp, x_{\mathrm{e}_N})$, $E_{x_{\mathrm{e}_N}^\perp}$, $E_{x_{\mathrm{e}_N}}$, $g_{x_{\mathrm{e}_N}^\perp}$, $g_{x_{\mathrm{e}_N}}$.

3 Basic properties of BV-functions and a coarea formula

Let $u \in L^1_{\mathrm{loc}}(\Omega)$; we say that u has an *approximate limit* at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon(x)} |u(y) - z|\, dx = 0\,,$$

where $\fint_{B_\varepsilon(x)}$ stands for $\frac{1}{\mathcal{L}^N(B_\varepsilon(x))}\int_{B_\varepsilon(x)}$. Let S_u be the set of points where the previous property does not hold, the so-called *approximate discontinuity set.* Note that it is a Borel set. If $x \notin S_u$, z is uniquely determined, it is called the *approximate limit* of u at x and it is denoted by $\widetilde{u}(x)$. We recall that $\widetilde{u} : \Omega \setminus S_u \to \mathbb{R}$ is a Borel function.

We say that u is *approximately continuous* at x if $x \notin S_u$ and $u(x) = \widetilde{u}(x)$. Clearly, x is a point of approximate continuity of u if and only if is a Lebesgue point of u and since $\mathcal{L}^N$-almost every $x \in \Omega$ is a Lebesgue point, $\mathcal{L}^N(S_u) = 0$. Notice that in general the above definition of approximate continuity is stronger than the usual one given by Federer (see [13]). However, if $u \in L^\infty_{\mathrm{loc}}(\Omega)$ the two notions agree.

We say that $x_0 \in S_u$ is an *approximate jump point* of u if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{N-1}$, such that $a \neq b$ and

$$\lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon^+(x_0,\nu)} |u(x) - a|\, dx = 0, \qquad \lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon^-(x_0,\nu)} |u(x) - b|\, dx = 0,$$

where $B_\varepsilon^\pm(x_0, \nu) = x_0 + \varepsilon B_\nu^\pm$ and $B_\nu^\pm = \{x \in B_1(0) : \langle x, \nu)\rangle \gtrless 0\}$. The triplet (a, b, ν), uniquely determined by the previous definition up to a permutation of a, b and a change of sign of ν_u, is denoted by $(u^+(x_0), u^-(x_0), \nu_u(x_0))$. We adopt the convention that $u^+(x_0) > u^-(x_0)$. The set of approximate jump points is denoted by J_u and is a Borel set. The quantity $u^+ - u^-$ is the jump of u across the interface J_u and ν_u is the direction of the jump.

The space $\mathrm{BV}(\Omega)$ is defined as the space of all functions $u : \Omega \to \mathbb{R}$ belonging to $L^1(\Omega)$ whose distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure (i.e., $Du \in \mathcal{M}(\Omega; \mathbb{R}^N)$) with total variation $|Du|$ bounded in Ω. We indicate by $D^a u$ and $D^s u$ the *absolutely continuous* and the *singular part* of the measure Du with respect to the Lebesgue measure. We recall that $D^a u$ and $D^s u$ are mutually singular, moreover we can write

$$Du = D^a u + D^s u \qquad \text{and} \qquad D^a u = \nabla u \, \mathcal{L}^N,$$

where ∇u is the *Radon-Nikodým derivative* of $D^a u$ with respect to the Lebesgue measure. In particular,

$$D^s u = D^c u + (u^+ - u^-)\nu_u \, \mathcal{H}^{N-1} \lfloor J_u$$

and J_u is a *countably $\mathcal{H}^{N-1}$-rectifiable* Borel set (see [2, Definition 2.57]) contained in S_u, such that $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. The remaining part $D^c u$ is called the *Cantor part* of Du.

Remark 3.1 Since, for every $u \in \mathrm{BV}(\Omega)$, J_u is a countably $\mathcal{H}^{N-1}$-rectifiable set, hence σ-finite with respect to $\mathcal{H}^{N-1}$, it follows that the set

$$\{\nu \in \mathbb{S}^{N-1} \ : \ \mathcal{H}^{N-1}(\{x \in J_u \ : \ \nu_u(x) = \pm\nu\}) > 0\}$$

is at most countable.

Finally, we define the *precise representative* of a function $u \in \mathrm{BV}(\Omega)$ as

$$u^*(x) = \begin{cases} \widetilde{u}(x) & \text{if } x \in \Omega \setminus S_u \\ \dfrac{u^+(x) + u^-(x)}{2} & \text{if } x \in J_u \\ 0 & \text{if } x \in S_u \setminus J_u \ . \end{cases}$$

Clearly, $u^* : \Omega \to \mathbb{R}$ is a Borel function coinciding $\mathcal{L}^N$-almost everywhere with $\widetilde{u}$ and u.

Let us recall that if $S \subset \mathbb{R}^N$ is a countably $\mathcal{H}^{N-1}$-rectifiable set, then for $\mathcal{H}^{N-1}$-a.e. $x \in S$ there exists the *approximate tangent plane* π_x^S to S at x (see [2, Theorem 2.83]). A unit vector orthogonal to π_x^S is called an *approximate normal* to S at x and denoted by $\nu^S(x)$. Notice that if S is the jump set J_u of some BV function u, then ([2, Theorem 3.59]) $\nu^{J_u}(x) = \pm\nu_u(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$.
Let $1 \le k \le N-1$ be a given integer. By $\pi_{k,N} : \mathbb{R}^N \to \mathbb{R}^k$ we denote the projection of $\mathbb{R}^N$ over the first k components. The formula (3.1) below is a consequence of the general *coarea formula* for rectifiable set [2, Theorem 2.93] and will be used in the sequel. For the reader's convenience we give an explicit proof.

Theorem 3.2 *Let S be a countably $\mathcal{H}^{N-1}$-rectifiable subset of $\mathbb{R}^N$ and $g : \mathbb{R}^N \to [0, +\infty]$ a Borel function. Then, for any integer $1 \le k \le N-1$,*

$$\int_S g(x) \sqrt{\sum_{i=k+1}^{N} |\langle \nu^S(x), e_i \rangle|^2} \, d\mathcal{H}^{N-1}(x) = \int_{\mathbb{R}^k} dt \int_{\pi_{k,N}^{-1}(t) \cap S} g(x) \, d\mathcal{H}^{N-k-1}(x) \,. \tag{3.1}$$

Proof. From the coarea formula for rectifiable sets [2, Theorem 2.93] we have that

$$\int_S g(x) C_k L_x \, d\mathcal{H}^{N-1}(x) = \int_{\mathbb{R}^k} dt \int_{\pi_{k,N}^{-1}(t) \cap S} g(x) \, d\mathcal{H}^{N-k-1}(x) \,, \tag{3.2}$$

where

$$C_k L_x := \sqrt{\det(L_x \circ L_x^*)},$$

$L_x : \pi_x^S \to \mathbb{R}^k$ is the differential of $\pi_{k,N}$ on S at x and L_x^* is the adjoint of the linear map L_x.
Let us denote by $\{\tau_1, \ldots, \tau_{N-1}\}$ an orthonormal base for π_x^S; thus, $\{\tau_1, \ldots, \tau_{N-1}, \nu^S(x)\}$ is an orthonormal base for $\mathbb{R}^N$. Denoting by (a_{ij}) the $k \times k$ matrix representing the linear map $L_x \circ L_x^*$ with respect to the standard base in $\mathbb{R}^k$, we get that

$$a_{ij} = \sum_{h=1}^{N-1} \langle \tau_h, e_i \rangle \langle \tau_h, e_j \rangle \qquad \text{for all } i, j = 1, \ldots, k\,.$$

Writing, for all i, j, $e_i = \sum_{h=1}^{N-1} \langle \tau_h, e_i \rangle \tau_h + \langle \nu^S(x), e_i \rangle \nu^S(x)$, we get

$$\delta_{ij} = \langle e_i, e_j \rangle = \sum_{h=1}^{N-1} \langle \tau_h, e_i \rangle \langle \tau_h, e_j \rangle + \nu_i^S \nu_j^S\,,$$

hence

$$a_{ij} = \delta_{ij} - \nu_i^S \nu_j^S \qquad \text{for all } i, j = 1, \ldots, k\,.$$

From this formula it follows that

$$C_k L_x = \sqrt{\det (I - \widetilde{\nu} \otimes \widetilde{\nu})}\,.$$

where I is the $k \times k$ identity matrix and $\widetilde{\nu} = (\nu_1^S(x), \ldots, \nu_k^S(x))$. The assertion then follows if we show that

$$\det (I - \widetilde{\nu} \otimes \widetilde{\nu}) = \sum_{i=k+1}^{N} |\nu_i^S|^2\,. \tag{3.3}$$

To this aim, recalling that $\widetilde{\nu} \wedge \widetilde{\nu} = 0$, we notice that

$$\begin{aligned}
\det (I - \widetilde{\nu} \otimes \widetilde{\nu}) &= \big\langle (e_1 - \nu_1^S \widetilde{\nu}) \wedge \ldots \wedge (e_k - \nu_k^S \widetilde{\nu}), e_1 \wedge \ldots \wedge e_k \big\rangle \\
&= \langle e_1 \wedge \ldots \wedge e_k, e_1 \wedge \ldots \wedge e_k \rangle - \sum_{i=1}^{k} \nu_i^S \langle e_1 \wedge \ldots \wedge e_{i-1} \wedge \widetilde{\nu} \wedge e_{i+1} \wedge \ldots \wedge e_k, e_1 \wedge \ldots \wedge e_k \rangle \\
&= 1 - \sum_{i=1}^{k} |\nu_i^S|^2 = \sum_{i=k+1}^{N} |\nu_i^S|^2\,.
\end{aligned}$$

Thus, the proof is complete. □

For a general survey on measures and BV-functions we refer to [13], [22], [19], [23], [12], [2].

4 Sections of BV-functions

In this section we state some fine properties of BV-functions, which will be needed in the sequel.

Lemma 4.1 *Let $g : \Omega \to \mathbb{R}$ be a Borel function. Assume that $g \in L^1_{\text{loc}}(\Omega)$ and set*

$$G^* := \left\{ (x', y) \in \Omega \setminus S_g : \lim_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} |g(z', y) - g^*(x', y)|\, dz' = 0 \right\}\,,$$

where $Q'(x', \varepsilon) := x' + \varepsilon Q'$ and $Q' = (-1/2, 1/2)^{N-1}$. Then G^* is a Borel set.

Proof. Notice that $G^* = G_1 \cap G_2$, where

$$G_1 := \left\{(x', y) \in \Omega \setminus S_g : \text{ there exists } z \in \mathbb{R} \text{ such that } \lim_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} |g(z', y) - z| \, dz' = 0\right\}$$

and

$$G_2 := \left\{(x', y) \in \Omega \setminus S_g : \lim_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} g(z', y) \, dz' = g^*(x', y)\right\}.$$

We claim that G_1 is a Borel set. In fact, consider a dense sequence $\{q_i\} \subset \mathbb{R}$ and, for every $i, j \in \mathbb{N}$, set

$$G_{ij} = \left\{(x', y) \in \Omega : \limsup_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} |g(z', y) - q_i| \; dz' < \frac{1}{j}\right\}.$$

It is not difficult to check that if $h : \mathbb{R}^N \to \mathbb{R}$ is Borel function, then also $(x', y) \mapsto \fint_{Q'(x',\varepsilon)} h(z', y) dz'$ is a Borel function for any $\varepsilon > 0$. From this fact we get immediately that each G_{ij} is a Borel set, and since

$$G_1 = \bigcap_{j=1}^{\infty} \bigcup_{i=1}^{\infty} G_{ij} \,,$$

our claim follows. On the other hand, since for any $\varepsilon > 0$ the function $(x', y) \mapsto \fint_{Q'(x',\varepsilon)} g(z', y) dz'$ is Borel, we easily get that also G_2 is a Borel set. This concludes the proof. □

Lemma 4.2 ([2, Theorem 3.108]) *Let $g \in \mathrm{BV}(\Omega)$ be a given function. Then, for $\mathcal{L}^{N-1}$-almost every $x' \in \pi_{N-1}(\Omega)$, the function $g_{x'}$ belongs to $\mathrm{BV}(\Omega_{x'})$. Moreover $J_{g_{x'}} = (J_g)_{x'}$ and $(g^*)_{x'}(y) = (g_{x'})^*(y)$ for every $y \in \Omega_{x'} \setminus (J_g)_{x'}$. In particular, if $(J_g)_{x'} = \emptyset$, then $(g^*)_{x'}(y) = (g_{x'})^*(y)$ for every $y \in \Omega_{x'}$ and both functions $(g^*)_{x'}$ and $(g_{x'})^*$ are continuous in $\Omega_{x'}$.*

Lemma 4.3 *Let $g \in \mathrm{BV}(\Omega)$ be a given function. Then, for $\mathcal{L}^1$-almost every $y \in \pi_1(\Omega)$, the function g_y belongs to $\mathrm{BV}(\Omega_y)$.*

Proof. If $N = 2$, the property follows by Lemma 4.2. If $N > 2$, the property can be easily obtained following the proof of [2, Theorem 3.103]. □

In the next two lemmas we assume $N > 2$. For every $x = (x', y) \in \mathbb{R}^N$, we set

$$\hat{x}'_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N-1}) \in \mathbb{R}^{N-2}$$

and write, for the sake of simplicity, $x = (x_i, \hat{x}'_i, y)$. Moreover, if E is a given set, with $E_{\hat{x}'_i y}$ we denote $E_{x_\nu^\perp}$, with $\nu = \mathrm{e}_i$; a similar notation will be used also for functions.

Lemma 4.4 *Let $g \in \mathrm{BV}(\Omega)$ be a given function. Then there exists a set $N_0 \subset \mathbb{R}$ with $\mathcal{L}^1(N_0) = 0$ with the following property: for every $y \in \mathbb{R} \setminus N_0$, $g_y \in \mathrm{BV}(\Omega_y)$ and, for every $i = 1, \ldots, N-1$, there exists a set $N^{iy} \subset \mathbb{R}^{N-2}$ with $\mathcal{L}^{N-2}(N^{iy}) = 0$ such that, for every $\hat{x}'_i \in \mathbb{R}^{N-2} \setminus N^{iy}$ we have that $g_{\hat{x}'_i y} \in \mathrm{BV}(\Omega_{\hat{x}'_i y})$ and*

$$(S_{g_y})_{\hat{x}'_i} = J_{g_{\hat{x}'_i y}} \tag{4.1}$$

$$[(g_y)^*]_{\hat{x}'_i}(x_i) = (g_{\hat{x}'_i y})^*(x_i) \qquad \textit{for every } x_i \in \Omega_{\hat{x}'_i y} \setminus J_{g_{\hat{x}'_i y}} \,. \tag{4.2}$$

Proof. For the sake of simplicity, assume $\Omega = \mathbb{R}^N$. By Lemma 4.3 there exists $N_0 \subset \mathbb{R}$ with $\mathcal{L}^1(N_0) = 0$ such that for every $y \in \mathbb{R} \setminus N_0$ we have that $g_y \in BV(\mathbb{R}^{N-1})$, J_{g_y} is a countably $\mathcal{H}^{N-2}$-rectifiable set and $\mathcal{H}^{N-2}(S_{g_y} \setminus J_{g_y}) = 0$. Moreover, by Theorem 3.2, we have that, for every $i = 1, \ldots, N-1$,

$$0 = \int_{S_{g_y} \setminus J_{g_y}} |\langle \nu^{S_{g_y}}, e_i \rangle| \, d\mathcal{H}^{N-2} = \int_{\mathbb{R}^{N-2}} \mathcal{H}^0((S_{g_y} \setminus J_{g_y})_{\hat{x}'_i}) \, d\hat{x}'_i$$

so that there exists a set $N_1^{iy} \subset \mathbb{R}^{N-2}$ with $\mathcal{L}^{N-2}(N_1^{iy}) = 0$, such that for every $\hat{x}'_i \in \mathbb{R}^{N-2} \setminus N_1^{iy}$ we have $(S_{g_y} \setminus J_{g_y})_{\hat{x}'_i} = \emptyset$; i.e.,

$$(S_{g_y})_{\hat{x}'_i} = (J_{g_y})_{\hat{x}'_i} \, . \tag{4.3}$$

Moreover, by Lemma 4.2 applied to g_y, we have that for every $y \in \mathbb{R} \setminus N_0$ and for every $i = 1, \ldots, N-1$, there exists a set $N_2^{iy} \subset \mathbb{R}^{N-2}$ with $\mathcal{L}^{N-2}(N_2^{iy}) = 0$, such that for every $\hat{x}'_i \in \mathbb{R}^{N-2} \setminus N_2^{iy}$ we have that $g_{\hat{x}'_i y} \in BV(\mathbb{R})$ and

$$(J_{g_y})_{\hat{x}'_i} = J_{g_{\hat{x}'_i y}} \, . \tag{4.4}$$

$$[(g_y)^*]_{\hat{x}'_i}(x_i) = (g_{\hat{x}'_i y})^*(x_i) \qquad \text{for every } x_i \in \mathbb{R} \setminus J_{g_{\hat{x}'_i y}} \, . \tag{4.5}$$

Finally, for every $y \in \mathbb{R} \setminus N_0$ and for every $i = 1, \ldots, N-1$, set $N^{iy} = N_1^{iy} \cup N_2^{iy} \subset \mathbb{R}^{N-2}$, so that $\mathcal{L}^{N-2}(N^{iy}) = 0$ and for every $\hat{x}'_i \in \mathbb{R}^{N-2} \setminus N^{iy}$ we have that (4.3), (4.4) and (4.5) hold. Hence the assertion follows. □

Lemma 4.5 *Let $g \in \mathrm{BV}(\Omega)$ be a function which is approximately continuous in $\mathcal{H}^{N-1}$-almost every point of Ω. Then there exists a set $M_0 \subset \mathbb{R}$ with $\mathcal{L}^1(M_0) = 0$ with the following property: for every $y \in \mathbb{R} \setminus M_0$, $g_y \in \mathrm{BV}(\Omega_y)$ and, for every $i = 1, \ldots, N-1$, there exists a set $M^{iy} \subset \mathbb{R}^{N-2}$ with $\mathcal{L}^{N-2}(M^{iy}) = 0$ such that, for every $\hat{x}'_i \in \mathbb{R}^{N-2} \setminus M^{iy}$ we have that $g_{\hat{x}'_i y} \in \mathrm{BV}(\Omega_{\hat{x}'_i y})$ and*

$$J_{g_{\hat{x}'_i y}} = (S_g)_{\hat{x}'_i y} = (J_g)_{\hat{x}'_i y} = \emptyset \, , \tag{4.6}$$

$$(g_{\hat{x}'_i y})^*(x_i) = (g^*)_{\hat{x}'_i y}(x_i) \qquad \textit{for every } x_i \in \Omega_{\hat{x}'_i y} \tag{4.7}$$

and both functions are continuous.

Proof. As before, we assume for simplicity $\Omega = \mathbb{R}^N$. By assumption we have that $\mathcal{H}^{N-1}(S_g) = 0$, thus from Theorem 3.2 we get that for every $i = 1, \ldots, N-1$

$$0 = \int_{S_g} |\langle \nu^{S_g}, e_i \rangle| \, d\mathcal{H}^{N-1} = \int_{\mathbb{R}^{N-1}} \mathcal{H}^0((S_g)_{\hat{x}'_i y}) \, d\hat{x}'_i \, dy \, .$$

Therefore there exists $M_1^i \subset \mathbb{R}^{N-1}$, with $\mathcal{L}^{N-1}(M_1^i) = 0$, such that for every $(\hat{x}'_i, y) \in \mathbb{R}^{N-1} \setminus M_1^i$ we have $(S_g)_{\hat{x}'_i y} = \emptyset$. Set

$$M_1 = \{ y \in \mathbb{R} : \mathcal{L}^{N-2}((M_1^i)_y) > 0 \quad \text{for at least one index } i = 1, \ldots, N-1 \} \, .$$

Then, by Fubini's theorem

$$0 = \mathcal{L}^{N-1}(M_1^i) = \int_{\mathbb{R}} \mathcal{L}^{N-2}((M_1^i)_y) \, dy \, ,$$

so that $\mathcal{L}^{N-2}((M_1^i)_y) = 0$ for a.e. $y \in \mathbb{R}$; i.e. $\mathcal{L}^1(M_1) = 0$. Moreover, for every $y \in \mathbb{R} \setminus M_1$ and every $\hat{x}_i' \in \mathbb{R}^{N-2} \setminus (M_1^i)_y$, recalling that $(J_g)_{\hat{x}_i' y} \subseteq (S_g)_{\hat{x}_i' y}$, we have

$$(S_g)_{\hat{x}_i' y} = (J_g)_{\hat{x}_i' y} = \emptyset \,. \tag{4.8}$$

By Lemma 4.2 we have that for every $i = 1, \ldots, N-1$, there exists a set $M_2^i \subset \mathbb{R}^{N-1}$ with $\mathcal{L}^{N-1}(M_2^i) = 0$, such that for every $(\hat{x}_i', y) \in \mathbb{R}^{N-1} \setminus M_2^i$ we have that $g_{\hat{x}_i' y} \in BV(\mathbb{R})$ and

$$J_{g_{\hat{x}_i' y}} = (J_g)_{\hat{x}_i' y} \,, \tag{4.9}$$

$$(g^*)_{\hat{x}_i' y}(x_i) = (g_{\hat{x}_i' y})^*(x_i) \qquad \text{for every } x_i \in \mathbb{R} \setminus (J_g)_{\hat{x}_i' y} \,. \tag{4.10}$$

Set

$$M_2 = \{y \in \mathbb{R} : \mathcal{L}^{N-2}((M_2^i)_y) > 0 \quad \text{for at least one index } i = 1, \ldots, N-1\} \,.$$

As before, Fubini's theorem implies that $\mathcal{L}^1(M_2) = 0$. Thus for any $y \in \mathbb{R} \setminus M_2$ and any $\hat{x}_i' \in \mathbb{R}^{N-2} \setminus (M_2^i)_y$ we obtain that (4.9) and (4.10) hold. Finally, set $M_0 = M_1 \cup M_2 \subset \mathbb{R}$ and $M^{iy} = (M_1^i)_y \cup (M_2^i)_y$, so that $\mathcal{L}^1(M_0) = 0$ and, for every $y \in \mathbb{R} \setminus M_0$, $\mathcal{L}^{N-2}(M^{iy}) = 0$. Moreover, for every $y \in \mathbb{R} \setminus M_0$, by (4.8), (4.9) and (4.10) we have

$$\emptyset = (S_g)_{\hat{x}_i' y} = (J_g)_{\hat{x}_i' y} = J_{g_{\hat{x}_i' y}} \,,$$

$$(g^*)_{\hat{x}_i' y}(x_i) = (g_{\hat{x}_i' y})^*(x_i) \qquad \text{for every } x_i \in \mathbb{R}$$

and by Lemma 4.2 both functions are continuous. □

Next theorem state that given a $\mathcal{H}^{N-1}$-a.e. approximately continuous BV function g, its $(N-1)$-dimensional sections are still $\mathcal{H}^{N-1}$-a.e. approximately continuous along a countably $\mathcal{H}^{N-1}$-rectifiable set whose normals are "never" orthogonal to the direction in which the sections are taken.

Theorem 4.6 *Let $g \in \mathrm{BV}(\Omega)$ be a function which is approximately continuous in $\mathcal{H}^{N-1}$- almost every point of Ω. Set*

$$G = \left\{ (x', y) \in \Omega \setminus S_g : \lim_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} |g(z', y) - g(x', y)| \, dz' = 0 \right\} .$$

Let $S \subset \Omega$ be a countably $\mathcal{H}^{N-1}$-rectifiable set such that $\mathcal{H}^{N-1}(\{x \in S : \nu^S(x) = \pm e_N\}) = 0$. Then $\mathcal{H}^{N-1}(S \setminus G) = 0$.

Proof. Again, we assume for simplicity that $\Omega = \mathbb{R}^N$.
Since g is approximately continuous $\mathcal{H}^{N-1}$-a.e., the thesis will be achieved if we prove that $\mathcal{H}^{N-1}(S \setminus G^*) = 0$, where G^* is the set defined in Lemma 4.1. To this aim, let us first assume that $N > 2$.
Following the notation used in Lemma 4.4 and Lemma 4.5 let us take $y \in \mathbb{R} \setminus (N_0 \cup M_0)$ and for every $i = 1, \ldots, N-1$, $\hat{x}_i' \in \mathbb{R}^{N-2} \setminus (N^{iy} \cup M^{iy})$. Then by (4.1) and (4.6) we obtain

$$(S_g)_{\hat{x}_i' y} = (S_{g_y})_{\hat{x}_i'} = \emptyset$$

so that, for every $x_i \in \mathbb{R}$ we have that $x_i \notin (S_g)_{\hat{x}_i' y}$, (i.e., $x = (x_i, \hat{x}_i', y) \notin S_g$) and $x_i \notin (S_{g_y})_{\hat{x}_i'}$ (i.e., $x' = (x_i, \hat{x}_i') \notin S_{g_y}$). Hence, x is a point of approximate continuity for g and x' is a point of approximate continuity for g_y. Moreover by (4.2) and (4.7) it follows that $(g_y)^*(x') = (g^*)_{\hat{x}_i' y}(x_i) = g^*(x)$, i.e.

$$\lim_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} |g(z', y) - g^*(x)| \, dz' = \lim_{\varepsilon \to 0^+} \fint_{Q'(x',\varepsilon)} |g(z', y) - (g_y)^*(x')| \, dz' = 0 \,,$$

which implies that $x \in G^*$; i.e., $x_i \in G^*_{\hat{x}'_i y}$. In particular we obtain that, for every $y \in \mathbb{R} \setminus (N_0 \cup M_0)$, every $i = 1, \ldots, N-1$ and every $\hat{x}'_i \in \mathbb{R}^{N-2} \setminus (N^{iy} \cup M^{iy})$

$$G^*_{\hat{x}'_i y} = \mathbb{R}\,. \tag{4.11}$$

By Theorem 3.2 and (4.11) we obtain

$$\begin{aligned}\mathcal{H}^{N-2}((S \setminus G^*)_y) &= \sum_{i=1}^{N-1} \int_{(S\setminus G^*)_y} |\langle \nu^{S_y}, e_i \rangle|^2 \, d\mathcal{H}^{N-2} \leq \sum_{i=1}^{N-1} \int_{(S\setminus G^*)_y} |\langle \nu^{S_y}, e_i \rangle| \, d\mathcal{H}^{N-2} \\ &= \sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-2}} \mathcal{H}^0([(S \setminus G^*)_y]_{\hat{x}'_i}) \, d\hat{x}'_i = \sum_{i=1}^{N-1} \int_{\mathbb{R}^{N-2}} \mathcal{H}^0((S \setminus G^*)_{\hat{x}'_i y}) \, d\hat{x}'_i = 0\,,\end{aligned}$$

which implies $\mathcal{H}^{N-2}((S \setminus G^*)_y) = 0$ for every $y \in \mathbb{R} \setminus (N_0 \cup M_0)$. Finally, using again Theorem 3.2, we obtain

$$\int_{S\setminus G^*} \sqrt{1 - |\langle \nu^S, e_N \rangle|^2} \, d\mathcal{H}^{N-1} = \int_{\mathbb{R}} \mathcal{H}^{N-2}((S \setminus G^*)_y) \, dy = 0\,.$$

Therefore, taking into account the assumption made on S, we have $\mathcal{H}^{N-1}(S \setminus G^*) = 0$.
If $N = 2$, we apply the coarea formula (3.1) again, thus getting

$$\int_{S\setminus G^*} |\langle \nu^S, e_1 \rangle| \, d\mathcal{H}^1 = \int_{\mathbb{R}} \mathcal{H}^0((S \setminus G^*)_y) \, dy = 0\,,$$

where the last equality holds since $\mathcal{H}^1(S_g) = 0$ implies $(J_g)_y = \emptyset$ for $\mathcal{L}^1$-almost every $y \in \pi_1(\Omega)$ and, by Lemma 4.2, $(g^*)_y(x) = (g_y)^*(x)$ for all $x \in \Omega_y$ and for $\mathcal{L}^1$-almost every $y \in \pi_1(\Omega)$. Hence, the assertion follows. □

Remark 4.7 Clearly, Theorem 4.6 still holds if we replace e_N by a generic direction ν. More precisely, given any direction $\nu \in \mathbb{S}^{N-1}$, set

$$G_\nu = \left\{ x = (x_\nu^\perp, x_\nu) \in \Omega \setminus S_g : \lim_{\varepsilon \to 0^+} \fint_{Q_\nu^\perp(x,\varepsilon)} |g(z_\nu^\perp, x_\nu) - g(x_\nu^\perp, x_\nu)| \, dz_\nu^\perp = 0 \right\}\,,$$

where $Q_\nu^\perp(x,\varepsilon) = \pi_{\nu^\perp}(x + \varepsilon Q_\nu)$, $Q_\nu = R_\nu(-1/2, 1/2)^N$ and R_ν denotes a rotation such that $R_\nu e_N = \nu$. Then $\mathcal{H}^{N-1}(S \setminus G_\nu) = 0$ for every countably $\mathcal{H}^{N-1}$-rectifiable subset S of Ω, such that $\mathcal{H}^{N-1}(\{x \in S : \nu^S(x) = \pm\nu\}) = 0$.

5 Setting of the problem

Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Borel function satisfying the following conditions:

$$\begin{cases} (i) & f(\cdot, s, \xi) \in W^{1,1}(\Omega)\,, \text{ for every } (s,\xi) \in \mathbb{R} \times \mathbb{R}^N\,; \\ (ii) & f(\cdot, s, \xi) \text{ is approximately continuous } \mathcal{H}^{N-1}\text{-a.e. in } \Omega, \text{ for every } (s,\xi) \in \mathbb{R} \times \mathbb{R}^N\,; \\ (iii) & \text{for every bounded set } B \subset \mathbb{R} \times \mathbb{R}^N \text{ there exists a constant } L(B) \text{ such that} \\ & \displaystyle\int_\Omega |\nabla_x f|(x, s, \xi) \, dx < L(B) \qquad \forall (s,\xi) \in B\,. \end{cases} \tag{5.1}$$

Remark 5.1 Notice that assumption (ii) of (5.1) seems redundant, since every $W^{1,1}$-function admits a $\mathcal{H}^{N-1}$-a.e approximately continuous representative. Moreover, the functional in (5.5) is clearly not affected by the choice of the representative. However, functional (5.6) does depend on the particular representative chosen. Therefore, the representation formula provided by Theorem 6.1 below does not hold if we take a representative of f not satisfying (ii).

We will assume that

$$\text{(5.2)} \qquad f(x,s,\cdot) \quad \text{is convex for every } (x,s) \in \Omega \times \mathbb{R}\,;$$

$$\text{(5.3)} \qquad |f(x,s,\xi) - f(x,s_0,\xi)| \le \Lambda(1+|\xi|)|s-s_0| \quad \text{for every } (x,s,\xi),(x,s_0,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\,;$$

$$\text{(5.4)} \qquad 0 \le f(x,s,\xi) \le \Lambda(1+|\xi|) \quad \text{for every } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\,,$$

for some positive Λ. From (5.2) and (5.4), it follows that f is Lipschitz continuous in the last variable, uniformly with respect to (x,s).

For every $A \in \mathcal{A}(\Omega)$ and every $u \in W^{1,1}(\Omega)$, we define

$$\text{(5.5)} \qquad F(u,A) = \begin{cases} \displaystyle\int_A f(x,u,\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega) \\ +\infty & \text{if } u \in \mathrm{BV}(\Omega) \setminus W^{1,1}(\Omega). \end{cases}$$

Our aim is to prove an integral representation theorem for the relaxation $\overline{F}$ of F, with respect to the L^1-topology. We recall that the relaxation of F is the greatest lower semicontinuous functional not greater than F; i.e.,

$$\overline{F}(u,\Omega) := \inf\{\liminf_{n\to+\infty} F(u_n,\Omega) \;:\; u_n \in W^{1,1}(\Omega)\,,\; u_n \to u \text{ in } L^1(\Omega)\}\,.$$

Among the main properties of the relaxation, we recall the following ones:

(i) for every $A \in \mathcal{A}(\Omega)$, $\overline{F}(\cdot,A)$ is lower semicontinuous with respect to the L^1-topology;

(ii) for every $A \in \mathcal{A}(\Omega)$, $\overline{F}(\cdot,A)$ is local; i.e., for every $u,v \in \mathrm{BV}(\Omega)$, with $u=v$ on A, $\overline{F}(u,A) = \overline{F}(v,A)$;

(iii) for every $u \in \mathrm{BV}(\Omega)$, $\overline{F}(u,\cdot)$ is a σ-additive measure on $\mathcal{B}(\Omega)$.

For other properties of the relaxation we refer to [10], [11], [4], [5].

We set, for every $A \in \mathcal{A}(\Omega)$ and every $u \in BV(\Omega)$,

$$\text{(5.6)} \quad \mathcal{F}(u,A) = \int_A f(x,u,\nabla u)dx + \int_A f^\infty\left(x,\widetilde{u},\frac{D^c u}{|D^c u|}\right)d|D^c u| + \int_{J_u\cap A}\left(\int_{u^-(x)}^{u^+(x)} f^\infty(x,s,\nu_u)\,ds\right)d\mathcal{H}^{N-1}(x),$$

where $f^\infty : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is the so-called *recession function* of f, defined by

$$\text{(5.7)} \qquad f^\infty(x,s,\xi) = \lim_{t\to+\infty} \frac{f(x,s,t\xi)}{t} = \sup_{t>0} \frac{f(x,s,t\xi) - f(x,s,0)}{t}\,.$$

Notice that assumptions (5.2) and (5.4) imply that the limit in (5.7) exists for every $(x,s,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$ (since the function $t \mapsto \frac{f(x,s,t\xi)-f(x,s,0)}{t}$ is increasing). Moreover, the function f^∞ is convex and positively homogeneous of degree one in the last variable and, as a consequence of definition (5.7), we have that

$$\text{(5.8)} \qquad \frac{f(x,s,t\xi)}{t} \le f^\infty(x,s,\xi) + \frac{f(x,s,0)}{t} \qquad \text{for all } t > 0\,.$$

Notice also that f^∞ is a Borel function in $\Omega \times \mathbb{R} \times \mathbb{R}^N$. Thus, the functional $\mathcal{F}$ in (5.6) is well defined. By the assumptions made on f, it follows that

$$0 \le f^\infty(x,s,\xi) \le \Lambda|\xi| \qquad \text{for every } (x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\,, \tag{5.9}$$

$$|f^\infty(x,s,\xi) - f^\infty(x,s_0,\xi)| \le \Lambda|\xi||s-s_0| \qquad \text{for every } (x,s,\xi),(x,s_0,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\,. \tag{5.10}$$

In the sequel, we will assume also that $f^\infty(\cdot,s,\xi)$ is in $\mathrm{BV}(\Omega)$ for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, that for every bounded Borel set $B \subset \mathbb{R} \times \mathbb{R}^N$ there exists a constant $L(B)$ such that

$$|D_x f^\infty(\cdot,s,\xi)|(\Omega) < L(B) \qquad \text{for all } (s,\xi) \in B\,, \tag{5.11}$$

and that, for any $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$f^\infty(\cdot,s,\xi) \text{ is approximately continuous for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega\,. \tag{5.12}$$

Remark 5.2 Note that (5.11), (5.12) trivially follow from (5.1) when $f(x,s,\cdot)$ is positively 1-homogeneous (since in this case f^∞ coincides with f) and when $f(x,s,\xi) = a(x,s)b(\xi)$ or $f(x,s,\xi) = a(x)b(s,\xi)$ (since in this case the dependence on x is not involved in the limit (5.7)). However, in general, property (5.12) is not a consequence of (5.1)-(5.4), as the following example shows.

Example 5.3 Let $f : (-1,1) \times \mathbb{R} \to [0,+\infty)$ be a function defined by

$$f(x,\xi) = \begin{cases} \dfrac{5}{2}|\xi| - \dfrac{1}{2x} & \text{if } \ x \ge \dfrac{1}{|\xi|}, \\ \dfrac{x|\xi|^2}{2} + \dfrac{3}{2}|\xi| & \text{if } \ 0 \le x \le \dfrac{1}{|\xi|}, \\ \dfrac{3}{2}|\xi| & \text{if } \ x \le 0, \end{cases}$$

if $\xi \ne 0$ and $f(x,0) = 0$ for every $x \in (-1,1)$. It is easy to check that f is a Lipschitz function with respect to x satisfying (5.1)-(5.4). Nevertheless, condition (5.12) does not hold, since

$$f^\infty(x,\xi) = \begin{cases} \dfrac{5}{2}|\xi| & \text{if } \ x > 0, \\ \dfrac{3}{2}|\xi| & \text{if } \ x \le 0. \end{cases}$$

The following proposition shows that, under further assumptions on f, the recession function f^∞ necessarily satisfies (5.11) and (5.12).

Proposition 5.4 *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a function satisfying (5.1), (5.2) and (5.4). Assume that (i) $f_\xi(\cdot,s,\xi)$ is weakly differentiable in Ω for all (s,ξ) and that*

$$\int_\Omega |\nabla_x f_\xi(x,s,\xi)|\,dx \le c_0 \qquad \text{for all } (s,\xi) \in \mathbb{R} \times \mathbb{R}^N\,,$$

(ii) for every $(x_0, s) \in \Omega \times \mathbb{R}$ and for every $\varepsilon > 0$ there exist $\delta > 0$ and $L > 0$ such that

$$\left| f^\infty(x, s, \xi) - \frac{f(x, s, t\xi)}{t} \right| \leq \varepsilon \left(1 + \frac{f(x, s, t\xi)}{t} \right)$$

for any $x \in \Omega$, with $|x - x_0| \leq \delta$, any $\xi \in \mathbb{R}^N$ and any $t > L$.
Then f^∞ satisfies conditions (5.11) and (5.12).

Proof. Fix $s \in \mathbb{R}$ and $\varphi \in C_0^1(\Omega, \mathbb{R}^N)$ and set $\psi_s(\xi) = \int_\Omega \langle \nabla_x f(x, s, \xi), \varphi(x) \rangle dx$. From assumption (i) we get easily that $|\nabla \psi_s(\xi)| \leq c_0 \|\varphi\|_\infty$ for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Hence ψ_s is Lipschitz continuous and $\mathrm{Lip}(\psi_s) \leq c_0 \|\varphi\|_\infty$. Therefore, from (5.4) and (5.1), we get that for all (s, ξ)

$$\begin{aligned} \int_\Omega f^\infty(x, s, \xi) \mathrm{div}\varphi(x)\, dx &= \lim_{h \to \infty} \int_\Omega \frac{f(x, s, h\xi)}{h} \mathrm{div}\varphi(x)\, dx = \lim_{h \to \infty} \int_\Omega \langle \frac{\nabla_x f(x, s, h\xi)}{h}, \varphi(x) \rangle\, dx \\ &= \lim_{h \to \infty} \int_\Omega \frac{1}{h} \langle \nabla_x f(x, s, h\xi) - \nabla_x f(x, s, 0), \varphi(x) \rangle\, dx \leq c_0 \|\varphi\|_\infty |\xi| \end{aligned}$$

and from this inequality we get at once that $f^\infty(\cdot, s, \xi) \in \mathrm{BV}(\Omega)$ and $|D_x f^\infty(\cdot, s, \xi)|(\Omega) \leq c_0|\xi|$.
Fix $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Recalling that $f(\cdot, s, \xi) \in W^{1,1}(\Omega)$, from (5.1) we get that there exists a set $N_{s,\xi}$, with $\mathcal{H}^{N-1}(N_{s,\xi}) = 0$, such that for any $h \in \mathbb{N}$ the function $f(\cdot, s, h\xi)$ is approximately continuous in $\Omega \setminus N_{s,\xi}$. Let us fix $x_0 \in \Omega \setminus N_{s,\xi}$ and $\varepsilon > 0$ and let δ and L be the quantities provided by the assumption (ii). Then for any $\rho \in (0, \delta)$ and $h > L \vee 1$ we get, recalling also (5.4),

$$\fint_{B_\rho(x_0)} |f^\infty(x, s, \xi) - f^\infty(x_0, s, \xi)|\, dx \leq \frac{1}{h} \fint_{B_\rho(x_0)} |f(x, s, h\xi) - f(x_0, s, h\xi)|\, dx + 2\varepsilon(1 + \Lambda(1 + |\xi|))\,.$$

Thus, letting first $\rho \to 0$ and then $\varepsilon \to 0$ in the inequality above, we get that $f^\infty(\cdot, s, \xi)$ is approximately continuous at x_0. Hence, the assertion follows. □

Lemma 5.5 *Assume that $f, f^\infty : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy (5.2)–(5.4) and (5.12). Then, there exists $N_0 \subset \Omega$ (indipendent of (s, ξ)), with $\mathcal{H}^{N-1}(N_0) = 0$, such that*

$$\lim_{\varepsilon \to 0^+} \fint_{B_\varepsilon(x)} |f^\infty(y, s, \xi) - f^\infty(x, s, \xi)|\, dy = 0 \tag{5.13}$$

for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and every $x \in \Omega \setminus N_0$.

Proof. From (5.12), we get that (5.13) holds for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and every $x \in \Omega \setminus N_{s,\xi}$, with $\mathcal{H}^{N-1}(N_{s,\xi}) = 0$. Now, let $\Sigma_0 = \{(s_k, \xi_k)\}$ be a countable dense set in $\mathbb{R} \times \mathbb{R}^N$ and set $N_0 = \bigcup_k N_{s_k,\xi_k}$. Clearly, $\mathcal{H}^{N-1}(N_0) = 0$ and (5.13) holds for every $x \in \Omega \setminus N_0$ and every $(s_k, \xi_k) \in \Sigma_0$. Therefore, observing that (5.9) implies that $\xi \mapsto f^\infty(x, s, \xi)$ is Lipschitz continuous with a constant not depending on (x, s), and recalling (5.10), it follows that (5.13) actually holds for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and every $x \in \Omega \setminus N_0$. □

Lemma 5.6 *Let f, f^∞ satisfy the same assumptions as in Lemma 5.5 and (5.11). Then, for every $\nu \in \mathbb{S}^{N-1}$, there exists $N_\nu \subset \pi_{\nu^\perp}(\Omega)$ (indipendent of s), with $\mathcal{H}^{N-1}(N_\nu) = 0$, such that the function $x_\nu \mapsto f^\infty(x_\nu^\perp, x_\nu, s, \nu)$ is continuous for every $x_\nu^\perp \in \pi_{\nu^\perp}(\Omega) \setminus N_\nu$ and every $s \in \mathbb{R}$.*

Proof. By Lemma 4.2, it follows that for every $(s,\nu) \in \mathbb{R} \times \mathbb{S}^{N-1}$ there exists a set $N_{s,\nu} \subset \pi_{\nu^\perp}(\Omega)$, with $\mathcal{H}^{N-1}(N_{s,\nu}) = 0$, such that for every $x_\nu^\perp \in \pi_{\nu^\perp}(\Omega) \setminus N_{s,\nu}$ the function $x_\nu \mapsto f^\infty(x_\nu^\perp, x_\nu, s, \nu)$ is continuous in $\Omega_{x_\nu^\perp}$. Now, let $\Sigma_0 = \{s_k\}$ a countable dense set in $\mathbb{R}$ and set $N_\nu = \bigcup_k N_{s_k,\nu}$. Clearly, $\mathcal{H}^{N-1}(N_\nu) = 0$ and, for every $x_\nu^\perp \in \pi_{\nu^\perp}(\Omega) \setminus N_\nu$ and every $s_k \in \Sigma_0$, the function $x_\nu \mapsto f^\infty(x_\nu^\perp, x_\nu, s_k, \nu)$ is continuous in $\Omega_{x_\nu^\perp}$. By using (5.10) as in the previous proof, it follows that the function $x_\nu \mapsto f^\infty(x_\nu^\perp, x_\nu, s, \nu)$ actually is continuous in $\Omega_{x_\nu^\perp}$, for every $s \in \mathbb{R}$ and every $x_\nu^\perp \in \pi_{\nu^\perp}(\Omega) \setminus N_\nu$. □

Lemma 5.7 *Let f, f^∞ be as in Lemma 5.6. Let $D_0 = \{\nu_j\}$ be a countable sequence of directions in $\mathbb{S}^{N-1}$. There exists a set $\mathcal{G} \subset \Omega$ such that, for every $s \in \mathbb{R}$ and every $\nu_j \in D_0$, each point $x = (x_{\nu_j}^\perp, x_{\nu_j}) \in \mathcal{G}$ is a point of approximate continuity for $f^\infty(\cdot, s, \nu_j)$ and the function $f^\infty(\cdot, x_{\nu_j}, s, \nu_j)$ is approximately continuous at $x_{\nu_j}^\perp \in \Omega_{x_{\nu_j}}$. Moreover, $\mathcal{H}^{N-1}(S \setminus \mathcal{G}) = 0$ for any countably $\mathcal{H}^{N-1}$-rectifiable set $S \subset \Omega$ such that*

$$\mathcal{H}^{N-1}(\{x \in S \ : \ \nu^S(x) = \pm\nu_j\}) = 0 \qquad \text{for all } \nu_j \in D_0 .$$

Proof. For every $\nu_j \in D_0$ and every $s \in \mathbb{R}$, set

$$G_{\nu_j}^s = \{x = (x_{\nu_j}^\perp, x_{\nu_j}) \in \Omega \setminus N_0 : \lim_{\varepsilon \to 0^+} \fint_{Q_{\nu_j}^\perp(x,\varepsilon)} |f^\infty(z_{\nu_j}^\perp, x_{\nu_j}, s, \nu_j) - f^\infty(x_{\nu_j}^\perp, x_{\nu_j}, s, \nu_j)| \, dz_{\nu_j}^\perp = 0\} ,$$

where $Q_{\nu_j}^\perp(x,\varepsilon) = \pi_{\nu_j^\perp}(Q_{\nu_j}(x,\varepsilon))$ and N_0 is the set given by Lemma 5.5. By Remark 4.7, it follows that $\mathcal{H}^{N-1}(S \setminus G_{\nu_j}^s) = 0$. We consider the set $\mathcal{G}^s = \bigcap_j G_{\nu_j}^s$; then, for every $s \in \mathbb{R}$, $\mathcal{H}^{N-1}(S \setminus \mathcal{G}^s) = 0$. Now, let $\{s_k\}$ be a countable dense subset of $\mathbb{R}$ and, for every $k \in \mathbb{N}$, $\mathcal{G}^{s_k}$ be the corresponding set, constructed as above. Finally, set $\mathcal{G} = \bigcap_k \mathcal{G}^{s_k}$. Clearly, $\mathcal{H}^{N-1}(S \setminus \mathcal{G}) = 0$. Moreover, as a consequence (5.10) and the density of $\{s_k\}$, we have that, for every $\nu_j \in D_0$ and every $x \in \mathcal{G}$,

$$\lim_{\varepsilon \to 0^+} \fint_{Q_{\nu_j}^\perp(x,\varepsilon)} |f^\infty(z_{\nu_j}^\perp, x_{\nu_j}, s, \nu_j) - f^\infty(x_{\nu_j}^\perp, x_{\nu_j}, s, \nu_j)| \, dz_{\nu_j}^\perp = 0$$

for every $s \in \mathbb{R}$. □

6 Main result

As we pointed out in the introduction, it has been already proven, for istance in [6] and [14], under more regularity assumptions on the integrand function f, that the functional defined in (5.6) provides a "natural" extension of the functional (5.5) from $W^{1,1}(\Omega)$ to $BV(\Omega)$. In the next theorem we state that the same result still holds under the weaker assumptions on f considered here.

Theorem 6.1 *Assume that $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Borel function satisfying (5.1)–(5.4). Let $F : \mathrm{BV}(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty]$ be the functional defined in (5.5) and $\overline{F}$ be the relaxation of F. Assume also that (5.11) and (5.12) hold. Then, $\overline{F}(u, \cdot)$ is the trace of a finite Radon measure on $\mathcal{A}(\Omega)$, and*

$$\overline{F}(u, A) = \int_A f(x, u, \nabla u) \, dx + \int_A f^\infty \left(x, \widetilde{u}, \frac{D^c u}{|D^c u|} \right) d|D^c u| + \int_{J_u \cap A} \left(\int_{u^-(x)}^{u^+(x)} f^\infty(x, s, \nu_u) \, ds \right) d\mathcal{H}^{N-1}(x)$$

for every $A \in \mathcal{A}(\Omega)$ and every $u \in \mathrm{BV}(\Omega)$.

We start by observing that under the assumptions of Theorem 6.1 above it is well known that for any $u \in \mathrm{BV}(\Omega)$ the function $\overline{F}(u,\cdot)$ is the trace of a finite Radon measure on $\mathcal{A}(\Omega)$ and that for all $A \in \mathcal{A}(\Omega)$

$$0 \le \overline{F}(u,A) \le c(\mathcal{L}^N(A) + |Du|(A)) \,.$$

Hence, to prove Theorem 6.1 we have to establish the two inequalities

$$(i) \qquad \mathcal{F}(u,A) \le \overline{F}(u,A) \qquad \text{for all } A \in \mathcal{A}(\Omega) \text{ and } u \in \mathrm{BV}(\Omega)\,,$$

$$(ii) \qquad \mathcal{F}(u,A) \ge \overline{F}(u,A) \qquad \text{for all } A \in \mathcal{A}(\Omega) \text{ and } u \in \mathrm{BV}(\Omega)\,.$$

The first one is an immediate consequence of next theorem which, in turn, follows from a more general lower semicontinuity result [7, Theorem 1.1].

Theorem 6.2 *Assume that $f : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \to [0,+\infty)$ is a locally bounded Borel function, satisfying (5.1)–(5.3). Then, for every $A \in \mathcal{A}(\Omega)$ the functional $\mathcal{F}(\cdot, A) : \mathrm{BV}(\Omega) \to [0,+\infty)$ defined in (5.6) is lower semicontinuous with respect to the L^1-topology.*

Inequality (ii) is established in the next theorem.

Theorem 6.3 *Assume that $f : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \to \mathrm{I\!R}$ is a Borel function satisfying (5.2)–(5.4), (5.11) and (5.12). Then, for every $A \in \mathcal{A}(\Omega)$ and every $u \in \mathrm{BV}(\Omega)$, $\mathcal{F}(u,A) \ge \overline{F}(u,A)$.*

Following [14, Proof of Theorem 1.3], we fix $u \in \mathrm{BV}(\Omega)$ and consider the Radon-Nikodým derivatives of $\overline{F}(u,\cdot)$ with respect to the Lebesgue measure $\mathcal{L}^N$, to the total variation of the Cantor measure $|D^c u|$ and to the Hausdorff measure $\mathcal{H}^{N-1}\lfloor S_u$, respectively. In order to obtain Theorem 6.3, we will prove that

$$(L) \qquad \frac{d\overline{F}(u,\cdot)}{d\mathcal{L}^N}(x_0) \le f(x_0, u(x_0), \nabla u(x_0)) \qquad \text{for } \mathcal{L}^N\text{-almost every } x_0 \in \Omega\,,$$

$$(C) \qquad \frac{d\overline{F}(u,\cdot)}{d|D^c u|}(x_0) \le f^\infty\left(x_0, \widetilde{u}(x_0), \frac{D^c u}{|D^c u|}(x_0)\right) \qquad \text{for } |D^c u|\text{-almost every } x_0 \in \Omega\,,$$

$$(J) \qquad \frac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) \le \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu_u(x_0))\,ds \qquad \text{for } \mathcal{H}^{N-1}\text{-almost every } x_0 \in J_u\,.$$

Inequality (L) is proven in [14, Theorem 1.3, part (i)], under the only assumptions (5.2)–(5.4), hence, we have to prove (C) and (J). To this purpose let us define the following coercive functional associated to $\overline{F}$ by setting

$$\overline{F}_1(u,A) := \overline{F}(u,A) + |Du|(A)\,.$$

Proposition 6.4 *Assume that $f : \Omega \times \mathrm{I\!R} \times \mathrm{I\!R}^N \to \mathrm{I\!R}$ is a Borel function satisfying (5.2)–(5.4) and (5.12). Then, (C) holds; i.e., for every $u \in \mathrm{BV}(\Omega)$,*

$$(C) \qquad \frac{d\overline{F}(u,\cdot)}{d|D^c u|}(x_0) \le f^\infty\left(x_0, \widetilde{u}(x_0), \frac{D^c u}{|D^c u|}(x_0)\right) \qquad \textit{for } |D^c u|\textit{-almost every } x_0 \in \Omega\,.$$

Proof. By Lemma 3.9 of [3] for $|D^c u|$-almost every $x_0 \in \Omega$, there exists a double indexed sequence $\{t^k_\varepsilon, u^k_\varepsilon\}$ such that, for every $k \in \mathbb{N}$,

$$t^k_\varepsilon \to +\infty, \quad \varepsilon t^k_\varepsilon \to 0^+, \quad u^k_\varepsilon \to \widetilde{u}(x_0) \quad \text{as } \varepsilon \to 0^+ \,, \tag{6.1}$$

$$\begin{aligned}\frac{d\overline{F}_1(u,\cdot)}{d|D^c u|}(x_0) &= \frac{d\overline{F}(u,\cdot)}{d|D^c u|}(x_0) + 1 \\ &= \lim_{k\to+\infty} \limsup_{\varepsilon\to 0^+} \frac{\inf\{\overline{F}_1(v, Q^k_\nu(x_0,\varepsilon)) \colon v \in \mathrm{BV}(Q^k_\nu(x_0,\varepsilon)), v|_{\partial Q^k_\nu(x_0,\varepsilon)} = u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle\}}{k^{N-1}\varepsilon^N t^k_\varepsilon}\,,\end{aligned}$$

where $\nu = \dfrac{dD^c u}{d|D^c u|}(x_0)$, $|\nu| = 1$, and $Q^k_\nu(x_0,\varepsilon) := x_0 + \varepsilon Q^k_\nu$, with

$$Q^k_\nu := R_\nu\Big((-k/2, k/2)^{N-1} \times (-1/2, 1/2)\Big)\,, \tag{6.2}$$

and R_ν denotes a rotation such that $R_\nu e_N = \nu$. Fix $x_0 \in \Omega$ so that all the limits above exist and are finite. Moreover, since by Lemma 5.5 there exists $N_0 \subset \Omega$, with $\mathcal{H}^{N-1}(N_0) = 0$ (hence, $|D^c u|(N_0) = 0$), so that $f^\infty(\cdot, s, \xi)$ is approximately continuous at x_0 for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, we may assume with no loss of generality that $x_0 \in \Omega \setminus N_0$. Then, taking into account (5.8), (5.4) and (6.1), we have

$$\begin{aligned}\frac{d\overline{F}(u,\cdot)}{d|D^c u|}(x_0) + 1 &\le \liminf_{k\to+\infty} \limsup_{\varepsilon\to 0^+} \frac{1}{k^{N-1}\varepsilon^N t^k_\varepsilon} \overline{F}_1(u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, Q^k_\nu(x_0,\varepsilon)) \\ &\le \liminf_{k\to+\infty} \limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} \frac{f(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, t^k_\varepsilon \nu)}{t^k_\varepsilon}\, dx + 1 \\ &\le \liminf_{k\to+\infty} \limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} \left(f^\infty(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, \nu) + \frac{f(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, 0)}{t^k_\varepsilon} \right) dx + 1 \\ &\le \liminf_{k\to+\infty} \limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} f^\infty(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, \nu)\, dx + 1\,,\end{aligned}$$

which implies

$$\frac{d\overline{F}(u,\cdot)}{d|D^c u|}(x_0) \le \liminf_{k\to+\infty} \limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} f^\infty(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, \nu)\, dx\,.$$

Hence, in order to conclude, it is enough to prove that for all $k \in \mathbb{N}$

$$\limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} f^\infty(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, \nu)\, dx \le f^\infty(x_0, \widetilde{u}(x_0), \nu)\,. \tag{6.3}$$

By (5.10), the approximate continuity of the function $f^\infty(\cdot, \widetilde{u}(x_0), \nu)$ in $x_0 \in \Omega$ and (6.1), it follows that for every $k \in \mathbb{N}$

$$\begin{aligned}&\limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} f^\infty(x, u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle, \nu)\, dx \\ &\quad\le \limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} f^\infty(x, \widetilde{u}(x_0), \nu)\, dx + \limsup_{\varepsilon\to 0^+} \fint_{Q^k_\nu(x_0,\varepsilon)} \Lambda |u^k_\varepsilon + \langle t^k_\varepsilon \nu, x - x_0\rangle - \widetilde{u}(x_0)|\, dx \\ &\quad\le f^\infty(x_0, \widetilde{u}(x_0), \nu) + \Lambda \limsup_{\varepsilon\to 0^+} \left[|u^k_\varepsilon - \widetilde{u}(x_0)| + \fint_{Q^k_\nu(x_0,\varepsilon)} t^k_\varepsilon |x - x_0|\, dx \right] \\ &\quad\le f^\infty(x_0, \widetilde{u}(x_0), \nu) + \Lambda \limsup_{\varepsilon\to 0^+} \left[|u^k_\varepsilon - \widetilde{u}(x_0)| + \varepsilon t^k_\varepsilon \sqrt{(N-1)k^2 + 1} \right] = f^\infty(x_0, \widetilde{u}(x_0), \nu)\,.\end{aligned}$$

Thus (6.3) is proved. Hence, the assertion follows. □

Proposition 6.5 *Let f satisfy the assumptions of Theorem 6.3. Then,*

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) \leq \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu_u(x_0))\, ds \tag{6.4}$$

for every $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and for $\mathcal{H}^{N-1}$-almost every $x_0 \in J_u$.

Proof. Let $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$. We will prove that for $\mathcal{H}^{N-1}$-a.e. $x_0 \in J_u$

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) = \lim_{r\to 0^+} \frac{\overline{F}(u, B_r(x_0))}{\mathcal{H}^{N-1}\lfloor J_u(B_r(x_0))} \leq \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu_u(x_0))\, ds\,.$$

By Remark 3.1 the set Φ_u, defined by

$$\Phi_u = \{\nu \in \mathbb{S}^{N-1} : \mathcal{H}^{N-1}(\{x \in J_u : \nu_u(x) = \pm\nu\}) > 0\}\,,$$

is at most countable. Given a countable dense subset D_0 of directions in $\mathbb{S}^{N-1} \setminus \Phi_u$, we apply Lemma 5.7, with $S = J_u$. Thus, there exists a set $\mathcal{G} \subset \Omega$, such that, for every $s \in \mathbb{R}$ and every $\nu \in D_0$, if $x = (x_\nu^\perp, x_\nu) \in \mathcal{G}$, then $f^\infty(\cdot, s, \nu)$ is approximately continuous at x, $f^\infty(\cdot, x_\nu, s, \nu)$ is approximately continuous at $x_\nu^\perp$ and $\mathcal{H}^{N-1}(J_u \setminus \mathcal{G}) = 0$.
By Theorem 3.7 of [3], for $\mathcal{H}^{N-1}$-almost every $x_0 \in J_u \cap \mathcal{G}$ we have

$$\begin{aligned}\frac{d\overline{F}_1(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) &= \frac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) + |u^+(x_0) - u^-(x_0)| \\ &= \limsup_{\varepsilon\to 0^+} \frac{\inf\{\overline{F}_1(v, Q_\nu(x_0,\varepsilon)) \ :\ v \in \mathrm{BV}(Q_\nu(x_0,\varepsilon)),\ v|_{\partial Q_\nu(x_0,\varepsilon)} = w_\nu\}}{\varepsilon^{N-1}}\,,\end{aligned} \tag{6.5}$$

where $\nu = \nu_u(x_0)$, $Q_\nu(x_0,\varepsilon) = x_0 + \varepsilon Q_\nu$, Q_ν is defined as in (6.2) with $k = 1$, and w_ν is the jump function which takes the value $u^+(x_0)$ if $\langle x - x_0, \nu\rangle > 0$ and $u^-(x_0)$ if $\langle x - x_0, \nu\rangle \leq 0$. Let $\{\nu_j\}$ be a sequence of directions contained in D_0 converging to ν. Let us fix $\delta > 0$; then $Q_\nu(0, 1-\delta) \subset Q_{\nu_j} \subset Q_\nu(0, 1+\delta)$ for every j sufficiently large.
Let $\phi \in \mathcal{C}_0^\infty(Q_\nu(0; 1+\delta))$ be a cut-off function such that $\phi(x) = 1$ in $Q_\nu(0; 1-2\delta)$, $\phi(x) = 0$ on $Q_\nu(0; 1+\delta) \setminus Q_\nu(0; 1-\delta)$ and $|\nabla\phi| \leq c/\delta$. For every $\varepsilon > 0$, set $\phi_\varepsilon(x) = \phi(\frac{x - x_0}{\varepsilon})$, so that $|\nabla\phi_\varepsilon| \leq c/\varepsilon\delta$ and, for every $j \in \mathbb{N}$ sufficiently large, set $w_{\varepsilon,\nu,\nu_j}(x) = \phi_\varepsilon(x) w_{\nu_j}(x) + (1 - \phi_\varepsilon(x)) w_\nu(x)$, where w_{ν_j} is defined as w_ν, with ν replaced by ν_j. We note that $w_{\varepsilon,\nu,\nu_j}$ satisfies the boundary condition $w_{\varepsilon,\nu,\nu_j}|_{\partial Q_\nu(x_0,\varepsilon)} = w_\nu$, so that, by (6.5), we obtain

$$\frac{d\overline{F}_1(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) \leq \limsup_{\varepsilon\to 0^+} \left[\frac{\overline{F}(w_{\varepsilon,\nu,\nu_j}, Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} + \frac{|Dw_{\varepsilon,\nu,\nu_j}|(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}}\right]. \tag{6.6}$$

Clearly, for every $\varepsilon > 0$ and $j \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}&\frac{|Dw_{\varepsilon,\nu,\nu_j}|(Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} \\ &\leq \frac{1}{\varepsilon^{N-1}}\left[\int_{Q_\nu(x_0,\varepsilon)} |\nabla\phi_\varepsilon||w_{\nu_j} - w_\nu|\, dx + |Dw_{\nu_j}|(Q_\nu(x_0,\varepsilon)) + |Dw_\nu|(Q_\nu(x_0,\varepsilon) \setminus Q_\nu(x_0,(1-2\delta)\varepsilon))\right] \\ &\leq \frac{c}{\varepsilon^N\delta}\int_{Q_\nu(x_0,\varepsilon)} |w_{\nu_j} - w_\nu|\, dx + \frac{|u^+(x_0) - u^-(x_0)|}{|\langle\nu,\nu_j\rangle|} + c\delta \leq \frac{c}{\delta}\sin(\widehat{\nu\nu_j}) + \frac{|u^+(x_0) - u^-(x_0)|}{|\langle\nu,\nu_j\rangle|} + c\delta\,.\end{aligned} \tag{6.7}$$

Moreover, let us approximate the jump function $w_{\varepsilon,\nu,\nu_j}$ by means of a sequence of $W^{1,1}$-functions, given by $u^n_{\varepsilon,\nu,\nu_j}(x) = \phi_\varepsilon(x)u^n_{\varepsilon,\nu_j}(x) + (1-\phi_\varepsilon(x))u^n_{\varepsilon,\nu}(x)$, where

$$u^n_{\varepsilon,\nu}(x) := \begin{cases} u^+(x_0) & \text{if } \varepsilon/2n \le \langle x - x_0, \nu\rangle \ , \\ (u^+(x_0) - u^-(x_0))\dfrac{n}{\varepsilon}\langle x - x_0, \nu\rangle + \dfrac{u^+(x_0)+u^-(x_0)}{2} & \text{if } -\varepsilon/2n \le \langle x - x_0, \nu\rangle \le \varepsilon/2n \ , \\ u^-(x_0) & \text{if } \langle x - x_0, \nu\rangle \le -\varepsilon/2n \ , \end{cases}$$

and u^n_{ε,ν_j} is defined similarly by replacing ν with ν_j. Clearly, $\|u^n_{\varepsilon,\nu,\nu_j} - w_{\varepsilon,\nu,\nu_j}\|_{L^1(Q_\nu(x_0,\varepsilon))} \to 0$ as $n \to +\infty$, hence, using the lower semicontinuity of $\overline{F}$, (5.4), (5.8), it follows

$$\begin{aligned}
(6.8)\quad \frac{\overline{F}(w_{\varepsilon,\nu,\nu_j}, Q_\nu(x_0,\varepsilon))}{\varepsilon^{N-1}} &\le \liminf_{n\to+\infty} \frac{1}{\varepsilon^{N-1}} \int_{Q_\nu(x_0,\varepsilon)} f(x, u^n_{\varepsilon,\nu,\nu_j}, \nabla u^n_{\varepsilon,\nu,\nu_j})\, dx \\
&\le \liminf_{n\to+\infty} \frac{1}{\varepsilon^{N-1}} \Bigg[\int_{Q_\nu(x_0,\varepsilon(1-2\delta))} f(x, u^n_{\varepsilon,\nu_j}, \nabla u^n_{\varepsilon,\nu_j})\, dx + c\int_{Q_\nu(x_0,\varepsilon)} |\nabla\phi_\varepsilon||u^n_{\varepsilon,\nu} - u^n_{\varepsilon,\nu_j}|\, dx \\
&\qquad + c\int_{Q_\nu(x_0,\varepsilon)\setminus Q_\nu(x_0,(1-2\delta)\varepsilon)} \Big(\phi_\varepsilon|\nabla u^n_{\varepsilon,\nu_j}| + (1-\phi_\varepsilon)|\nabla u^n_{\varepsilon,\nu}| + 1\Big) dx \Bigg] \\
&\le \liminf_{n\to+\infty} \fint_{Q_{\nu_j}(x_0,\varepsilon)} \varepsilon f(x, u^n_{\varepsilon,\nu_j}, \nabla u^n_{\varepsilon,\nu_j}) dx + \frac{c}{\delta}\sin(\widehat{\nu\nu_j}) + c\delta \\
&\le \liminf_{n\to+\infty} \fint_{Q_{\nu_j}(x_0,\varepsilon)} \Big[f^\infty(x, u^n_{\varepsilon,\nu_j}, \varepsilon\nabla u^n_{\varepsilon,\nu_j}) + \varepsilon f(x, u^n_{\varepsilon,\nu_j}, 0)\Big]\, dx + \frac{c}{\delta}\sin(\widehat{\nu\nu_j}) + c\delta \\
&\le \liminf_{n\to+\infty} \fint_{Q_{\nu_j}(x_0,\varepsilon)} f^\infty(x, u^n_{\varepsilon,\nu_j}, \varepsilon\nabla u^n_{\varepsilon,\nu_j})\, dx + \Lambda\varepsilon + \frac{c}{\delta}\sin(\widehat{\nu\nu_j}) + c\delta \\
&\le (u^+(x_0) - u^-(x_0)) \liminf_{n\to+\infty} \fint_{Q^\perp_{\nu_j}(x_0,\varepsilon)} \left[\frac{n}{\varepsilon} \int_{(x_0)_{\nu_j}-\varepsilon/2n}^{(x_0)_{\nu_j}+\varepsilon/2n} f^\infty(x^\perp_{\nu_j}, x_{\nu_j}, u^n_{\varepsilon,\nu_j}(x), \nu_j)\, dx_{\nu_j} \right] dx^\perp_{\nu_j} \\
&\qquad + \Lambda\varepsilon + \frac{c}{\delta}\sin(\widehat{\nu\nu_j}) + c\delta \,,
\end{aligned}$$

where $Q^\perp_{\nu_j}(x_0,\varepsilon) = \pi_{\nu_j^\perp}(Q_{\nu_j}(x_0,\varepsilon))$. We note that the function u^n_{ε,ν_j} actually depends only on x_{ν_j}, so that, by the change of variable $s = u^n_{\varepsilon,\nu_j}(x_{\nu_j})$, we obtain

$$x_{\nu_j} = \rho_{n,\varepsilon}(s) = \frac{\varepsilon}{n(u^+(x_0) - u^-(x_0))}\left(s - \frac{u^+(x_0)+u^-(x_0)}{2}\right) + (x_0)_{\nu_j}$$

and $\rho_{n,\varepsilon}(s) \to (x_0)_{\nu_j}$, when $n \to +\infty$; so that, by dominated convergence theorem and Lemma 5.6, we have

$$\begin{aligned}
(6.9)\quad (u^+(x_0) - u^-(x_0)) \limsup_{\varepsilon\to0^+} \liminf_{n\to+\infty} \fint_{Q^\perp_{\nu_j}(x_0,\varepsilon)} &\left[\frac{n}{\varepsilon} \int_{(x_0)_{\nu_j}-\varepsilon/2n}^{(x_0)_{\nu_j}+\varepsilon/2n} f^\infty(x^\perp_{\nu_j}, x_{\nu_j}, u^n_{\varepsilon,\nu_j}(x_{\nu_j}), \nu_j)\, dx_{\nu_j} \right] dx^\perp_{\nu_j} \\
&= \limsup_{\varepsilon\to0^+} \liminf_{n\to+\infty} \fint_{Q^\perp_{\nu_j}(x_0,\varepsilon)} \left[\int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x^\perp_{\nu_j}, \rho_{n,\varepsilon}(s), s, \nu_j)\, ds \right] dx^\perp_{\nu_j}
\end{aligned}$$

$$\leq \limsup_{\varepsilon\to 0^+} \fint_{Q^{\perp}_{\nu_j}(x_0,\varepsilon)} \left[\int_{u^-(x_0)}^{u^+(x_0)} \lim_{n\to+\infty} f^\infty(x^{\perp}_{\nu_j}, \rho_{n,\varepsilon}(s), s, \nu_j)\, ds\right] dx^{\perp}_{\nu_j}$$

$$\leq \limsup_{\varepsilon\to 0^+} \fint_{Q^{\perp}_{\nu_j}u(x_0,\varepsilon)} \left[\int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x^{\perp}_{\nu_j}, (x_0)_{\nu_j}, s, \nu_j)\, ds\right] dx^{\perp}_{\nu_j}$$

$$= \int_{u^-(x_0)}^{u^+(x_0)} \left[\lim_{\varepsilon\to 0^+} \fint_{Q^{\perp}_{\nu_j}(x_0,\varepsilon)} f^\infty(x^{\perp}_{\nu_j}, (x_0)_{\nu_j}, s, \nu_j)\, dx^{\perp}_{\nu_j}\right] ds = \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu_j)\, ds\,,$$

where the last equality is due to the approximate continuity at $(x_0)^{\perp}_{\nu_j}$ of $f^\infty(\cdot, (x_0)_{\nu_j}, s, \nu_j)$, for every $s \in \mathbb{R}$. By (6.5), (6.6), (6.7), (6.8) and (6.9), we obtain, letting $\varepsilon \to 0^+$,

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) + |u^+(x_0) - u^-(x_0)| \leq \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu_j)\ ds + \frac{c}{\delta}\sin(\widehat{\nu\nu_j}) + c\delta + \frac{|u^+(x_0) - u^-(x_0)|}{|\langle \nu, \nu_j\rangle|}\,.$$

Now, taking into account the Lipschitz continuity of f^∞ with respect to the last variable and letting first $j \to +\infty$ and then $\delta \to 0^+$ we get

$$\frac{d\overline{F}(u,\cdot)}{d\mathcal{H}^{N-1}\lfloor J_u}(x_0) + |u^+(x_0) - u^-(x_0)| \leq \int_{u^-(x_0)}^{u^+(x_0)} f^\infty(x_0, s, \nu_u(x_0))\, ds + |u^+(x_0) - u^-(x_0)|\,.$$

Hence, the assertion follows. □

We are now in position to give the proof of Theorem 6.1.

Proof of Theorem 6.1. Taking into account Theorem 6.2, [14, Theorem 1.3, part (i)] and Propositions 6.4, 6.5, we obtain the assertion for any function $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and any $A \in \mathcal{A}(\Omega)$.
The general case $u \in \mathrm{BV}(\Omega)$, follows by a standard truncation argument. Indeed, let $u \in \mathrm{BV}(\Omega)$ and, for every $k > 0$, set $u_k = T_k(u) := (u\wedge k)\vee(-k)$. Then, $u_k \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, $\|u_k\|_\infty \leq k$ and (see, for instance, [2, Theorem 3.99])

$$\nabla u_k \mathcal{L}^N + D^c u_k = \begin{cases} \nabla u \mathcal{L}^N + D^c u & \text{if } |\widetilde{u}| < k\,, \\ 0 & \text{if } |\widetilde{u}| \geq k\,, \end{cases} \qquad \widetilde{u}_k(x) = \widetilde{u}(x) \text{ for } |Du_k|\text{-a.e. } x \in \Omega \setminus S_{u_k}\,,$$

while

$$(u_k^+ - u_k^-)\nu_{u_k} d\mathcal{H}^{N-1}\lfloor J_{u_k} = \big(T_k(u^+) - T_k(u^-)\big)\,\nu_u d\mathcal{H}^{N-1}\lfloor J_u\,.$$

Recalling that the representation formula for $\overline{F}$ holds true for all the functions u_k and that $u_k \to u$ in $L^1(\Omega)$ we have that for any $A \in \mathcal{A}(\Omega)$

$$\begin{aligned}(6.10)\ \overline{F}(u,A)\ &\leq\ \liminf_{k\to+\infty} \overline{F}(u_k, A) \\ &= \liminf_{k\to+\infty}\left[\int_A f(x,u_k,\nabla u_k)\,dx + \int_{A\cap\{|\widetilde{u}|<k\}} f^\infty\left(x, \widetilde{u}(x), \frac{D^c u}{|D^c u|}\right)|D^c u| \right. \\ &\qquad \left. + \int_{J_u\cap A}\left(\int_{T_k(u^-(x))}^{T_k(u^+(x))} f^\infty(x,s,\nu_u)\,ds\right) d\mathcal{H}^{N-1}(x)\right] \\ &\leq \int_A f(x,u,\nabla u)dx + \int_A f^\infty\left(x,\widetilde{u},\frac{D^c u}{|D^c u|}\right) d|D^c u| + \int_{J_u\cap A}\left(\int_{u^-(x)}^{u^+(x)} f^\infty(x,s,\nu_u)ds\right) d\mathcal{H}^{N-1}.\end{aligned}$$

The result then follows from this inequality and from Theorem 6.2. □

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