ADAPTIVE FRAME METHODS FOR MAGNETOHYDRODYNAMIC FLOWS

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ABSTRACT. In this paper we develop adaptive numerical schemes for certain nonlinear variational problems. The discretization of the variational problems is done by a suitable frame decomposition of the solution, i.e., a complete, stable, and redundant expansion. The discretization yields an equivalent nonlinear problem on $\ell_2(\mathcal{N})$, the space of frame coefficients. The discrete problem is then adaptively solved using approximated nested fixed point and Richardson type iterations. We investigate the convergence, stability, and optimal complexity of the scheme. This constitutes a theoretical advantage, for example, with respect to adaptive finite element schemes for which convergence and complexity results are usually hard to prove. The use of frames is further motivated by their redundancy, which, at least numerically, has been shown to improve the conditioning of the corresponding discretization matrices. Also frames are usually easier to construct than Riesz bases. Finally, we show how to apply the adaptive scheme we propose for finding an approximation to the solution of the PDE governing magnetohydrodynamic (MHD) flows, once suitable frames are constructed.

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1. INTRODUCTION

Adaptive numerical methods have yielded very promising results [2, 3, 4, 6, 8, 9, 12, 20, 45] when applied to a large class of operator equations, in particular, PDE and integral equations. In classical schemes the adaptivity is realized at the level of the discretization and in the finite element space. The finite element space is refined and enriched locally at each iteration step depending on some (a posteriori) error estimators [21, 35]. A novel paradigm for adaptive schemes has been recently proposed by Cohen, Dahmen, and DeVore in [8, 9], where the discretization via wavelet decompositions is fixed at the beginning. The adaptivity is indeed realized at the level of the solver of the equivalent bi-infinite system of linear equations. The basic idea is to transform the original problem of PDE into a discrete (bi-infinite) linear problem on $\ell_2(\mathcal{N})$, the space of wavelet coefficients. The discrete problem is then solved with the help of approximate iterative schemes. The advantage of the latter approach is the fact that its convergence and stability can be proved and its complexity can be estimated asymptotically in terms of the number of algebraic operations needed. On the contrary, it has been a very hard technical problem to obtain such nice theoretical estimates for classical finite element methods, although some important theoretical results has recently appeared in [5, 39] for linear elliptic equations. A version of the paradigm in [8, 9] has been recently proposed also for nonlinear problems in [10]. It is again based on (wavelet) bases discretizations.

One drawback of the wavelet approach is the construction of the wavelet system itself especially on domains with complicated geometry or manifolds [14, 15, 16]. The wavelet bases constructed so far exhibit relatively high condition numbers and limited smoothness. In particular, the patching used to construct global smooth wavelets by domain decomposition techniques appears complicated and, in most cases, makes the conditioning even worse. In fact the global smoothness of the basis, when implementing adaptive schemes in [8, 9], is a necessary condition for getting compressibility (i.e., finitely banded approximations) of (bi-infinite) stiffness matrices, especially for high order operators. This bottleneck has led to generalizations of Cohen, Dahmen and DeVore approach.

These generalizations are based on *frame discretizations* , i.e., stable, redundant, non-orthogonal expansions [13, 38], which are much more flexible and simpler to construct even on domains of complicated geometry. Frame construction is usually implemented by Overlapping Domain Decompositions (ODD) so that patching at the interfaces is no more needed to obtain global smoothness. Moreover, the use of frames, due to their intrinsic redundancy, improves the conditioning of the corresponding discretization matrices. Certainly, ODD generates regions of the domain where the side effect of the redundancy is that functions are no longer uniquely representable by the global frame system. At first sight, it may seem that redundancy contradicts the minimality requirement on the amount of information being used to approximate the solution. Especially in fluid mechanics, accurate simulations already require processing of huge amounts of data. How can one attempt such computations if the information is also made redundant? A figurative answer to this question is the so-called "dictionary example": The larger and richer is my dictionary the *shorter* are the phrases I compose. The use of proper terminology avoids long circumlocutions for describing an object. Of course, the key point is the capability to choose the right terminology. Back to mathematical terms, the combination of adaptivity (i.e., the capability to choose the right terminology) and redundancy (i.e., the richness or non-uniqueness of representations) indeed gives rise to fast and accurate approximations [18, 25, 31, 40, 41]. Numerical experiments in [46] show that frames improve conditioning without increasing the effective dimension of the problem, i.e. without increasing the number of the relevant quantities needed for computations.

This encourages us to present a generalization of the approach proposed in [10] to (wavelet) frame discretizations for some specific nonlinear PDE, in particular those describing certain magnetohydrodynamics problems. Magnetohydrodynamics [7, 30, 32, 33, 34, 36, 37, 47] studies macroscopic interactions between magnetic field and fluid conductors of electricity. In particular, the following physical phenomena are studied: A flow of an electrically conducting fluid across magnetic lines causes an electric current in the fluid. The electric current alters the electromagnetic state of the system modifying the total magnetic field, which creates the current. The flow of electric current across magnetic lines is associated with a body force - Lorentz force - which influences the fluid flow. To model the behavior of electrically conducting fluid, the stationary, incompressible Navier-Stokes and Maxwell equations, coupled via Ohm's law and Lorentz force are being used. We refer to [33, 34] for a rigorous analysis of a non-adaptive finite-element scheme for the simulation of MHD flows confined to a cubic domain only and arising during electromagnetic purification of molten metals before the casting stage.

We present a way for transforming the nonlinear magnetohydrodynamics problem, possibly defined even on more general domains, into an equivalent nonlinear discrete problem on $\ell_2(\mathcal{N})$ by using suitable frame expansions. We show how the discrete problem can be solved *adaptively* by means of nested fixed point and approximated Richardson iterations. We also discuss convergence, stability, and, under certain additional assumptions, computational cost (quasi-optimal complexity) of the proposed adaptive procedure.

The paper is organized as follows: in Section 2 we recall the mathematical model and the equations governing MHD flows, specify the boundary conditions, the solution and test function spaces. In Section 3, the corresponding weak formulation of the physical problem is derived. Its equivalence to a variational nonlinear problem on a suitable subspace of the solution function space follows from the standard LBB (Ladyzhenskaya-Babuška-Brezzi) theory. The proofs of the results listed in this section can be found in [7, 32, 34]. In Section 4, the nonlinear variational problem is re-formulated as an equivalent fixed point iteration scheme, where, at each iteration step, a linear (non-symmetric) elliptic operator equation is to be solved. Next, we present the way for discretizing the fixed point iteration associated to an abstract nonlinear problem arising in MHD. We translate the original nonlinear variational problem into a problem on $\ell_2(\mathcal{N})$, the space of suitable *frame* coefficients. The concept of frames, i.e., stable, redundant, and complete expansions, is recalled in Subsection 4.1. In Subsection 4.2 we show how a linear (non-symmetric) elliptic operator equation is discretized by means of frame expansions and how Algorithm 1 is used to approximate its solution adaptively, up to any prescribed accuracy. Using Algorithm 1 as the main building block, we formulate in Section 5 a fully discrete and finite version of the fixed point iteration. We show that this discrete version of the fixed point iteration converges to some frame coefficients of the true solution of the original problem. In Section 6 we discuss under which conditions on the building block procedures in Algorithm 1 the suggested scheme performs quasi-optimally with respect to suitable *sparseness classes* of frame coefficients. In particular, we show how the flexibility and redundancy of frames lead to technical difficulties, which do not arise in case of Riesz bases, when showing complexity estimates. In Section 7, we show that the relevant solution space for the MHD problem is a product of suitable divergence-free vector spaces. We shortly discuss the existence and the construction of suitable divergence-free frames for such spaces. This allows for applications of the general adaptive scheme we propose.

Throughout this paper '$a \sim b$' means that both quantities are uniformly bounded by some constant multiple of each other. Likewise, '$a \lesssim b$' means that there exists a positive constant C such that $a \leq Cb$. We determine the constants explicitly only if their value is crucial for further analysis. The symbol $\|\cdot\|$, when applied to bounded operators, denotes the operator norm from its domain space to its image space, these are not always explicitly specified for notational simplicity.

2. Equations and Boundary Conditions

In this section, we model the stationary flow of a viscous, incompressible, electrically conducting fluid occupying a 3-dimensional bounded region $\Omega \subset \mathbb{R}^3$. For later analysis, we assume that Ω is a Lipschitz domain. Assume also that $\mathbf{F}$, a body force, and $\boldsymbol{E}$, an externally generated electric field are given. Denote by η the viscosity, ρ the density, σ^{-1} the electrical resistivity and μ the magnetic permeability of the fluid (some positive constants).

To describe the interaction of the magnetic field and the electrically conducting fluid in Ω mathematically we combine the equations of the fluid dynamics and electromagnetic field equations. We use Navier-Stokes equations to model the fluid flow and include Lorenz force $\boldsymbol{J} \times \boldsymbol{B}$ to express the influence of the flow of electric current across magnetic lines on the fluid motion

$$-\eta\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \boldsymbol{J} \times \boldsymbol{B} = \mathbf{F}. \tag{1}$$

The flow of the electrically conducting fluid across magnetic lines causes an electric current in the fluid. The electric current alters the electromagnetic state of the system modifying the total magnetic field, which creates the current in the fluid. This phenomenon is expressed by means of Ohm's law

$$\sigma^{-1}\boldsymbol{J} + \nabla\phi - \mathbf{u} \times \boldsymbol{B} = \boldsymbol{E}. \tag{2}$$

The total magnetic field $\boldsymbol{B} = \boldsymbol{B}_0 + \mathcal{B}(\boldsymbol{J})$ is decomposed into a sum of a given externally generated magnetic field $\boldsymbol{B}_0$ and the induced magnetic field $\mathcal{B}(\boldsymbol{J})$, which is induced in the fluid by the electric current $\boldsymbol{J}$ caused by $\boldsymbol{B}$. $\mathcal{B}(\boldsymbol{J})$ is a unique solution (see Lemma 2.2. in [32]) of the quasi-stationary form of Maxwell's equations

$$\nabla \times \mu^{-1}\mathcal{B}(\boldsymbol{J}) = \boldsymbol{J} \quad \text{and} \quad \nabla \cdot \mathcal{B}(\boldsymbol{J}) = 0.$$

It is also given by the Biot-Savart law

$$\mathcal{B}(\boldsymbol{J})(\mathbf{x}) = -\nabla \times \left(\frac{\mu}{4\pi}\int_\Omega \frac{\boldsymbol{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\right) = -\frac{\mu}{4\pi}\int_\Omega \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \times \boldsymbol{J}(\mathbf{y})d\mathbf{y}, \tag{3}$$

for $\mathbf{x} \in \mathbb{R}^3$. Note that using (3), we are able to eliminate $\boldsymbol{B}$ from (1)-(2) and solve for $\boldsymbol{J}$ instead. Solving for $\boldsymbol{B}$ we would be facing the problem that the magnetic field is defined on all space and satisfies different equations inside and outside the region Ω: The Navier-Stoke's equations are posed inside the region occupied by the fluid, whereas the Maxwell's equations have to be solved in all of space. Thus, the boundary conditions for $\boldsymbol{B}$ on the surface of the region must be specified, which is possible only for perfect conductors or perfect insulators. This makes the prescribed boundary conditions somewhat artificial.

We also consider the continuity equations

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \boldsymbol{J} = 0 \tag{4}$$

to describe the incompressibility of the fluid and preservation of charge.

Denote by $\mathbf{n}$ the outward unit normal vector field on the boundary of Ω and the stress tensor by

$$T(\mathbf{u}, p) := -p\mathcal{I} + \eta \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) =: -p\mathcal{I} + \eta D(\mathbf{u}).$$

Let the boundary, denoted by Γ, of the domain Ω consist of several relatively open and pairwise disjoint components Γ_i's, $i = 1, \ldots, 4$, i.e. $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3 \cup \bar{\Gamma}_4$. Different boundary conditions are prescribed on each component. The boundary conditions we consider are
Dirichlet type (prescribed velocity)

$$\mathbf{u} = \mathbf{g}_1 \ \text{ on } \ \Gamma_1, \tag{5}$$

Neumann type (prescribed stress)

$$T\mathbf{n} = \mathbf{h}_2 \ \text{ on } \ \Gamma_2 \tag{6}$$

and mixed type for velocity and stress

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} = g_3 \quad &\text{and} \quad (T\mathbf{n} - ((\mathbf{n}T) \cdot \mathbf{n})\mathbf{n}) = \mathbf{h}_3 \ \text{ on } \ \Gamma_3, \\ \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n} = \mathbf{g}_4 \quad &\text{and} \quad (T\mathbf{n}) \cdot \mathbf{n} = h_4 \ \text{ on } \ \Gamma_4. \end{aligned} \tag{7}$$

These boundary conditions are helpful in modeling the free boundary value problems, when dealing with the artificially truncated computational domains and the boundary conditions on the artificial boundaries.

Let also the boundary Γ consist of two relatively open and pairwise disjoint components Σ_1 and Σ_2, and consider
Neumann type (prescribed electric current flux through the walls) boundary condition

$$\boldsymbol{J} \cdot \mathbf{n} = j \ \text{ on } \ \Sigma_1 \tag{8}$$

and Dirichlet type (prescribed electric potential) boundary condition

$$\phi = k \ \text{ on } \ \Sigma_2. \tag{9}$$

This helps us to model two different cases: the external magnetic field is given and Σ_1 is not electrically insulating then $j \neq 0$; the magnetic field is generated by the external conductors embedded into Σ_2. Note that incorporating the electric potential is useful for various control problems.

2.1. **Function spaces.** Denote by $H^s(\Omega)$ the Sobolev space of square integrable functions v on Ω with square integrable distributional derivatives $D^\alpha v$ up to order s with the norm

$$||v||_{H^s(\Omega)} = \Big(\sum_{|\alpha| \le s} ||D^\alpha v||^2_{L_2(\Omega)} \Big)^{\frac{1}{2}}.$$

For vector-valued functions $\mathbf{v} = (v_1, v_2, v_3)$ define

$$\begin{aligned} \mathbf{H}^s(\Omega) &:= \{\mathbf{v} : v_i \in H^s(\Omega), i = 1 \ldots 3\}, \\ \mathbf{L}_2(\Omega) &:= \{\mathbf{v} : v_i \in L_2(\Omega), i = 1 \ldots 3 \ \}. \end{aligned}$$

Let $H^{1/2}(\Gamma)$ denote the fractional order Sobolev space and its dual $H^{1/2}(\Gamma)'$.

The minimal regularity assumptions on the given data are

$$\mathbf{F} \in \mathbf{H}^1(\Omega)', \quad \boldsymbol{E} \in \boldsymbol{L}_2(\Omega), \quad \boldsymbol{B}_0 \in \mathbf{H}^1(\Omega),$$

$$\mathbf{g}_1 \in \mathbf{H}^{1/2}(\Gamma_1), \quad g_3 \in H^{1/2}(\Gamma_3), \quad \mathbf{g}_4 \in \mathbf{H}^{1/2}(\Gamma_4) \ \text{ with } \ \mathbf{g}_4 \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_4,$$

$$\mathbf{h}_2 \in H^{1/2}(\Gamma_2)', \quad \mathbf{h}_3 \in \mathbf{H}^{1/2}(\Gamma_3)' \quad h_4 \in H^{1/2}(\Gamma_4)' \ \text{ with } \ \mathbf{h}_3 \cdot \mathbf{n} = 0 \ \text{ on } \ \Gamma_3.$$

Note that (4) imposes the following compatibility conditions on the boundary data

$$\int_{\Gamma_1} \mathbf{g}_1 \cdot \mathbf{n} + \int_{\Gamma_3} g_3 = 0, \quad \text{if } \ \Gamma_2 \cup \Gamma_4 = \emptyset, \quad \text{and} \quad \int_{\Sigma_1} j = 0, \quad \text{if } \quad \Sigma_2 = \emptyset. \tag{10}$$

The solution of (1)-(9) is not unique if $\Gamma_2 \cup \Gamma_4 = \emptyset$ and/or $\Sigma_2 = \emptyset$. This is due to the fact that (1)-(9) then only involve the derivatives of p and/or ϕ. To avoid the non-uniqueness of the solution we solve for $p \in \dot{L}_2(\Omega)$ (subspace of $L_2(\Omega)$ consisting of functions with mean zero) and $\phi \in \dot{H}^1(\Omega)$ (subspace of $H^1(\Omega)$ consisting of functions with mean zero). Therefore, we seek the solution

$$\mathbf{u} \in \mathbf{H}^1(\Omega) \quad \text{satisfying (1)-(7)},$$

$$\boldsymbol{J} \in \boldsymbol{L}_2 \quad \text{satisfying} \quad \boldsymbol{J} \cdot \mathbf{n} = j \ \text{ on } \ \Sigma_1,$$

$$p \in M_p := \begin{cases} \dot{L}_2(\Omega) & , \ \text{if } \ \Gamma_2 \cup \Gamma_4 = \emptyset, \\ L_2(\Omega) & , \quad \text{otherwise} \end{cases}$$

$$\phi \in M_\phi := \begin{cases} \dot{H}^1(\Omega) & , \ \text{if } \ \Sigma_2 = \emptyset, \\ \{\phi \in H^1(\Omega) : \phi \text{ satisfying } \ (9)\} & , \ \text{otherwise.} \end{cases}$$

To derive an equivalent to (1)-(9) variational formulation, we choose the test functions $\mathbf{v}$ in the space

$$\mathbf{H}^1_\Gamma(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\Gamma_1} = 0, \ \mathbf{v} \cdot \mathbf{n}|_{\Gamma_3} = 0, \ \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}|_{\Gamma_4} = 0\},$$

$\boldsymbol{K} \in \boldsymbol{L}_2(\Omega)$, $q \in M_q := M_p$ and

$$\psi \in M_\psi := \begin{cases} \dot{H}^1(\Omega) & , \ \text{if } \ \Sigma_2 = \emptyset, \\ \{q \in H^1(\Omega) : q|_{\Sigma_2} = 0\} & , \ \text{otherwise.} \end{cases}$$

To simplify the notation we introduce the product spaces

$$\mathbf{X}_{(\mathbf{u},\boldsymbol{J})} := \mathbf{H}^1(\Omega) \times \boldsymbol{L}_2(\Omega), \quad \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} := \mathbf{H}^1_\Gamma(\Omega) \times \boldsymbol{L}_2(\Omega),$$
$$M_{(p,\phi)} := M_p \times M_\phi \ \text{ and } \ M_{(q,\psi)} := M_q \times M_\psi.$$

3. Weak Formulation

In this section we shortly recall how to derive a variational problem equivalent to (1)-(9), (see [7] for details).

Define a bilinear form $a_0 : \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \to \mathbb{R}$, a trilinear form $a_1 : \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \to \mathbb{R}$, a bilinear form $b : \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \big(L_2(\Omega) \times H^1(\Omega)\big) \to \mathbb{R}$ and a linear form $\chi : \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \to \mathbb{R}$ by

$$\begin{aligned}
a_0\big((\mathbf{v}_1, \boldsymbol{K}_1), (\mathbf{v}_2, \boldsymbol{K}_2)\big) &:= \frac{\eta}{2} \int_\Omega D(\mathbf{v}_1) : D(\mathbf{v}_2) + \sigma^{-1} \int_\Omega \boldsymbol{K}_1 \cdot \boldsymbol{K}_2 \\
&\quad + \int_\Omega \big((\boldsymbol{K}_2 \times \boldsymbol{B}_0) \cdot \mathbf{v}_1 - (\boldsymbol{K}_1 \times \boldsymbol{B}_0) \cdot \mathbf{v}_2\big), \\
a_1\big((\mathbf{v}_1, \boldsymbol{K}_1), (\mathbf{v}_2, \boldsymbol{K}_2), (\mathbf{v}_3, \boldsymbol{K}_3)\big) &:= \rho \int_\Omega \big((\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\big) \cdot \mathbf{v}_3 \\
&\quad + \int_\Omega \Big(\big(\boldsymbol{K}_3 \times \mathcal{B}(\boldsymbol{K}_1)\big) \cdot \mathbf{v}_2 - \big(\boldsymbol{K}_2 \times \mathcal{B}(\boldsymbol{K}_1)\big) \cdot \mathbf{v}_3\Big), \\
b\big((\mathbf{v}, \boldsymbol{K})(q, \psi)\big) &:= -\int_\Omega (\nabla \cdot \mathbf{v}) q + \int_\Omega \boldsymbol{K} \cdot (\nabla \psi)
\end{aligned}$$

and

$$\chi(\mathbf{v}, \boldsymbol{K}) = \int_\Omega \mathbf{F} \cdot \mathbf{v} + \int_\Omega \boldsymbol{E} \cdot \boldsymbol{K} + \int_{\Gamma_2} \mathbf{h}_2 \cdot \mathbf{v} + \int_{\Gamma_3} \mathbf{h}_3 \cdot \mathbf{v} + \int_{\Gamma_4} h_4 \mathbf{n} \cdot \mathbf{v}.$$

Let $\mathbf{v} \in \mathbf{H}^1_\Gamma(\Omega)$ and $\boldsymbol{K} \in \boldsymbol{L}_2(\Omega)$ be arbitrary test functions. Multiplying (1), (2) and (4) by corresponding test functions, integrating by parts and using some of the boundary conditions on $\mathbf{u}$ and $\boldsymbol{J}$ we get the following equivalent variational problem.

Problem 1: *Find* $(\mathbf{u}, \boldsymbol{J}) \in \mathbf{X}_{(\mathbf{u},\boldsymbol{J})}$ *s.th.* $\mathbf{u}|_{\Gamma_1} = \mathbf{g}_1$, $(\mathbf{u}\cdot\mathbf{n})|_{\Gamma_3} = g_3$, $\big(\mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\big)|_{\Gamma_4} = \mathbf{g}_4$ *and* $(p,\phi) \in M_{(p,\phi)}$ *such that*

$$a_0\big((\mathbf{u},\boldsymbol{J}),(\mathbf{v},\boldsymbol{K})\big) + a_1\big((\mathbf{u},\boldsymbol{J}),(\mathbf{u},\boldsymbol{J}),(\mathbf{v},\boldsymbol{K})\big) + b\big((\mathbf{v},\boldsymbol{K}),(p,\phi)\big) = \chi(\mathbf{v},\boldsymbol{K}),$$

and

$$b\big((\mathbf{u},\boldsymbol{J}),(q,\psi)\big) = \int_{\Sigma_1} j\psi$$

for all $(\mathbf{v},\boldsymbol{K}) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$ *and* $(q,\psi) \in M_{(q,\psi)}$.

Next, we list some properties (proved in [32, Lemma 2.2] and [7, Corollary 1]) of a_0, a_1 and b .

Lemma 3.1.
a) The forms a_0, a_1, b *are bounded on* $\mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \mathbf{X}_{(\mathbf{u},\boldsymbol{J})}$, $\mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times \mathbf{X}_{(\mathbf{u},\boldsymbol{J})}$ *and* $\mathbf{X}_{(\mathbf{u},\boldsymbol{J})} \times M_{(p,\phi)}$, *respectively, with*

$$\|a_1\| \lesssim \max\{\rho,\mu\}.$$

b) If the intersection of $\mathbf{H}^1_\Gamma(\Omega)$ *and the null space* $N(D)$ *of the deformation tensor* D *is* $\{0\}$, *then the form* a_0 *is positive definite on* $\mathbf{X}_{(\mathbf{v},\boldsymbol{K})} \times \mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$, *i.e.*

$$a_0\big((\mathbf{v},\boldsymbol{K}),(\mathbf{v},\boldsymbol{K})\big) \geq \alpha_0 \|(\mathbf{v},\boldsymbol{K})\|_{\mathbf{X}_{(\mathbf{v},\boldsymbol{K})}}$$

with $\alpha_0 := c(\Omega)\min\{\eta,\sigma^{-1}\}$, $c(\Omega)$ *some positive constant depending only on the domain.*
c) The bilinear form b *satisfies the inf-sup condition*

$$\inf_{(q,\psi)\in M_{(q,\psi)}} \sup_{(\mathbf{v},\boldsymbol{K})\in\mathbf{X}_{(\mathbf{v},\boldsymbol{K})}} \frac{b\big((\mathbf{v},\boldsymbol{K}),(q,\psi)\big)}{\|(\mathbf{v},\boldsymbol{K})\|_{\mathbf{X}_{(\mathbf{v},\boldsymbol{K})}} \|(q,\psi)\|_{M_{(q,\psi)}}} > 0.$$

Note that $\mathbf{H}^1_\Gamma(\Omega) \cap N(D) = \{0\}$, for example, if $\Gamma_1 \neq \emptyset$. Otherwise, the subsequent analysis would require the introduction of a suitable quotient space of $\mathbf{H}^1_\Gamma(\Omega)$.

Due to [7, Corollary 1] there exist the liftings of the boundary data

$$\begin{aligned} &\mathbf{u}_0 \in \mathbf{H}^1(\Omega) \quad \text{with } \nabla\cdot\mathbf{u}_0 = 0 \text{ and } \mathbf{u}_0|_{\Gamma_1} = \mathbf{g}_1, \mathbf{u}_0\cdot\mathbf{n}|_{\Gamma_3} = g_3 \text{ and } \mathbf{u}_0 - (\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}|_{\Gamma_4} = \mathbf{g}_4, \\ &\boldsymbol{J}_0 \in \boldsymbol{L}_2(\Omega) \quad \text{with } \nabla\cdot\boldsymbol{J}_0 = 0 \text{ and } \quad \boldsymbol{J}_0\cdot\mathbf{n} = j \text{ on } \Sigma_1. \end{aligned}$$

Define $\hat{\mathbf{u}} := \mathbf{u} - \mathbf{u}_0$, $\hat{\boldsymbol{J}} := \boldsymbol{J} - \boldsymbol{J}_0$. Also set $\hat{\phi} := \phi - \phi_0$ and $\hat{p} := p - p_0$, where $p_0 = 0$, if $\Gamma_2 \cup \Gamma_4 \neq \emptyset$, and $p_0 = \frac{1}{|\Omega|}\int_\Omega p$, otherwise, and ϕ_0 is the H^1-lifting of the boundary data k on Σ_2, if $\Sigma_2 \neq \emptyset$, and $\phi_0 = \frac{1}{|\Omega|}\int_\Omega \phi$, otherwise. Define the bilinear form $a : \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} \times \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} \to \mathbb{R}$ and the linear form $\ell : \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} \to \mathbb{R}$ by

$$\begin{aligned} a\big((\mathbf{v}_1,\boldsymbol{K}_1),(\mathbf{v}_2,\boldsymbol{K}_2)\big) := {}& a_0\big((\mathbf{v}_1,\boldsymbol{K}_1),(\mathbf{v}_2,\boldsymbol{K}_2)\big) + a_1\big((\mathbf{v}_1,\boldsymbol{K}_1),(\mathbf{u}_0,\boldsymbol{J}_0),(\mathbf{v}_2,\boldsymbol{K}_2)\big) \\ &+ a_1\big((\mathbf{u}_0,\boldsymbol{J}_0),(\mathbf{v}_1,\boldsymbol{K}_1),(\mathbf{v}_2,\boldsymbol{K}_2)\big) \end{aligned}$$

and

$$\ell(\mathbf{v},\boldsymbol{K}) := \chi(\mathbf{v},\boldsymbol{K}) - a_0\big((\mathbf{u}_0,\boldsymbol{J}_0),(\mathbf{v},\boldsymbol{K})\big) - a_1\big((\mathbf{u}_0,\boldsymbol{J}_0),(\mathbf{u}_0,\boldsymbol{J}_0),(\mathbf{v},\boldsymbol{K})\big) - b((\mathbf{v},\boldsymbol{K}),(p_0,\phi_0)).$$

Note that if $(\mathbf{u}_0,\boldsymbol{J}_0)$ have small norm in $\mathbf{X}_{(\mathbf{u},\boldsymbol{J})}$, then the form a **is coercive on** $\mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$, i.e.

$$a((\mathbf{v},\boldsymbol{K}),(\mathbf{v},\boldsymbol{K})) \geq \Big(\alpha_0 - 2\|a_1\| \, \|(\mathbf{u}_0,\boldsymbol{J}_0)\|_{\mathbf{X}_{(\mathbf{u},\boldsymbol{J})}}\Big)\|(\mathbf{v},\boldsymbol{K})\|^2_{\mathbf{X}_{(\mathbf{v},\boldsymbol{K})}} \tag{11}$$

for all $(\mathbf{v},\boldsymbol{K}) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$. Substituting $\mathbf{u} = \hat{\mathbf{u}} + \mathbf{u}_0$, $\boldsymbol{J} = \hat{\boldsymbol{J}} + \boldsymbol{J}_0$, $\phi = \hat{\phi} + \phi_0$ and $p = \hat{p} + p_0$ into **Problem 1** we get its equivalent formulation.

Problem 2: *Find* $(\hat{\mathbf{u}}, \hat{\boldsymbol{J}}) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$ *and* $(\hat{p}, \hat{\phi}) \in M_{(q,\psi)}$ *satisfying*

$$a\big((\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\mathbf{v}, \boldsymbol{K})\big) + a_1\big((\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\mathbf{v}, \boldsymbol{K})\big) + b\big((\mathbf{v}, \boldsymbol{K}), (\hat{p}, \hat{\phi})\big) = \ell(\mathbf{v}, \boldsymbol{K})$$

for all $(\mathbf{v}, \boldsymbol{K}) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$ *and* $b\big((\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (q, \psi)\big) = 0$ *for all* $(q, \psi) \in M_{(q,\psi)}$.

Next, define

$$\mathbf{V} := \big\{(\mathbf{v}, \boldsymbol{K}) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} : b\big((\mathbf{v}, \boldsymbol{K}), (q, \psi)\big) = 0 \text{ for all } (q, \psi) \in M_{(q,\psi)}\big\}$$

and consider

Problem 3: *Find* $(\hat{\mathbf{u}}, \hat{\boldsymbol{J}}) \in \mathbf{V}$ *such that*

$$a\big((\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\mathbf{v}, \boldsymbol{K})\big) + a_1\big((\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\mathbf{v}, \boldsymbol{K})\big) = \ell(\mathbf{v}, \boldsymbol{K})$$

for all $(\mathbf{v}, \boldsymbol{K}) \in \mathbf{V}$.
The standard nonlinear version of the classical LBB (Ladyzhenskaya-Babushka-Brezzi) theory (see Chapter 4.1 in [24]) tells us that **Problem 2** and **Problem 3** are equivalent, if the form b satisfies the *inf-sup* condition, and that **Problem 3** is uniquely solvable, if the form a is coercive and bounded. In other words, due to Lemma 3.1 and (11), the LBB-theory allows us to further transform **Problem 2** and solve **Problem 3** for the unknown velocity $\hat{\mathbf{u}}$ and electric current density $\hat{\boldsymbol{J}}$ on a subspace $\mathbf{V}$ of $\mathbf{X}_{(\mathbf{v},\boldsymbol{K})}$. Thus, if $(\hat{\mathbf{u}}, \hat{\boldsymbol{J}}) \in \mathbf{V}$ is a solution of **Problem 3**, then there exist a unique pair $(\hat{p}, \hat{\phi}) \in M_{(q,\psi)}$ such that $(\hat{\mathbf{u}}, \hat{\boldsymbol{J}}, \hat{p}, \hat{\phi})$ solves **Problem 2**. And, given any $((\hat{\mathbf{u}}, \hat{\boldsymbol{J}}), (\hat{p}, \hat{\phi})) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} \times M_{(q,\psi)}$, solution of **Problem 2**, then $(\hat{\mathbf{u}}, \hat{\boldsymbol{J}})$ is in $\mathbf{V}$ and solves **Problem 3**. By [7, Theorem in Section 3] **Problem 3** is uniquely solvable if

$$\|(\mathbf{u}_0, \boldsymbol{J}_0)\|_{\mathbf{X}_{(\mathbf{u},\boldsymbol{J})}} < \frac{\alpha_0}{2\|a_1\|} \tag{12}$$

and

$$\|\ell\|_{\mathbf{V}'} < \frac{(\alpha_0 - 2\|a_1\|\|(\mathbf{u}_0, \boldsymbol{J}_0)\|_{\mathbf{X}_{(\mathbf{u},\boldsymbol{J})}})^2}{4\|a_1\|}. \tag{13}$$

The definition of $\alpha_0 = c(\Omega)\min\{\eta, \sigma^{-1}\}$ implies that any given set of data will satisfy (12)-(13) if the viscosity η and the electric resistivity σ^{-1} are large enough.

4. From weak formulation to frame discretization

We start by reformulating **Problem 3** to fit the following abstract setting: There exists a separable Hilbert space $\mathcal{H}$ such that $\mathbf{V} \subset \mathcal{H} \subset \mathbf{V}'$ with bounded and dense inclusions. The triple $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$ is then called a Gelfand triple. The duality between $\mathbf{V}'$ and $\mathbf{V}$ is identified on $\mathcal{H}$ using the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ of $\mathcal{H}$. There exists an operator $A : \mathbf{V} \to \mathbf{V}'$ such that $\langle Av, w\rangle_{\mathbf{V}'\times\mathbf{V}} := a(v, w)$ defines an elliptic bilinear form, i.e., there exist positive constants α, β such that $\alpha\|v\|_{\mathbf{V}} \leq a(v, v) \leq \beta\|v\|_{\mathbf{V}}$ for all $v \in \mathbf{V}$. The ellipticity of a implies that $\|Av\|_{\mathbf{V}'} \sim \|v\|_{\mathbf{V}}$ and that A is a boundedly invertible operator with $\|A^{-1}\| \leq \alpha^{-1}$. We also assume that there exists a trilinear form a_1 inducing a bounded bilinear operator $A_1 : \mathbf{V} \times \mathbf{V} \to \mathbf{V}'$, defined by $\langle A_1(v, w), z\rangle_{\mathbf{V}'\times\mathbf{V}} := a_1(v, w, z)$ for $v, w, z \in \mathbf{V}$. It has been shown in [7] that **Problem 3** translates into the following abstract problem:

Find $u := (\hat{\mathbf{u}}, \hat{\boldsymbol{J}}) \in \mathbf{V}$ *such that*

$$Au + A_1(u, u) = \ell, \tag{14}$$

where $\ell \in \mathbf{V}'$ *is a functional on* $\mathbf{V}$.

The solvability of (14) is ensured by [7, Lemma 2] and summarized by the following theorem.

Theorem 4.1. *Let* $H : \mathbf{V} \to \mathbf{V}$ *be given by* $H(v) := A^{-1}(\ell - A_1(v, v))$. *The following hold*

(a) *If* $0 < r < \frac{\alpha}{2\|A_1\|}$, *then* $H_{|B_r}$ *is a contraction with Lipschitz constant* $L := \frac{2\|A_1\|r}{\alpha}$, *where* $B_r \subset \mathbf{V}$ *is the closed ball of radius* r *centered at the origin;*

(b) *If* $0 < r < \frac{\alpha}{\|A_1\|}$ *and* $\|\ell\|_{\mathbf{V}'} \leq r(\alpha - \|A_1\|r)$, *then* $H_{|B_r} : B_r \to B_r$;

(c) *If* $\|\ell\|_{\mathbf{V}'} \leq \frac{\alpha^2}{4\|A_1\|}$, *then* (14) *has a unique solution* u *with* $\|u\|_{\mathbf{V}} < \frac{\alpha}{2\|A_1\|}$ *and the solution is given by the fixed point iteration*

$$\begin{aligned} u_{n+1} &= Hu_n, \quad u_0 = 0, \quad n \in \mathbb{N}_0, \\ u &= \lim_{n\to\infty} u_n. \end{aligned} \tag{15}$$

Note that (15) is equivalent to

$$Au_{n+1} = \ell - A_1(u_n, u_n), \quad n \in \mathbb{N}_0, \quad u_0 = 0. \tag{16}$$

Under our assumptions on A, the equations in (16) are elliptic operator equations. We show in this section how to derive a discrete problem equivalent to the abstract nonlinear problem in (14) and prove a result similar to Theorem 4.1 showing that the solution of the discrete problem exists and is unique under certain assumptions on the parameters of the original MHD problem. The discrete problem is obtained using suitable *stable*, *redundant*, and *nonorthogonal* expansions, so-called Gelfand frames for the Gelfand triple $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$. We also show that the corresponding discrete fixed point iteration can be numerically realized efficiently. In particular, we present the realization of the key numerical routine **SOLVE** used in our discrete fixed point iteration to approximate adaptively the solution of the elliptic problems in (16).

4.1. **Gelfand Frames.** In the following, the sequence space $\ell_2(\mathcal{N})$ on the countable index set $\mathcal{N} \subset \mathbb{R}^d$ is induced by the norm

$$\|\vec{\mathbf{c}}\|_{\ell_2(\mathcal{N})} := \left(\sum_{n\in\mathcal{N}} |c_n|^2\right)^{1/2}, \quad \vec{\mathbf{c}} = \{c_n\}_{n\in\mathcal{N}} \in \ell_2(\mathcal{N}).$$

The space $\ell_0(\mathcal{N}) \subset \ell_2(\mathcal{N})$ is the subspace of sequences with compact support. Denote $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ the inner product and the norm on the separable Hilbert space $\mathcal{H}$, respectively. A sequence $\mathcal{F} := \{f_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if

$$\|f\|_{\mathcal{H}}^2 \sim \sum_{n\in\mathcal{N}} \left|\langle f, f_n\rangle_{\mathcal{H}}\right|^2, \quad \text{for all } f \in \mathcal{H}. \tag{17}$$

Due to (17) the corresponding operators of analysis and synthesis given by

$$F : \mathcal{H} \to \ell_2(\mathcal{N}), \quad f \mapsto \left(\langle f, f_n\rangle_{\mathcal{H}}\right)_{n\in\mathcal{N}}, \tag{18}$$

$$F^* : \ell_2(\mathcal{N}) \to \mathcal{H}, \quad \vec{\mathbf{c}} \mapsto \sum_{n\in\mathcal{N}} c_n f_n, \tag{19}$$

are bounded. The composition $S := F^*F$ is a boundedly invertible (positive and self–adjoint) operator called the *frame operator* and $\tilde{\mathcal{F}} := \{S^{-1}f_n\}_{n\in\mathcal{N}}$ is again a frame for $\mathcal{H}$, called the *canonical dual frame*, with corresponding analysis and synthesis operators

$$\tilde{F} := F(F^*F)^{-1}, \quad \tilde{F}^* := (F^*F)^{-1}F^*. \tag{20}$$

In particular, one has the following orthogonal decomposition of $\ell_2(\mathcal{N})$

$$\ell_2(\mathcal{N}) = \operatorname{ran}(F) \oplus \ker(F^*), \tag{21}$$

and

$$\mathbf{Q} := F(F^*F)^{-1}F^* : \quad \ell_2(\mathcal{N}) \to \operatorname{ran}(F), \tag{22}$$

is the orthogonal projection onto $\operatorname{ran}(F)$.

The frame $\mathcal{F}$ is a Riesz basis for $\mathcal{H}$ if and only if $\ker(F^*) = \{0\}$. In general, we assume that $\{0\}$ is a proper subspace of $\ker(F^*)$. In other words, due to the redundancy of the frame there may exist sequences $\vec{\mathbf{c}} = \{c_n\}_{n\in\mathcal{N}} \neq \vec{\mathbf{d}} = \{d_n\}_{n\in\mathcal{N}}$ in $\ell_2(\mathcal{N})$ such that $\sum_{n\in\mathcal{N}} c_n f_n = \sum_{n\in\mathcal{N}} d_n f_n$. In particular, the redundancy may lead to the situation when a small perturbation $\vec{\mathbf{d}}$ of the coefficient sequence $\vec{\mathbf{c}}$ has no effect on the synthesis operator. This possible reduction effect on errors, noise, and numerical round-offs is the motivation for using frames for the applications, where tolerance to errors is required. The intrinsic stability of frames is also expected to play a role in the conditioning of the discretizations of operator equations and leads to additional robustness that the discretization inherits from the frame. In fact, it has been recently confirmed by numerical experiments in [46] that increasing (uniform) redundancy of the frame one improves the conditioning of the corresponding discretization matrices.

It is somewhat perplexing that different sets of coefficients yield equivalent representations of the *same* element of $\mathcal{H}$. It is not clear then what are the "good and computable coefficients": The importance of the canonical dual frame is its use in reconstruction of any $f \in \mathcal{H}$, i.e.

$$f = SS^{-1}f = \sum_{n\in\mathcal{N}} \langle f, S^{-1}f_n\rangle_{\mathcal{H}} f_n = S^{-1}Sf = \sum_{n\in\mathcal{N}} \langle f, f_n\rangle_{\mathcal{H}} S^{-1}f_n. \tag{23}$$

Since a frame is typically overcomplete in the sense that the coefficient functionals $\vec{\mathbf{c}} = \{c_n(f)\}_{n\in\mathcal{N}} \in \ell_2(\mathcal{N})$ in the representation

$$f = \sum_{n\in\mathcal{N}} c_n(f) f_n \tag{24}$$

are in general not unique ($\ker(F^*) \neq \{0\}$), there exist many possible non–canonical duals $\{\tilde{f}_n\}_{n\in\mathcal{N}}$ in $\mathcal{H}$ for which

$$f = \sum_{n\in\mathcal{N}} \langle f, \tilde{f}_n\rangle_{\mathcal{H}} f_n. \tag{25}$$

A more general definition of frames is required for Banach spaces. For details on Banach frames we refer, for example, to [13, 22, 23, 26, 28]. Throughout this paper we make use of *Gelfand frames* that are particular instances of Banach frames. So we do not introduce the latter here in full generality. Assuming that $\mathcal{B}$ is a Banach space continuously and densely embedded in $\mathcal{H}$ we get

$$\mathcal{B} \subseteq \mathcal{H} \simeq \mathcal{H}' \subseteq \mathcal{B}'. \tag{26}$$

If the right inclusion is dense, then $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ is called a *Gelfand triple*. The symbol $\simeq$ stands for the canonical Riesz identification of $\mathcal{H}$ with its dual $\mathcal{H}'$.

Definition 1. A frame $\mathcal{F}$ (here $\tilde{\mathcal{F}}$ is the canonical dual frame) for $\mathcal{H}$ is called a *Gelfand frame* for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$, if $\mathcal{F} \subset \mathcal{B}$, $\tilde{\mathcal{F}} \subset \mathcal{B}'$ and there exists a Gelfand triple $\big(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\big)$ of sequence spaces such that

$$F^* : \mathcal{B}_d \to \mathcal{B},\ F^*\vec{\mathbf{c}} = \sum_{n\in\mathcal{N}} c_n f_n \quad \text{and} \quad \tilde{F} : \mathcal{B} \to \mathcal{B}_d,\ \tilde{F}f = \big(\langle f, \tilde{f}_n\rangle_{\mathcal{B}\times\mathcal{B}'}\big)_{n\in\mathcal{N}} \tag{27}$$

are bounded operators.

REMARKS:

1. If $\mathcal{F}$ (again $\tilde{\mathcal{F}}$ is the canonical dual frame) is a Gelfand frame for the Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ with respect to the Gelfand triple of sequences $\big(\mathcal{B}_d, \ell_2(\mathcal{N}), \mathcal{B}_d'\big)$, then by duality also the operators

$$\tilde{F}^* : \mathcal{B}_d' \to \mathcal{B}',\ \tilde{F}^*\vec{\mathbf{c}} = \sum_{n\in\mathcal{N}} c_n \tilde{f}_n \quad \text{and} \quad F : \mathcal{B}' \to \mathcal{B}_d',\ Ff = \big(\langle f, f_n\rangle_{\mathcal{B}'\times\mathcal{B}}\big)_{n\in\mathcal{N}} \tag{28}$$

are bounded, see, e.g., [29] for details.

2. If $\mathcal{B} = \mathcal{H}$ then Definition 1 becomes the definition of frames for Hilbert spaces.

3. Even if the solution space $\mathbf{V} \subset \mathcal{H} = \mathbf{L}^2(\Omega) \subset \mathbf{V}'$ is a Hilbert space, by using the notation "$\mathcal{B}$" here we want to emphasize that the frame $\mathcal{F}$ we consider is *not* a Hilbert space frame for $\mathbf{V}$. It is a frame for $\mathcal{H}$, which also characterizes $\mathbf{V}$ (as a subspace of $\mathcal{H}$) with frame coefficients $\langle f, \tilde{f}_i \rangle_{\mathcal{H}}$ belonging to some suitable sequence space $\mathbf{V}_d \subset \ell_2(\mathcal{N})$ and being computed using $\mathcal{H}$-inner product. Of course, we can consider genuine Hilbert frame expansions for $\mathbf{V}$ (as it is done, for example, in [38]). This enforces though the use of $\mathbf{V}-$inner product when determining the duality pair. This, in most of the cases, is not compatible with numerical implementations.

4. Definition 1 generalizes the following case to pure frames: Consider a wavelet system $\Psi := \{\psi_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j}$ on Ω ($\mathcal{J}_j$ is a suitable set of indexes depending on the scale j, see [44] for details), $\mathcal{B} = H^s(\Omega)$, $\mathcal{H} = L_2(\Omega)$ and

$$\mathcal{B}_d = \ell_{2,2^{s\cdot}} := \{\vec{\mathbf{d}} := \{d_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j} : \left(\sum_{j\geq -1} \sum_{k\in\mathcal{J}_j} 2^{2sj} |d_{j,k}|^2 \right)^{1/2} < \infty\}.$$

It is well known that if Ψ is a Riesz basis for $L_2(\Omega)$ and its elements, together with those of its biorthogonal dual basis $\tilde{\Psi} := \{\tilde{\psi}_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j}$, are compactly supported, smooth enough, and with a sufficient number of vanishing moments, then $H^s(\Omega)$ is fully characterized by Ψ in the sense that $f \in H^s(\Omega)$ if and only if

$$f = \sum_{j\geq -1} \sum_{k\in\mathcal{J}_j} \langle f, \tilde{\psi}_{j,k} \rangle_{L_2(\Omega)} \psi_{j,k} \tag{29}$$

and

$$\|f\|_{H^s(\Omega)} \sim \left(\sum_{j\geq -1} \sum_{k\in\mathcal{J}_j} 2^{2sj} |\langle f, \tilde{\psi}_{j,k} \rangle_{L_2(\Omega)}|^2 \right)^{1/2} . \tag{30}$$

See [13] for the same characterization by using pure wavelet frames, constructed by Overlapping Domain Decomposition. Note that there exists a natural unitary isomorphism from $\ell_{2,2^{s\cdot}}$ into ℓ_2 given by

$$D_{H^s(\Omega)} : \ell_{2,2^{js\cdot}} \to \ell_2, \quad \vec{\mathbf{d}} := \{d_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j} \mapsto D_{H^s(\Omega)}\vec{\mathbf{d}} := \{2^{js} d_{j,k}\}_{j\geq -1, k\in\mathcal{J}_j}. \tag{31}$$

Keeping in mind the example in the above REMARK 4., we proceed to numerical treatment of abstract elliptic operator equations by means of Gelfand frame discretizations.

4.2. **Adaptive Numerical Frame Schemes for Elliptic Operator Equations.** To implement the fixed point iteration described in Theorem 4.1, we first study the solvability (for fixed $u^{(n)}$) of the linear operator equations in (16). Generally, such equations are of the form

$$Au = f, \tag{32}$$

where A, as before, is a boundedly invertible operator from Hilbert space $\mathbf{V}$ into its dual $\mathbf{V}'$,

$$\|Au\|_{\mathbf{V}'} \sim \|u\|_{\mathbf{V}}, \quad u \in \mathbf{V}. \tag{33}$$

We also have that

$$a(v, w) := \langle Av, w \rangle_{\mathbf{V}'\times\mathbf{V}} \ , \tag{34}$$

defines a bilinear form on $\mathbf{V}$, where $\langle \cdot, \cdot \rangle_{\mathbf{V}\times\mathbf{V}'}$ defines the dual pairing of $\mathbf{V}$ and $\mathbf{V}'$. The form a is *elliptic*, i.e., there exist positive constants α, β such that

$$\alpha\|v\|_{\mathbf{V}} \leq a(v, v) \leq \beta\|v\|_{\mathbf{V}} \tag{35}$$

for all $v \in \mathbf{V}$, and a is *non-symmetric*. The assumption on the non-symmetry of a is motivated by the MHD example presented in Sections 2-3.

Here and throughout the rest of the paper we assume that $\mathcal{F} = \{f_n\}_{n\in\mathcal{N}}$ is a Gelfand frame for the Gelfand triple $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$ with $(\mathbf{V}_d, \ell_2(\mathcal{N}), \mathbf{V}'_d)$ being the corresponding Gelfand triple of sequence spaces. Moreover, keeping in mind REMARK 4. in Subsection 4.1, we also assume that there exists a unitary isomorphism $D_{\mathbf{V}} : \mathbf{V}_d \to \ell_2(\mathcal{N})$, so that its $\ell_2(\mathcal{N})$–adjoint $D^*_{\mathbf{V}} : \ell_2(\mathcal{N}) \to \mathbf{V}'_d$ is also an isomorphism.

We show, next, how the Gelfand frame setting can be used for the adaptive numerical treatment of elliptic operator equations (32). Following, e.g. [8, 13, 38], one uses frame expansions to convert the problem (32) into an operator equation on $\ell_2(\mathcal{N})$. The problem that arises is that the redundancy of the frame leads to a singular discretization matrix. Nevertheless, in Theorem 4.3 below we show that this can be handled in practice and that the solution of (32) can be computed by a version of Richardson iteration applied to the associated normal equations. The resulting scheme is not directly implementable since one has to deal with infinite matrices and vectors. Therefore, similarly to [8, 9, 38], we also show how the scheme can be transformed into an implementable scheme using "finite" versions of the building blocks procedures we introduce in Subsection 4.2.2. The result is a convergent adaptive frame algorithm.

4.2.1. *A series representation.* We start by generalizing Lemma 4.1 and Theorem 4.2 given in [13] to the case of non-symmetric a. We give the detailed proofs to emphasize the difference between the symmetric and non-symmetric cases.

Lemma 4.2. *Under the assumptions* (34), (35) *on A, the operator*

$$\mathbf{A} := (D^*_{\mathbf{V}})^{-1} F A F^* D^{-1}_{\mathbf{V}} \tag{36}$$

is a bounded operator from $\ell_2(\mathcal{N})$ to $\ell_2(\mathcal{N})$. Moreover $\mathbf{A}$ is boundedly invertible on its range $\operatorname{ran}(\mathbf{A}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F)$.

Proof. Since $\mathbf{A}$ is a composition of bounded operators $D^{-1}_{\mathbf{V}} : \ell_2(\mathcal{N}) \to \mathbf{V}_d$, $F^* : \mathbf{V}_d \to \mathbf{V}$, $A : \mathbf{V} \to \mathbf{V}'$, $F : \mathbf{V}' \to \mathbf{V}'_d$ and $(D^*_{\mathbf{V}})^{-1} : \mathbf{V}'_d \to \ell_2(\mathcal{N})$, $\mathbf{A}$ is a bounded operator from $\ell_2(\mathcal{N})$ to $\ell_2(\mathcal{N})$. Moreover, from the decomposition (36) we get

$$\ker(\mathbf{A}) = \ker(F^* D^{-1}_{\mathbf{V}}), \quad \operatorname{ran}(\mathbf{A}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F). \tag{37}$$

Define $\mathbf{L} := (D^*_{\mathbf{V}})^{-1} F F^* D^{-1}_{\mathbf{V}}$. Note that $\ker(\mathbf{L}) = \ker(F^* D^{-1}_{\mathbf{V}})$ and $\operatorname{ran}(\mathbf{L}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F)$. The fact that $\ell_2(\mathcal{N}) = \ker(\mathbf{L}^*) \oplus \operatorname{ran}(\mathbf{L})$ implies, due to the self-adjointness of $\mathbf{L}$, that

$$\ell_2(\mathcal{N}) = \ker(F^* D^{-1}_{\mathbf{V}}) \oplus \operatorname{ran}((D^*_{\mathbf{V}})^{-1}F). \tag{38}$$

Therefore,

$$\mathbf{A}_{|\operatorname{ran}(\mathbf{A})} : \operatorname{ran}(\mathbf{A}) \to \operatorname{ran}(\mathbf{A}) \tag{39}$$

is boundedly invertible. ■

Denote by $\mathbf{P} : \ell_2(\mathcal{N}) \to \operatorname{ran}(\mathbf{A})$ the orthogonal projection of $\ell_2(\mathcal{N})$ onto $\operatorname{ran}(\mathbf{A})$.

Theorem 4.3. *Let A satisfy* (34) *and* (35)*. Denote*

$$\vec{\mathbf{f}} := (D^*_{\mathbf{V}})^{-1} F f \tag{40}$$

and $\mathbf{A}$ as in (36)*. Then the solution u of* (32) *can be computed by*

$$u = F^* D^{-1}_{\mathbf{V}} \mathbf{P}\vec{\mathbf{u}} \tag{41}$$

where $\vec{\mathbf{u}}$ solves

$$\mathbf{P}\vec{\mathbf{u}} = \left(\alpha^* \sum_{n=0}^{\infty} (\operatorname{id} - \alpha^* \mathbf{A}^* \mathbf{A})^n_{|\operatorname{ran}(\mathbf{A})} \right) \mathbf{A}^* \vec{\mathbf{f}}, \tag{42}$$

with $0 < \alpha^* < 2/\lambda_{\max}$, where $\lambda_{max} = \|\mathbf{A}^*\mathbf{A}\|_2$ with $\|\cdot\|_2$ being the usual spectral norm.

Proof. We have $u = \sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathcal{H}} f_n$ in $\mathcal{H}$. Since $\mathcal{F}$ is a Gelfand frame, $F^*\tilde{F} : \mathbf{V} \to \mathbf{V}$ is bounded and implies $u = F^*\tilde{F}u = \sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathbf{V}\times\mathbf{V}'} f_n$ in $\mathbf{V}$. Moreover, one can show that (32) is equivalent to the following system of equations

$$\sum_{n\in\mathcal{N}} \langle u, \tilde{f}_n\rangle_{\mathbf{V}\times\mathbf{V}'} \langle Af_n, f_m\rangle_{\mathbf{V}'\times\mathbf{V}} = \langle f, f_m\rangle_{\mathbf{V}'\times\mathbf{V}}, \quad m \in \mathcal{N}. \tag{43}$$

Denote $\vec{\mathbf{u}} := D_{\mathbf{V}}\tilde{F}u$ and $\vec{\mathbf{f}}$, $\mathbf{A}$ as in (40) and (36). Then (43) can be rewritten as

$$\mathbf{A}\vec{\mathbf{u}} = \vec{\mathbf{f}}. \tag{44}$$

Multiplying both side of (44) by $\mathbf{A}^*$ we get the normal equation

$$(\mathbf{A}^*\mathbf{A})\,\vec{\mathbf{u}} = \mathbf{A}^*\vec{\mathbf{f}}. \tag{45}$$

Note that $\mathbf{A}^*\mathbf{A}$ is self-adjoint and positive-definite by the hypothesis. Note also that $\ker(\mathbf{A}^*)$ is orthogonal to $\operatorname{ran}(\mathbf{A})$ and (38) implies that $\ker(\mathbf{A}^*) = \ker(\mathbf{A})$. This and the invertibility of $\mathbf{A}$ on its range implies that $\mathbf{A}^*\mathbf{A}$ is boundedly invertible on $\operatorname{ran}(\mathbf{A})$. Therefore, the solution of (44) is equivalent to the solution of (45).

Note that, for $0 < \alpha^* < 2/\lambda_{\max}$, the operator

$$\mathbf{B} := \alpha^* \sum_{n=0}^{\infty} (\mathrm{id} - \alpha^*\mathbf{A}^*\mathbf{A})^n_{|\operatorname{ran}(\mathbf{A})}. \tag{46}$$

is well-defined and bounded on $\operatorname{ran}(\mathbf{A})$, since $\rho(\alpha^*) := \|\,(\mathrm{id} - \alpha^*\mathbf{A}^*\mathbf{A})_{|\operatorname{ran}(\mathbf{A})}\,\|_2 = \max\{\alpha^*\lambda_{\max} - 1, 1 - \alpha^*\lambda_{\min}\} < 1$, where $\lambda_{\min} := \|(\mathbf{A}^*\mathbf{A}_{|\operatorname{ran}(\mathbf{A})})^{-1}\|_2$. The function ρ is minimal at $\alpha^* := 2/(\lambda_{\max} + \lambda_{\min})$. Moreover,

$$\mathbf{B} \circ (\mathbf{A}^*\mathbf{A}_{|\operatorname{ran}(\mathbf{A})}) = (\mathbf{A}^*\mathbf{A}) \circ \mathbf{B}_{|\operatorname{ran}(\mathbf{A})} = \mathrm{id}_{|\operatorname{ran}(\mathbf{A})}. \tag{47}$$

Since $\mathbf{A}(\mathrm{id} - \mathbf{P}) = 0$,

$$\mathbf{A}\vec{\mathbf{u}} = \mathbf{A}\mathbf{P}\vec{\mathbf{u}} = \vec{\mathbf{f}}. \tag{48}$$

Therefore $\mathbf{P}\vec{\mathbf{u}} \in \operatorname{ran}(\mathbf{A})$ is the unique solution of (44) in $\operatorname{ran}(\mathbf{A})$ and by (47)

$$\mathbf{P}\vec{\mathbf{u}} = \mathbf{B}\mathbf{A}^*\vec{\mathbf{f}}. \tag{49}$$

By construction

$$\begin{aligned}
\langle f, f_m\rangle_{\mathbf{V}'\times\mathbf{V}} &= \langle \tilde{F}^*Ff, f_m\rangle_{\mathbf{V}'\times\mathbf{V}} \\
&= \langle \tilde{F}^*D^*_{\mathbf{V}}\vec{\mathbf{f}}, f_m\rangle_{\mathbf{V}'\times\mathbf{V}} \\
&= \langle \tilde{F}^*D^*_{\mathbf{V}}\mathbf{A}\mathbf{P}\vec{\mathbf{u}}, f_m\rangle_{\mathbf{V}'\times\mathbf{V}} \\
&= \langle AF^*D^{-1}_{\mathbf{V}}\mathbf{P}\vec{\mathbf{u}}, f_m\rangle_{\mathbf{V}'\times\mathbf{V}}, \quad m \in \mathcal{N},
\end{aligned}$$

so that $u = F^*D^{-1}_{\mathbf{V}}\mathbf{P}\vec{\mathbf{u}}$ solves (32). ∎

4.2.2. *Numerical realization.* Now we turn to the numerical treatment of (44). Due to Theorem 4.3, the computation of $\vec{\mathbf{u}}$ solving (44) amounts to an application of the following damped Richardson iteration

$$\vec{\mathbf{u}}^{(i+1)} = \vec{\mathbf{u}}^{(i)} - \alpha^*\mathbf{A}^*(\mathbf{A}\vec{\mathbf{u}}^{(i)} - \vec{\mathbf{f}}), \quad i \in \mathbb{N}_0, \quad \vec{\mathbf{u}}^{(0)} = \mathbf{0}. \tag{50}$$

Certainly this iteration cannot be practically realized for infinite vectors $\vec{\mathbf{u}}^{(i)}$, $i \in \mathbb{N}_0$. To avoid this problem, we make use of the following procedures (see [8, 9, 10, 38] for details on their analysis and numerical realization) :

- $\mathbf{RHS}[\varepsilon, \vec{\mathbf{f}}] \to \vec{\mathbf{f}}_\varepsilon$: determines for $\vec{\mathbf{f}} \in \ell_2(\mathcal{N})$ a vector $\vec{\mathbf{f}}_\varepsilon \in \ell_0(\mathcal{N})$ such that

$$\|\vec{\mathbf{f}} - \vec{\mathbf{f}}_\varepsilon\|_{\ell_2(\mathcal{N})} \le \varepsilon; \tag{51}$$

- **APPLY**$[\varepsilon, \mathbf{A}, \vec{\mathbf{v}}] \to \vec{\mathbf{w}}_\varepsilon$: determines for a bounded linear operator $\mathbf{A}$ on $\ell_2(\mathcal{N})$ and for $\vec{\mathbf{v}} \in \ell_0(\mathcal{N})$ a vector $\vec{\mathbf{w}}_\varepsilon \in \ell_0(\mathcal{N})$ such that

$$\|\mathbf{A}\vec{\mathbf{v}} - \vec{\mathbf{w}}_\varepsilon\|_{\ell_2(\mathcal{N})} \leq \varepsilon; \tag{52}$$

- **COARSE**$[\varepsilon, \vec{\mathbf{v}}] \to \vec{\mathbf{v}}_\varepsilon$: determines for $\vec{\mathbf{v}} \in \ell_0(\mathcal{N})$ a vector $\vec{\mathbf{v}}_\varepsilon \in \ell_0(\mathcal{N})$ such that

$$\|\vec{\mathbf{v}} - \vec{\mathbf{v}}_\varepsilon\|_{\ell_2(\mathcal{N})} \leq \varepsilon. \tag{53}$$

We discuss in more details further properties of the routines **RHS**, **APPLY** and **COARSE** in Section 6, where we study the complexity and the computational cost required to approximate the solution of the original problem up to some prescribed tolerance.

Let $\rho := \rho(\alpha^*)$ be as in the proof of Theorem 4.3. We can define the following inexact version of the damped Richardson iteration (50):

Algorithm 1.

SOLVE$[\epsilon, \mathbf{A}, \vec{\mathbf{f}}] \to \vec{\mathbf{v}}_\epsilon$:
Let $\theta < 1/3$ and $K \in \mathbb{N}$ be fixed such that $3\rho^K < \theta$.
$j := 0,\ \vec{\mathbf{v}}^{(0)} := 0,\ \epsilon_0 := \|\mathbf{A}^{-1}_{|\operatorname{ran}(\mathbf{A})}\| \|\vec{\mathbf{f}}\|_{\ell_2(\mathcal{N})}$
While $\epsilon_j > \epsilon$ do
  $j := j + 1$
  $\epsilon_j := 3\rho^K \epsilon_{j-1}/\theta$
  $\vec{\mathbf{g}}^{(j)} := \mathbf{RHS}[\frac{\theta\epsilon_j}{12\alpha^* K\|\mathbf{A}^*\|}, \vec{\mathbf{f}}]$
  $\vec{\mathbf{f}}^{(j)} := \mathbf{APPLY}[\frac{\theta\epsilon_j}{12\alpha^* K}, \mathbf{A}^*, \vec{\mathbf{g}}^{(j)}]$
  $\vec{\mathbf{v}}^{(j,0)} := \vec{\mathbf{v}}^{(j-1)}$
  For $k = 1, ..., K$ do
    $\vec{\mathbf{w}}^{(j,k-1)} := \mathbf{APPLY}[\frac{\theta\epsilon_j}{12\alpha^* K\|\mathbf{A}^*\|}, \mathbf{A}, \vec{\mathbf{v}}^{(j,k-1)}]$
    $\vec{\mathbf{v}}^{(j,k)} := \vec{\mathbf{v}}^{(j,k-1)} - \alpha^* \left(\mathbf{APPLY}[\frac{\theta\epsilon_j}{12\alpha^* K}, \mathbf{A}^*, \vec{\mathbf{w}}^{(j,k-1)}] - \vec{\mathbf{f}}^{(j)}\right)$
  od
  $\vec{\mathbf{v}}^{(j)} := \mathbf{COARSE}[(1-\theta)\epsilon_j, \vec{\mathbf{v}}^{(j,K)}]$
od
$\vec{\mathbf{v}}_\epsilon := \vec{\mathbf{v}}^{(j)}$.

The parameter θ plays an important role in complexity estimates (given in Section 6) for **COARSE**.

The proof of the convergence of Algorithm 1 is analogous to that of [38, Proposition 2.1] and of [13, Theorem 4.2] except for the fact that here we make use of the damped Richardson iteration in (50) on normal equations, due to the non-symmetry of a. Nevertheless, the following result holds.

Theorem 4.4. *Under assumptions of Theorem 4.3, let $\vec{\mathbf{u}} \in \ell_2(\mathcal{N})$ be a solution of* (44)*. Then* **SOLVE**$[\epsilon, \mathbf{A}, \vec{\mathbf{f}}]$ *produces finitely supported vectors $\vec{\mathbf{v}}^{(j,K)}, \vec{\mathbf{v}}^{(j)}, \vec{\mathbf{v}}_\epsilon$ such that*

$$\left\|\mathbf{P}(\vec{\mathbf{u}} - \vec{\mathbf{v}}^{(j)})\right\|_{\ell_2(\mathcal{N})} \leq \epsilon_j, \quad j \in \mathbb{N}_0. \tag{54}$$

In particular,

$$\|u - F^* D_{\mathbf{V}}^{-1} \vec{\mathbf{v}}_\epsilon\|_{\mathbf{V}} \leq \|F^*\| \|D_{\mathbf{V}}^{-1}\| \epsilon. \tag{55}$$

Moreover, it holds that

$$\left\|\mathbf{P}\vec{\mathbf{u}} - (\operatorname{id} - \mathbf{P})\vec{\mathbf{v}}^{(j-1)} - \vec{\mathbf{v}}^{(j,K)}\right\|_{\ell_2(\mathcal{N})} \leq \frac{2\theta\epsilon_j}{3}, \quad j \geq 1. \tag{56}$$

Of course, the numerical implementation of the damped Richardson iteration on the normal equations (45) might exhibit a low convergence rate if the relaxation parameter α^* is small. To improve the efficiency of the proposed scheme, the generalizations of Algorithm 1 towards, e.g., (conjugate) gradient iterations as suggested in [17, 46] are now a matter of investigation.

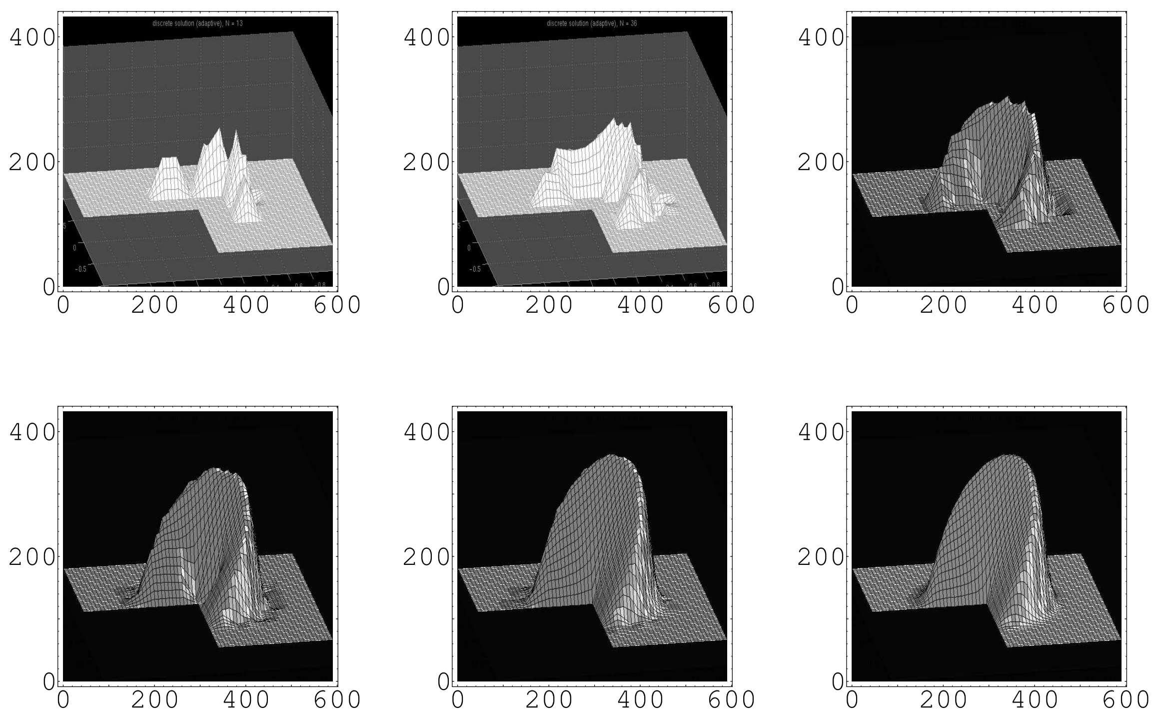

FIGURE 1. Example of application of the procedure **SOLVE** for the solution of the Poisson equation on the L-shape domain by means of overlapping square patches and the use of aggregate wavelet frames. The approximations of the solution are illustrated for successive iterations. We refer to [13, 46] for more details on the numerical implementation.

5. Numerical realization of the fixed point iteration

Now we have at hand the major building blocks needed to formulate an implementable fixed point iteration. In this section we show how the problem in (14) can be discretized and how Algorithm 1 can be used to implement the fixed point iteration in (15).

We denote

i) $\mathbf{A} := (D^*_{\mathbf{V}})^{-1} F A F^* D^{-1}_{\mathbf{V}} : \ell_2(\mathcal{N}) \to \ell_2(\mathcal{N})$, bounded and boundedly invertible on its range $\operatorname{ran}(\mathbf{A}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1} F)$;
ii) $\vec{\mathbf{l}} := (D^*_{\mathbf{V}})^{-1} F \ell \in \operatorname{ran}(\mathbf{A}) \subset \ell_2(\mathcal{N})$;
iii) $\mathbf{A}_1(\cdot) := (D^*_{\mathbf{V}})^{-1} F A_1(F^* D^{-1}_{\mathbf{V}} \cdot, F^* D^{-1}_{\mathbf{V}} \cdot) : \ell_2(\mathcal{N}) \to \operatorname{ran}(\mathbf{A}) \subset \ell_2(\mathcal{N})$;
iv) $\mathbf{H}(\cdot) := (\mathbf{A}_{|\operatorname{ran}\mathbf{A}})^{-1}(\vec{\mathbf{l}} - \mathbf{A}_1(\cdot)) : \ell_2(\mathcal{N}) \to \operatorname{ran}(\mathbf{A})$.

REMARK: Note that $\operatorname{ran}(\mathbf{A}) = \operatorname{ran}((D^*_{\mathbf{V}})^{-1} F)$ implies that the operators in ii)-iv) map $\ell_2(\mathcal{N})$ into $\operatorname{ran}(\mathbf{A})$. By definition of $\mathbf{P}$ we also have $\mathbf{H}(\mathbf{P}\vec{\mathbf{v}}) = \mathbf{P}\mathbf{H}(\vec{\mathbf{v}}) = \mathbf{H}(\vec{\mathbf{v}})$ for any $\vec{\mathbf{v}} \in \ell_2(\mathcal{N})$.

Then, it is easily verified that the variational problem in (14) is equivalent to the following discrete problem.

Problem 4: *Find* $\vec{\mathbf{u}} \in \mathrm{ran}(\mathbf{A}) \subset \ell_2(\mathcal{N})$ *such that*

$$\mathbf{A}\vec{\mathbf{u}} + \mathbf{A}_1(\vec{\mathbf{u}}) = \vec{\mathbf{l}}, \tag{57}$$

or, equivalently, such that $\vec{\mathbf{u}}$ *is a fixed point of* $\mathbf{H}$ *in* $\mathrm{ran}(\mathbf{A})$, *i.e*

$$\vec{\mathbf{u}} = \mathbf{H}(\vec{\mathbf{u}}), \quad \vec{\mathbf{u}} \in \mathrm{ran}(\mathbf{A}). \tag{58}$$

Define the closed subset of $\mathrm{ran}(\mathbf{A})$ by

$$\boldsymbol{B}_r := \{\vec{\mathbf{u}} \in \mathrm{ran}(\mathbf{A}) : \|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \le r\} \quad \text{for some} \quad r \in \mathbb{R}_+.$$

The next theorem is a discrete analogue of Lemma 2 and Corollary 2 in [7].

Theorem 5.1. *Under the assumptions and notations specified above, the following statements hold true:*

a) *If* $0 < r < \left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$, *then* $\mathbf{H}_{|\boldsymbol{B}_r}$ *is a contraction with Lipschitz constant* $L := r\left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right) < 1$;

b) *if* $0 < r < \left(\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$ *and* $\|\vec{\mathbf{l}}\| \le r\left(\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^{-1} - \|A_1\|\|F\|^3 r\right)$ *then* $\mathbf{H}(\boldsymbol{B}_r) \subseteq \boldsymbol{B}_r$;

c) *if* $\|\vec{\mathbf{l}}\| < \left(4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|\right)^{-1}$ *then* (58) *has a unique solution* $\vec{\mathbf{u}} \in \boldsymbol{B}_{r^*}$, *for some suitable* r^* *such that* $0 < r^* < \left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|\|A_1\|\right)^{-1}$.

Proof. Recall that $D_{\mathbf{V}}$ is assumed unitary and $\|D_{\mathbf{V}}\| = \|D_{\mathbf{V}}^{-1}\| = \|D_{\mathbf{V}}^*\| \equiv 1$. Moreover, since F maps $\mathbf{V}'$ into $\mathbf{V}'_d$ both being Hilbert spaces, thus $\|F\| = \|F^*\|$. Then, for $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \boldsymbol{B}_r$,

$$\begin{aligned}
\|\mathbf{H}\vec{\mathbf{u}} - \mathbf{H}\vec{\mathbf{v}}\| &\le \|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|\mathbf{A}_1\vec{\mathbf{u}} - \mathbf{A}_1\vec{\mathbf{v}}\|_{\ell_2(\mathcal{N})} \\
&= \|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|(D_{\mathbf{V}}^*)^{-1}F\left(A_1(F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}, F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}) - A_1(F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{v}}, F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{v}})\right)\|_{\ell_2(\mathcal{N})} \\
&\le \|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|\|A_1(F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}, F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}) - A_1(F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{v}}, F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{v}})\|_{\mathbf{V}'} \\
&= \|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|\|A_1(F^*D_{\mathbf{V}}^{-1}(\vec{\mathbf{u}} - \vec{\mathbf{v}}), F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}) - A_1(F^*D_{\mathbf{V}}^{-1}\vec{\mathbf{v}}, F^*D_{\mathbf{V}}^{-1}(\vec{\mathbf{v}} - \vec{\mathbf{u}}))\|_{\mathbf{V}'} \\
&\le \|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|\|A_1\|\|F^*\|^2\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\|_{\ell_2(\mathcal{N})}\left(\|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} + \|\vec{\mathbf{v}}\|_{\ell_2(\mathcal{N})}\right) \\
&\le 2r\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\|\vec{\mathbf{u}} - \vec{\mathbf{v}}\|_{\ell_2(\mathcal{N})}.
\end{aligned}$$

Thus, as $L < 1$ we have that $\mathbf{H}$ is a contraction. To show b), just observe that by definition of $\mathbf{H}$ and the estimate above

$$\|\mathbf{H}\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \le \|\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}}^{-1}\|\left(\|\vec{\mathbf{l}}\| + \|A_1\|\|F\|^3 r^2\right) \le r.$$

c) We have to show that if $\|\vec{\mathbf{l}}\| < \left(4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|\right)^{-1}$ then there exists r^* with $0 < r^* < \left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$ such that $\|\vec{\mathbf{l}}\| < r^*\left(\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^{-1} - \|A_1\|\|F\|^3 r^*\right)$. Then, using parts a) and b) of this Lemma we get that $H|_{\boldsymbol{B}_{r^*}}$ is a contractive mapping of $\boldsymbol{B}_{r^*}$ into itself and, thus, has a unique fixed point. To see that such an r^* exists, consider

$$h(r) = r\left(\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^{-1} - \|A_1\|\|F\|^3 r\right),$$

a quadratic mapping, and note that h assumes values from 0 to $\left(4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|\right)^{-1}$ as r varies from 0 to $\left(2\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|\|F\|^3\|A_1\|\right)^{-1}$. ■

Corollary 5.2. *If* $\|\vec{\mathbf{l}}\| < (4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$, *then the solution* $\vec{\mathbf{u}}$ *in* $\boldsymbol{B}_{r^*}$ *of* (58) *is given by the following discrete fixed point iteration*

$$\vec{\mathbf{u}}_{n+1} = \mathbf{H}\vec{\mathbf{u}}_n, \quad n \ge 0, \quad \vec{\mathbf{u}}_0 = \mathbf{0} \in \mathrm{ran}(\mathbf{A}), \tag{59}$$

$$\vec{\mathbf{u}} = \lim_{n\to\infty} \vec{\mathbf{u}}_n. \tag{60}$$

The discrete fixed point iteration cannot be implemented for infinite vectors. For this reason we propose the following new approximating adaptive scheme **FIXPT**, where the procedure **SOLVE** introduced in Subsection 4.2.2 replaces the exact computation of $\vec{\mathbf{u}}_{n+1}$ in (59).

Algorithm 2.

$\mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}_1, \vec{\mathbf{l}}] \to \vec{\mathbf{u}}_\varepsilon$:

$i := 0,\ \vec{\mathbf{v}}_0 := 0,\ 0 < \varepsilon_0 < r^*;\ \varepsilon_0 \neq L;$

While $\left(\varepsilon_i > \frac{\varepsilon_0 - L}{\varepsilon_0\left(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^i\right)}(\varepsilon - L^i r^*)\right)$ do

$\quad \varepsilon_{i+1} := \varepsilon_0^{i+1}$

$\quad \vec{\mathbf{v}}_{i+1} = \mathbf{SOLVE}[\varepsilon_{i+1}, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A_1}(\vec{\mathbf{v}}_i)]$

$\quad i := i + 1$

od

$\vec{\mathbf{u}}_\varepsilon := \vec{\mathbf{v}}_i$.

REMARK: The Algorithm 2 is a perturbation of the exact fixed point iteration creating sequences $\{\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}_0}$ of finite vectors. In general such vectors will not belong to $\mathrm{ran}(\mathbf{A})$ and there is not much hope that $\lim_{n\to\infty} \vec{\mathbf{v}}_n = \vec{\mathbf{u}}$. However, one might try to see whether $\lim_{n\to\infty} \mathbf{P}\vec{\mathbf{v}}_n = \vec{\mathbf{u}}$. A priori it might even happen that, because of an accumulation of the perturbation errors, there exists some $n \in \mathbb{N}$ large enough such that $\mathbf{P}\vec{\mathbf{v}}_n \notin \boldsymbol{B}_{r^*}$! In order to show the convergence of Algorithm 2 to the solution of problem (58) we prove the following Lemma.

Lemma 5.3. *If $\|\vec{\mathbf{l}}\| < (4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$ and if the quantities*

$$\begin{aligned}\mathcal{E}_n \quad := \quad & \sum_{k=2}^{n+1}\varepsilon_k + (1+L)\sum_{h=0}^{n-3}\sum_{k=3}^{n-h}\varepsilon_k L^{n-h-k} + \Big(\varepsilon_1 + L(\varepsilon_0 + \|(\mathbf{A}_{|\,\mathrm{ran}(\mathbf{A})})^{-1}\|\|\vec{\mathbf{l}}\|)\Big)\sum_{k=0}^{n-1}L^k \\ + \quad & \varepsilon_0 + \|\mathbf{A}^{-1}_{|\,\mathrm{ran}(\mathbf{A})}\|\|\vec{\mathbf{l}}\| \leq r^*, \ \textit{for all } n \in \mathbb{N}_0,\end{aligned}$$

with L and r^* as in Theorem 5.1 a) and c), respectively, then the sequence $\{\mathbf{P}\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}}$ resulting from the application of Algorithm 2 all lies in $\boldsymbol{B}_{r^*}$.

Proof. We show by induction on n that

$$\|\mathbf{P}(\vec{\mathbf{v}}_{n+1} - \vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})} \leq \varepsilon_{n+1} + (1+L)\sum_{k=3}^{n}\varepsilon_k L^{n-k} + L^{n-1}(\varepsilon_1 + L\|\mathbf{P}\vec{\mathbf{v}}_1\|_{\ell_2(\mathcal{N})}). \tag{61}$$

and

$$\begin{aligned}\|\mathbf{P}(\vec{\mathbf{v}}_{n+1})\|_{\ell_2(\mathcal{N})} \quad := \quad & \sum_{k=2}^{n+1}\varepsilon_k + (1+L)\sum_{h=0}^{n-3}\sum_{k=3}^{n-h}\varepsilon_k L^{n-h-k} \\ + \quad & \Big(\varepsilon_1 + L(\varepsilon_0 + \|(\mathbf{A}_{|\,\mathrm{ran}(\mathbf{A})})^{-1}\|\|\vec{\mathbf{l}}\|)\Big)\sum_{k=0}^{n-1}L^k + \varepsilon_0 + \|\mathbf{A}^{-1}_{|\,\mathrm{ran}(\mathbf{A})}\|\|\vec{\mathbf{l}}\|\end{aligned} \tag{62}$$

Set $n = 1$. We get

$$\begin{aligned}\|\mathbf{P}\vec{\mathbf{v}}_1\|_{\ell_2(\mathcal{N})} = \|\mathbf{P}\Big(\mathbf{SOLVE}[\varepsilon_\mathbf{0}, \mathbf{A}, \vec{\mathbf{l}}] - \mathbf{H}(0)\Big) + \mathbf{H}(0)\|_{\ell_2(\mathcal{N})} \quad & \leq \quad \varepsilon_0 + \|\mathbf{A}^{-1}_{|\,\mathrm{ran}(\mathbf{A})}\vec{\mathbf{l}}\| \\ & \leq \quad \varepsilon_0 + \|(\mathbf{A}_{|\,\mathrm{ran}(\mathbf{A})})^{-1}\|\|\vec{\mathbf{l}}\| \leq r^*.\end{aligned}$$

Moreover, since $\mathbf{H}(\vec{\mathbf{v}}) = \mathbf{H}(\mathbf{P}\vec{\mathbf{v}}) = \mathbf{P}\mathbf{H}(\vec{\mathbf{v}})$, we get

$$\mathbf{P}\Big(\mathbf{SOLVE}[\varepsilon_\mathbf{2}, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A_1}(\vec{\mathbf{v}}_1)]\Big) - \mathbf{H}(\vec{\mathbf{v}}_1) = \mathbf{P}\Big(\mathbf{SOLVE}[\varepsilon_\mathbf{2}, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A_1}(\vec{\mathbf{v}}_\mathbf{1})]\Big) - \mathbf{P}\ \mathbf{H}(\vec{\mathbf{v}}_1).$$

By (54) it holds that $\|\mathbf{P}\,\mathbf{SOLVE}[\varepsilon_{n+1},\mathbf{A},\vec{\mathbf{l}}-\mathbf{A}_1(\vec{\mathbf{v}}_n)]-\mathbf{P}\,\mathbf{H}(\vec{\mathbf{v}}_n)\| \le \varepsilon_{n+1}$ and, using the assumption on $\|\vec{\mathbf{l}}\|$ ensuring that $\mathbf{H}$ is a contraction with the Lipschitz constant L, we get

$$\begin{aligned}
\|\mathbf{P}(\vec{\mathbf{v}}_2-\vec{\mathbf{v}}_1)\|_{\ell_2(\mathcal{N})} &= \left\|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_2,\mathbf{A},\vec{\mathbf{l}}-\mathbf{A}_1(\vec{\mathbf{v}}_1)]\right)-\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_1,\mathbf{A},\vec{\mathbf{l}}-\mathbf{A}_1(\vec{\mathbf{v}}_0)]\right)\right\|_{\ell_2(\mathcal{N})}\\
&= \|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_2,\mathbf{A},\vec{\mathbf{l}}-\mathbf{A}_1(\vec{\mathbf{v}}_1)]-\mathbf{H}(\vec{\mathbf{v}}_1)\right)+\mathbf{H}(\vec{\mathbf{v}}_1)-\mathbf{H}(\vec{\mathbf{v}}_0)\\
&+ \mathbf{H}(\vec{\mathbf{v}}_0)-\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_1,\mathbf{A},\vec{\mathbf{l}}-\mathbf{A}_1(\vec{\mathbf{v}}_0)]\right)\|_{\ell_2(\mathcal{N})}\\
&\le \varepsilon_2+\varepsilon_1+L\|\mathbf{P}\vec{\mathbf{v}}_1\|_{\ell_2(\mathcal{N})}.
\end{aligned}$$

We assume now that all $\mathbf{P}\vec{\mathbf{v}}_1,\dots,\mathbf{P}\vec{\mathbf{v}}_n\in\boldsymbol{B}_{r^*}$ and that formulas (61) and (62) are valid replacing n with $n-1$. Then

$$\begin{aligned}
\|\mathbf{P}(\vec{\mathbf{v}}_{n+1}-\vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})} &\le \varepsilon_{n+1}+\varepsilon_n+L\|\mathbf{P}(\vec{\mathbf{v}}_n-\vec{\mathbf{v}}_{n-1})\|_{\ell_2(\mathcal{N})}\\
&\le \varepsilon_{n+1}+\varepsilon_n+L\left(\varepsilon_n+(1+L)\sum_{k=3}^{n-1}\varepsilon_k L^{n-1-k}+L^{n-2}(\varepsilon_1+L\|\mathbf{P}\vec{\mathbf{v}}_1\|_{\ell_2(\mathcal{N})})\right)\\
&= \varepsilon_{n+1}+(1+L)\sum_{k=3}^{n}\varepsilon_k L^{n-k}+L^{n-1}(\varepsilon_1+L\|\mathbf{P}\vec{\mathbf{v}}_1\|_{\ell_2(\mathcal{N})}).
\end{aligned}$$

Using the above estimate for $\|\mathbf{P}(\vec{\mathbf{v}}_{n+1}-\vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})}$, the triangular inequality, i.e.

$$\|\mathbf{P}\vec{\mathbf{v}}_{n+1}\|_{\ell_2(\mathcal{N})}\le\|\mathbf{P}(\vec{\mathbf{v}}_{n+1}-\vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})}+\|\mathbf{P}\vec{\mathbf{v}}_n\|_{\ell_2(\mathcal{N})}$$

and (62) for $\mathbf{P}\vec{\mathbf{v}}_n$ we have immediately (62) for $\mathbf{P}\vec{\mathbf{v}}_{n+1}$. Thus, by hypothesis on $\mathcal{E}_n$ we obtain $\|\mathbf{P}\vec{\mathbf{v}}_{n+1}\|_{\ell_2(\mathcal{N})}\le r^*$. This implies by induction that $\mathbf{P}\vec{\mathbf{v}}_n\in\boldsymbol{B}_{r^*}$ for all $n\in\mathbb{N}$. ■

REMARK: Note that the assumption on $\mathcal{E}_n$ in the above Theorem is not restrictive. In fact, it holds that

- $\displaystyle\sum_{k=2}^{n+1}\varepsilon_k=\sum_{k=0}^{n+1}\varepsilon_0^k-1-\varepsilon_0=\frac{1-\varepsilon_0^{n+2}}{1-\varepsilon_0}-1-\varepsilon_0\le\frac{\varepsilon_0^2}{1-\varepsilon_0}\to 0$ for $\varepsilon_0\to 0$;
- the map
$$n\mapsto\sum_{h=0}^{n-3}\sum_{k=3}^{n-h}\varepsilon_k L^{n-h-k}\le C(\varepsilon_0,L),$$
where $C(\varepsilon_0,L)\to 0$ for $\varepsilon_0\to 0$, uniformly with respect to n;
- $\left(\varepsilon_1+L(\varepsilon_0+\|\mathbf{A}^{-1}_{|\,\mathrm{ran}(\mathbf{A})}\|\|\vec{\mathbf{l}}\|\right)\sum_{k=0}^{\infty}L^k+(\varepsilon_0+\|\mathbf{A}^{-1}_{|\,\mathrm{ran}(\mathbf{A})}\|\|\vec{\mathbf{l}}\|)$ is small as one wants whenever ε_0 and $\|\vec{\mathbf{l}}\|$ are chosen small enough. Note that $\|\vec{\mathbf{l}}\|\lesssim\|F\|\|\ell\|$, where $\|\ell\|$, at least for the MHD problem discussed above, depends on the size of the norms of the forcing terms and boundary data in (1)-(9). Controlling the size of $\|\vec{\mathbf{l}}\|$ we can ensure that the assumption on $\mathcal{E}_n$ in Lemma 5.3 holds.

We are now ready to show our main convergence result.

Theorem 5.4. *If* $\|\vec{\mathbf{l}}\|<(4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$ *and* $\mathcal{E}_n\le r^*$ *for all* $n\in\mathbb{N}$, *then*

$$\mathbf{H}(\vec{\mathbf{u}})=\vec{\mathbf{u}}=\lim_{n\to\infty}\mathbf{P}\vec{\mathbf{v}}_n=\mathbf{P}\left(\mathbf{FIXPT}[0,\mathbf{A},\mathbf{A}_1,\vec{\mathbf{l}}]\right), \tag{63}$$

where $\{\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}}$ *are obtained by Algorithm 2. After* n *iterations of Algorithm 2 one gets*

$$\|\mathbf{P}\vec{\mathbf{v}}_{n+1}-\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})}\le\varepsilon_0\frac{\varepsilon_0^n(\varepsilon_0-L\left(\frac{L}{\varepsilon_0}\right)^n)}{\varepsilon_0-L}+L^n\|\vec{\mathbf{u}}\|\le\varepsilon_0\frac{\left(\varepsilon_0^n(\varepsilon_0-L\left(\frac{L}{\varepsilon_0}\right)^n\right)}{\varepsilon_0-L}+L^n r^*. \tag{64}$$

Therefore, for $\varepsilon > 0$ one has

$$\|\vec{\mathbf{u}} - \mathbf{P}\left(\mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}_1, \vec{\mathbf{l}}]\right)\| \leq \varepsilon, \tag{65}$$

and the number N of iterations to achieve the accuracy $\varepsilon > 0$ is estimated by

$$N \sim -\log(\varepsilon). \tag{66}$$

Proof. First we want to show that $\{\mathbf{P}\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\boldsymbol{B}_{r^*}$. To do that, we use the estimation (61) and get

$$\begin{aligned}\|\mathbf{P}(\vec{\mathbf{v}}_{n+m} - \vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})} &\leq \sum_{k=n+1}^{n+m} \varepsilon_k + (1+L)\sum_{h=1}^{m}\sum_{k=3}^{n+m-h} \varepsilon_k L^{n+m-h-k} \\ &+ \left(\varepsilon_1 + L(\varepsilon_0 + \|(\mathbf{A}_{|\,\mathrm{ran}(\mathbf{A})})^{-1}\|\|\vec{\mathbf{l}}\|)\right)\sum_{k=n-1}^{n+m-2} L^k.\end{aligned}$$

A straight forward computation yields that the expression on the right in the estimate above goes to zero as n goes to infinity. Therefore, for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ large enough such that for all $m \in \mathbb{N}$, $m > n$ it holds that $\|\mathbf{P}(\vec{\mathbf{v}}_{n+m} - \vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})} \leq \varepsilon$. Due to Lemma 5.3 the sequence $\{\mathbf{P}\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}} \in \boldsymbol{B}_{r^*}$. Therefore, $\{\mathbf{P}\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\boldsymbol{B}_{r^*}$ and then (because $\boldsymbol{B}_{r^*}$ is a closed subset) there exists a unique $\vec{\mathbf{v}} \in \boldsymbol{B}_{r^*}$ such that $\vec{\mathbf{v}} = \lim_{n\to\infty} \mathbf{P}\vec{\mathbf{v}}_n$. Moreover, we have

$$\begin{aligned}\mathbf{P}\vec{\mathbf{v}}_{n+1} &= \mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_{n+1}, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A}_1(\vec{\mathbf{v}}_n)]\right) \\ &= \left(\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_{n+1}, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A}_1(\vec{\mathbf{v}}_n)]\right) - \mathbf{H}(\mathbf{P}\vec{\mathbf{v}}_n)\right) + (\mathbf{H}(\mathbf{P}\vec{\mathbf{v}}_n) - \mathbf{H}(\vec{\mathbf{u}})) + \vec{\mathbf{u}}.\end{aligned}$$

This implies that

$$\begin{aligned}\|\mathbf{P}\vec{\mathbf{v}}_{n+1} - \vec{\mathbf{u}}\| &= \|\mathbf{P}\left(\mathbf{SOLVE}[\varepsilon_{n+1}, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A}_1(\vec{\mathbf{v}}_n)]\right) - \mathbf{H}(\mathbf{P}\vec{\mathbf{v}}_n)\|_{\ell_2(\mathcal{N})} + \|\mathbf{H}(\mathbf{P}\vec{\mathbf{v}}_n) - \mathbf{H}(\vec{\mathbf{u}})\|_{\ell_2(\mathcal{N})} \\ &\leq \varepsilon_{n+1} + L\|\mathbf{P}\vec{\mathbf{v}}_n - \vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \leq \varepsilon_{n+1} + L(\varepsilon_n + L\|\mathbf{P}\vec{\mathbf{v}}_{n-1} - \vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})}) \\ &\leq \varepsilon_0 \sum_{k=0}^{n} \varepsilon_0^{n-k} L^k + L^n\|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \\ &= \varepsilon_0 \frac{(\varepsilon_0^{n+1} - L^{n+1})}{\varepsilon_0 - L} + L^n\|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \\ &= \varepsilon_0 \frac{\varepsilon_0^n\left(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n\right)}{\varepsilon_0 - L} + L^n\|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \to 0, \quad n \to \infty.\end{aligned}$$

Therefore, one has $\mathbf{H}(\vec{\mathbf{u}}) = \vec{\mathbf{u}} = \lim_{n\to\infty} \mathbf{P}\vec{\mathbf{v}}_n = \mathbf{P}\left(\mathbf{FIXPT}[0, \mathbf{A}, \mathbf{A}_1, \vec{\mathbf{l}}]\right)$. Moreover, since $\|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})} \leq r^*$, then the second inequality of (64) is valid and one has

$$\varepsilon_0 \frac{\varepsilon_0^n(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n)}{\varepsilon_0 - L} + L^n r^* \leq \varepsilon \tag{67}$$

if and only if

$$\varepsilon_n \leq \frac{\varepsilon_0 - L}{\varepsilon_0\left(\varepsilon_0 - L\left(\frac{L}{\varepsilon_0}\right)^n\right)}(\varepsilon - L^n r^*), \tag{68}$$

which is the criterion to exit the loop in Algorithm 2. This implies immediately (65). From (67) one shows that the necessary number of iterations of achieve accuracy $\varepsilon > 0$ is given by (66). ■

Similarly to the proof of formula (55) (see [13, Theorem 4.2]) one finally shows the following.

Corollary 5.5. *If* $\|\vec{\mathbf{l}}\| < (4\|(\mathbf{A}_{|\,\mathrm{ran}\,\mathbf{A}})^{-1}\|^2\|F\|^3\|A_1\|)^{-1}$ *and* $\mathcal{E}_n \leq r^*$ *for all* $n \in \mathbb{N}$*, then*

$$u = \sum_{n\in\mathcal{N}} \left(D_{\mathbf{V}}^{-1}\mathbf{FIXPT}[0, \mathbf{A}, \mathbf{A}_1, \vec{\mathbf{l}}]\right)_n f_n, \tag{69}$$

is a solution in $\mathbf{V}$ *of the fixed point problem* $H(u) = u$ *and one has that for* $\varepsilon > 0$

$$\|u - \sum_{n\in\mathcal{N}} \left(D_{\mathbf{V}}^{-1}\mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}_1, \vec{\mathbf{l}}]\right)_n f_n\|_{\mathbf{V}} \lesssim \varepsilon. \tag{70}$$

Proof. Since $\ker(F^* D_{\mathbf{V}}^{-1}) = \ker(\mathbf{A}) = \ker(\mathbf{P})$ and $F^*\tilde{F} = \mathrm{id}$, we finally verify

$$\begin{aligned} \|u - F^* D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}_\varepsilon\|_{\mathbf{V}} &= \left\|F^*(\tilde{F}u - D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}_\varepsilon)\right\|_{\mathbf{V}} \\ &= \left\|F^* D_{\mathbf{V}}^{-1}(\mathbf{P}\vec{\mathbf{u}} - \vec{\mathbf{u}}_\varepsilon)\right\|_{\mathbf{V}} \\ &= \left\|F^* D_{\mathbf{V}}^{-1}\mathbf{P}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_\varepsilon)\right\|_{\mathbf{V}} \\ &\leq \|F^*\|\|D_{\mathbf{V}}^{-1}\|\left\|\mathbf{P}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_\varepsilon)\right\|_{\ell_2(\mathcal{N})}. \end{aligned}$$

The claim follows from the above Theorem. ■

REMARK: The fact that at each iteration we compute the approximation up to a perturbation/tolerance ε_i also means that the scheme is stable. In other words, not only can ε_i be interpreted as the numerical approximation accuracy we achieve at each step, but also as the error tolerance we can afford, without spoiling convergence. Moreover, the scheme is *fully adaptive* in the sense that the iterations are enforced (by the use of suitable implementation of **COARSE**, see below) to work only with minimal number of relevant quantities (frame coefficients), in order to keep the prescribed accuracy-complexity balance.

6. Quasi-optimal complexity of the algorithm

In this section we present the complexity analysis of Algorithm 2. From Theorem 5.4, in particular from formula (66), we already know that to achieve a prescribed accuracy $\varepsilon > 0$ one needs to execute $N \sim -\log(\varepsilon)$ iterations. Therefore, having an estimation of the cost of each iteration, the asymptotic analysis of the complexity of the suggested adaptive scheme can be done. This is one of the very interesting theoretical advantages of the adaptive (wavelet) frame approach, together with the fact that one can prove both convergence and stability of the adaptive scheme as shown in the previous section.

Of course, the main ingredient of iterations in Algorithm 2 is the procedure **SOLVE**. As announced in Subsection 4.2.2, we discuss here an implementation of such a procedure and study its complexity. To this end, one should illustrate how the building block procedures **RHS**, **APPLY**, and **COARSE** can be implemented and estimate their computational cost. Therefore, in this section we focus on the main *properties and requirements* of these building blocks so that Algorithms 1 and 2 have certain complexity. We refer the interested reader to [9, 10, 13, 38] for the descriptions of procedures **RHS**, **APPLY**, and **COARSE** that fulfill the requirements stated below and needed for the complexity estimates. The complexity estimates for even more general algorithms than Algorithm 2 for linear and nonlinear variational problems, but under the more restrictive assumption that the discretizing frame $\mathcal{F}$ is a Riesz basis, have been given in [9, 10]. In order to describe the complexity in the more general case of pure frame discretizations a bit more (technical) effort and preparation is needed. We start by defining a so-called sparseness class $\mathcal{A}^s$ of vectors, $s \in \mathbb{R}_+$. $\mathcal{A}^s$ will turn out to be such that, if $\vec{\mathbf{u}} \in \mathcal{A}^s$, then the size of the support of $\vec{\mathbf{u}}_\varepsilon = \mathbf{FIXPT}[\varepsilon, \mathbf{A}, \mathbf{A}_1, \vec{\mathbf{l}}]$ and the computational cost for obtaining $\vec{\mathbf{u}}_\varepsilon$ can be estimated a priori.

An algorithm for computing a finite approximation $\vec{\mathbf{u}}_\varepsilon$ of $\vec{\mathbf{u}}$ up to ε, for $\vec{\mathbf{u}}$ given implicitly as a solution of some equation, is called *optimal* if $\#\,\mathrm{supp}(\vec{\mathbf{u}}_\varepsilon)$ (the number of elements of the support of $\vec{\mathbf{u}}_\varepsilon$) is not asymptotically larger (for $\varepsilon \to 0$) than the same quantity obtained by direct computation of any other approximation of $\vec{\mathbf{u}}$ using the same tolerance ε, for $\vec{\mathbf{u}}$ being given explicitly. In addition to this, the optimality is fully realized if the complexity to compute $\vec{\mathbf{u}}_\varepsilon$ does not exceed $\#\,\mathrm{supp}(\vec{\mathbf{u}}_\varepsilon)$ asymptotically (for $\varepsilon \to 0$). In other words, one does want the number of algebraic operations to be comparable to the size of what is being computed. For discretizations by means of Riesz bases optimal algorithms can be realized, see [8, 9, 10], for example. Analogous algorithms for frames may exhibit "arbitrarily small reductions" with respect to the expected optimality (see the following Theorem 6.1 for the precise statement). Since the techniques for estimating complexity we use are similar to those in [38, Theorem 3.12], we encounter similar difficulties to achieve the full optimality for **FIXPT** when dealing with pure frames. We call this situation *quasi-optimal*.

An optimal sparseness class is modeled as follows: For given $s > 0$ we define the space

$$\mathcal{A}^s_{weak} := \{\vec{\mathbf{c}} \in \ell_2(\mathcal{N}) : \ \|\vec{\mathbf{c}}\|_{\mathcal{A}^s_{weak}} := \sup_{n\in\mathbb{N}} n^{1/2+s}|\gamma_n(\vec{\mathbf{c}})| < \infty\}, \tag{71}$$

where $\gamma_n(\vec{\mathbf{c}})$ is the $n-$th largest coefficient in modulus of $\vec{\mathbf{c}}$. It turns out that $\|\cdot\|_{\mathcal{A}^s_{weak}}$ is a quasi–norm and we refer to [8, 19] for further details on the quasi–Banach spaces $\mathcal{A}^s_{weak}$. Such spaces can be usually found in literature under the notation ℓ^w_τ (weak-ℓ_τ) where $\tau = (1/2+s)^{-1} \in (0,2)$, and they are nothing but particular instances of Lorentz sequence spaces. Let us only mention that

$$\|\vec{\mathbf{c}}\|_{\mathcal{A}^s_{weak}} \sim \sup_{N\in\mathbb{N}} N^s \|\vec{\mathbf{c}} - \vec{\mathbf{c}}_N\|_{\ell_2(\mathcal{N})}, \tag{72}$$

where $\mathbf{c}_N$ is the best $N-$term approximation of $\vec{\mathbf{c}}$, i.e., the subsequence of $\vec{\mathbf{c}}$ consisting of the N largest coefficients in modulus of $\vec{\mathbf{c}}$. In particular, (72) implies that for all $\varepsilon > 0$ there exists $N_\varepsilon > 0$ large enough such that for all $\vec{\mathbf{c}}$ the best N_ε-term approximation $\vec{\mathbf{c}}_{N_\varepsilon}$ has the following properties

(i) $\|\vec{\mathbf{c}} - \vec{\mathbf{c}}_{N_\varepsilon}\|_{\ell_2(\mathcal{N})} \le \varepsilon$;
(ii) $\#\,\mathrm{supp}(\vec{\mathbf{c}}_{N_\varepsilon}) \lesssim \varepsilon^{-1/s}\|\vec{\mathbf{c}}\|^{1/s}_{\mathcal{A}^s_{weak}}$;
(iii) $\|\vec{\mathbf{c}}_{N_\varepsilon}\|_{\mathcal{A}^s_{weak}} \le \|\vec{\mathbf{c}}\|_{\mathcal{A}^s_{weak}}$.

Furthermore, from (72) one gets the following useful technical estimate

$$\|\vec{\mathbf{c}}\|_{\mathcal{A}^{\tilde{s}}_{weak}} \lesssim (\#\,\mathrm{supp}(\vec{\mathbf{c}}))^{\tilde{s}-s}\|\vec{\mathbf{c}}\|_{\mathcal{A}^s_{weak}} \tag{73}$$

whenever $0 < s < \tilde{s}$ and $\vec{\mathbf{c}}$ has finite support. This optimal class of vectors perfectly fits with complexity estimates for adaptive schemes for elliptic linear equations [8, 9, 13, 38]. For more general nonlinear problems a "weaker" version of the sparseness class has been introduced and denoted by $\mathcal{A}^s_{tree}$ in [10].

It is not known whether there exist frames for which the solutions of generic nonlinear equations, particularly for MHD equations, can have frame coefficients belonging to the sparseness classes described above. Therefore, we discuss here the requirements, fulfilled by $\mathcal{A}^s_{weak}$ and $\mathcal{A}^s_{tree}$, that a generic sparseness class should have to ensure quasi-optimality of our scheme. And, in case the solution belongs to any sparseness class with such properties, the algorithm will behave as ensured theoretically. The numerical tests in [1, 44] for turbulent flows motivate our assumption that the (wavelet) frame coefficients of the corresponding solutions do belong to some of these generic sparseness classes (i.e., only few significant wavelet coefficients can be expected to be relevant in the representation of the solution). Our conceptual approach is also motivated by the need to simplify the presentation of our complexity result without going into rather technical details of the properties of the particular instances $\mathcal{A}^s_{weak}$ and $\mathcal{A}^s_{tree}$ of $\mathcal{A}^s$. We refer the reader to [10] for more specific details.

For $s > 0$ and for a nondecreasing function $T : \mathbb{N} \to \mathbb{N}$ such that $N \lesssim T(N)$ we call any space $\mathcal{A}^s$ a *T-sparseness class* if $\mathcal{A}^{\tilde{s}}_{weak} \subset \mathcal{A}^s \subseteq \mathcal{A}^s_{weak}$ and $\mathcal{A}^{\tilde{s}} \subset \mathcal{A}^s$ for all $\tilde{s} > s$ and if for all $\vec{\mathbf{u}} \in \mathcal{A}^s$ and for all $\varepsilon > 0$ there exists a finite vector $\vec{\mathbf{u}}_\varepsilon$ with the properties

a) $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_\varepsilon\| \le \varepsilon$;
b) $T(\#\,\mathrm{supp}(\vec{\mathbf{u}}_\varepsilon)) \lesssim \varepsilon^{-1/s}\|\vec{\mathbf{u}}\|^{1/s}_{\mathcal{A}^s}$;
c) $\|\vec{\mathbf{u}}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\vec{\mathbf{u}}\|_{\mathcal{A}^s}$.

In particular we assume that there exists a constant $C_1(s)$ such that $\|\vec{\mathbf{u}}+\vec{\mathbf{v}}\|_{\mathcal{A}^s} \leq C_1(s)(\|\vec{\mathbf{u}}\|_{\mathcal{A}^s} + \|\vec{\mathbf{v}}\|_{\mathcal{A}^s})$. Of course, $\mathcal{A}^s_{weak}$ itself is a sparseness class with $T = I$. Moreover, there exist other T-sparseness classes different from $\mathcal{A}^s_{weak}$, for example, the class $\mathcal{A}^s_{tree}$ defined in [10, Formula (6.7)], that also turns out to be relevant in our context.

Note that, for $\tilde{\tilde{s}} > \tilde{s} > s > 0$, by the inclusions $\mathcal{A}^{\tilde{\tilde{s}}}_{weak} \subset \mathcal{A}^{\tilde{s}} \subset \mathcal{A}^s \subseteq \mathcal{A}^s_{weak}$ and (73) we have

$$\|\vec{\mathbf{c}}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\vec{\mathbf{c}}\|_{\mathcal{A}^{\tilde{\tilde{s}}}_{weak}} \lesssim (\#\operatorname{supp}(\vec{\mathbf{c}}))^{\tilde{\tilde{s}}-s}\|\vec{\mathbf{c}}\|_{\mathcal{A}^s_{weak}} \lesssim (T(\#\operatorname{supp}(\vec{\mathbf{c}})))^{\tilde{\tilde{s}}-s}\|\vec{\mathbf{c}}\|_{\mathcal{A}^s}, \tag{74}$$

for all finite vectors $\vec{\mathbf{c}}$.

Now we are ready to formulate our main conceptual requirements. For a fixed $s > 0$

(A1) Let $\theta < 1/3$. We assume that for any $\varepsilon > 0$, $\vec{\mathbf{v}} \in \mathcal{A}^s$ and any finitely supported $\vec{\mathbf{w}}$ such that

$$\|\vec{\mathbf{v}} - \vec{\mathbf{w}}\| \leq \theta\varepsilon,$$

for $\vec{\mathbf{w}}^* = \mathbf{COARSE}[(1-\theta)\varepsilon, \vec{\mathbf{w}}]$ it holds that

$$T(\#\operatorname{supp}(\vec{\mathbf{w}}^*)) \lesssim \varepsilon^{-1/s}\|\vec{\mathbf{v}}\|^{1/s}_{\mathcal{A}^s},$$

and

$$\|\vec{\mathbf{w}}^*\|_{\mathcal{A}^s} \leq C_2(s)\|\vec{\mathbf{v}}\|_{\mathcal{A}^s},$$

for some costant $C_s(s) > 0$. Moreover, we assume that the number of algebraic operation needed to compute $\vec{\mathbf{w}}^* := \mathbf{COARSE}[\varepsilon, \vec{\mathbf{w}}]$ for any finite vector $\vec{\mathbf{w}}$ can be estimated by $\lesssim T(\#\operatorname{supp}(\vec{\mathbf{w}})) + o(\varepsilon^{-1/s}\|\vec{\mathbf{w}}\|^{1/s}_{\mathcal{A}^s})$. See [38, Proposition 3.2] and [10, Proposition 6.3] for examples of procedures **COARSE** with such properties. As it will be clear in the proof of Theorem 6.1 the use of the procedure **COARSE** is fundamental to ensure that the supports of the iterates generated by the algorithm can be controlled. Due to the redundancy of the system used for the discretization, one can expect that the introduction of greedy algorithms of matching pursuit type [18, 25, 31, 40, 41] can potentially allow even higher rate of compression of the active coefficients with respect to the current implementations of **COARSE**. We postpone this investigation to forthcoming work.

(A2) The vector $\vec{\mathbf{w}}_\varepsilon := \mathbf{APPLY}[\varepsilon, \mathbf{A}, \vec{\mathbf{v}}]$ is such that $\|\vec{\mathbf{w}}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\vec{\mathbf{v}}\|_{\mathcal{A}^s}$, $T(\#\operatorname{supp}(\vec{\mathbf{w}}_\varepsilon)) \lesssim \varepsilon^{-1/s}\|\vec{\mathbf{v}}\|^{1/s}_{\mathcal{A}^s}$, and it is computed with a number of algebraic operations estimable by $\lesssim \varepsilon^{-1/s}\|\vec{\mathbf{v}}\|^{1/s}_{\mathcal{A}^s} + T(\#\operatorname{supp}(\vec{\mathbf{v}}))$. See [38, Proposition 3.8] and [10, Corollary 7.5] for examples of procedures **APPLY** with such properties.

(A3) One of the most crucial procedures of the iterative approximate fixed point scheme is the realization of $\vec{\mathbf{g}}^{(j)}_i := \mathbf{RHS}[\frac{\theta\varepsilon_j}{12\alpha K}, \vec{\ell} - \mathbf{A}_1(\vec{\mathbf{v}}_i)]$ for each step i and j of the outer and inner loops, respectively. In particular, it requires computing efficiently a finite approximation to $\mathbf{A}_1(\vec{\mathbf{v}}_i)$, where $\mathbf{A}_1$ is some nonlinear operator and $\vec{\mathbf{v}}_i$ is a given finite vector. In the pioneering work [10, 11] Cohen, Dahmen, and DeVore have found an effective way to solve this problem, at least for multiscale and wavelet expansions. Here we only mention their results and refer to [10, 11] for technical details (in particular see [10, Theorem 7.3 and Theorem 7.4]): there exists a procedure **RHS** such that $\|\vec{\mathbf{g}}^{(j)}_i - (\vec{\mathbf{l}} - \mathbf{A}_1(\vec{\mathbf{v}}_i))\|_{\ell_2} \leq \frac{\theta\epsilon_j}{12\alpha K}$, $T(\#\operatorname{supp}(\vec{\mathbf{g}}^{(j)}_i)) \lesssim \epsilon_j^{-1/s}\|\vec{\mathbf{v}}_i\|^{1/s}_{\mathcal{A}^s}$, $\|\vec{\mathbf{g}}^{(j)}_i\|_{\mathcal{A}^s} \lesssim 1 + \|\vec{\mathbf{v}}_i\|_{\mathcal{A}^s}$, and the number of algebraic operations needed to compute $\vec{\mathbf{g}}^{(j)}_i$ is bounded by $\lesssim \epsilon_j^{-1/s}\|\vec{\mathbf{v}}_i\|^{1/s}_{\mathcal{A}^s} + T(\#\operatorname{supp}(\vec{\mathbf{v}}_i))$.

In the following the subscript index i refers to the iterations in the outer loop of the fixed point iteration and the superscript j refers to the inner loop iterations in **SOLVE**. Moreover, ε_i and ϵ_j refer to the outer and inner loop tolerances, respectively. All estimations below hold asymptotically for $\varepsilon \to 0$ (ε as in Algorithm 2).

Theorem 6.1. *For $0 < s < \tilde{s} < \tilde{\tilde{s}}$ let $\mathcal{A}^{\tilde{s}}$ be a T-sparseness class and $\vec{\mathbf{u}} \in \mathcal{A}^s$, the solution of* (57) *as in Corollary 5.2. Assume that*

(i) *(A1)-(A3) hold for all $s \in (0, \tilde{s}]$;*
(ii) $\mathbf{P}$ *is bounded on $\mathcal{A}^t$ for all $t \in (0, \tilde{s}]$;*

(iii) *$K>0$ and $0<\theta<1/3$ in Algorithm 1 are chosen so that*

$$C_1(s)C_2(s)\|\operatorname{id}-\mathbf{P}\|(3\rho^K/\theta)^{\tilde{\tilde{s}}/s-1}<1. \tag{75}$$

Here the norm $\|\operatorname{id}-\mathbf{P}\|$ is the norm of $\operatorname{id}-\mathbf{P}$ as an operator on $\mathcal{A}^s$;

(iv) *the constants L,ε_0 and r^* satisfy*

$$L<\varepsilon_0<r^* \quad \text{and} \quad \rho^{-K}\frac{L}{\varepsilon_0-L}+\delta_0\le\frac{1}{2}, \quad \text{for some } \delta_0>0. \tag{76}$$

Then for any $\varepsilon>0$ and $\delta>0$ such that $\tilde{\tilde{s}}/\tilde{s}=1+\delta$, the finite vector $\vec{\mathbf{u}}_\varepsilon:=\mathbf{FIXPT}[\varepsilon,\mathbf{A},\mathbf{A}_1,\vec{\mathbf{l}}]$ satisfies

a) $\|\vec{\mathbf{u}}-\mathbf{P}\vec{\mathbf{u}}_\varepsilon\|_{\ell_2(\mathcal{N})}\le\varepsilon$;

b) $\#\operatorname{supp}(\vec{\mathbf{u}}_\varepsilon)\lesssim\varepsilon^{-(1+\delta)/s}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{(1+\delta)/s}$;

c) *the number of algebraic operations needed to compute $\vec{\mathbf{u}}_\varepsilon$ is $\lesssim\varepsilon^{-(1+\delta)/s}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{(1+\delta)/s}$.*

Proof. For the proof of part a) see Theorem 5.4. Next, we show part b). Assume that $\{\vec{\mathbf{v}}_i\}_{i\in\mathbb{N}_0}$ is the sequence of vectors generated in Algorithm 2. We want to show that $T(\#\operatorname{supp}(\vec{\mathbf{v}}_i))\lesssim\varepsilon_i^{-(1+\delta)/s}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{(1+\delta)/s}$ for i large enough. Since $\mathbf{P}$ is bounded on $\mathcal{A}^t$ for all $t\in(0,\tilde{s}]$, it is also bounded on $\mathcal{A}^s$. Therefore $\mathbf{P}\vec{\mathbf{u}}\in\mathcal{A}^s$. Then for $\epsilon_j>0$ there exists a finite vector $(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}$ such that $\|\mathbf{P}\vec{\mathbf{u}}-(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}\|_{\ell_2(\mathcal{N})}\le\frac{\theta}{6}\epsilon_j$ and $\#\operatorname{supp}((\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j})\lesssim T(\#\operatorname{supp}((\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}))\lesssim\epsilon_j^{-1/s}\|\mathbf{P}\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{1/s}\lesssim\epsilon_j^{-1/s}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{1/s}$. Therefore, by (74) we have

$$\epsilon_j^{\tilde{\tilde{s}}/s-1}\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}\|_{\mathcal{A}^{\tilde{s}}}\lesssim\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s-1}\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}\|_{\mathcal{A}^s}\lesssim\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s-1}\|\mathbf{P}\vec{\mathbf{u}}\|_{\mathcal{A}^s}\lesssim\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s-1}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}=\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s} \tag{77}$$

for any $\tilde{\tilde{s}}>\tilde{s}>s>0$. From (56) we get that

$$\left\|\mathbf{P}\vec{\mathbf{v}}_i^{ex}-(\operatorname{id}-\mathbf{P})\vec{\mathbf{v}}_i^{(j-1)}-\vec{\mathbf{v}}_i^{(j,K)}\right\|_{\ell_2(\mathcal{N})}\le\frac{2\theta\epsilon_j}{3} \tag{78}$$

with $\vec{\mathbf{v}}_i^{ex}:=\mathbf{H}(\vec{\mathbf{v}}_{i-1})$. Due to (76) and (64), we obtain for any i large enough and some $\delta_0>0$ that

$$\begin{aligned}
\|\mathbf{P}\vec{\mathbf{u}}-\mathbf{P}\vec{\mathbf{v}}_i^{ex}\|_{\ell_2(\mathcal{N})} &\le \|\mathbf{H}(\vec{\mathbf{u}})-\mathbf{H}(\vec{\mathbf{v}}_{i-1})\|_{\ell_2(\mathcal{N})}\\
&\le L\|\mathbf{P}\vec{\mathbf{u}}-\mathbf{P}\vec{\mathbf{v}}_{i-1}\|_{\ell_2(\mathcal{N})}\\
&\le L\left(\frac{\varepsilon_0^{i-1}(\varepsilon_0-L(L/\varepsilon_0)^{i-2})}{\varepsilon_0-L}+L^{i-2}r^*\right)\\
&= \frac{L}{\varepsilon_0}\left(\frac{\varepsilon_0^{i}(\varepsilon_0-L(L/\varepsilon_0)^{i-2})}{\varepsilon_0-L}+L^{i-2}\varepsilon_0 r^*\right)\\
&= \varepsilon_0^i\frac{L}{\varepsilon_0}\left(\frac{(\varepsilon_0-L(L/\varepsilon_0)^{i-2})}{\varepsilon_0-L}+\left(\frac{L}{\varepsilon_0}\right)^{i-2}\frac{r^*}{\varepsilon_0}\right)\\
&\le \varepsilon_0^i\left(\frac{L}{\varepsilon_0-L}+\delta_0\right).
\end{aligned}$$

Due to the stopping criterion of Algorithm 1 we have for all i that for the last $j-$th iteration $\varepsilon_0^i\le\frac{\theta}{3\rho^K}\epsilon_j$. Therefore, for all i large enough, due to (76) we get

$$\|\mathbf{P}\vec{\mathbf{u}}-\mathbf{P}\vec{\mathbf{v}}_i^{ex}\|_{\ell_2(\mathcal{N})}\le\frac{\theta}{6}\epsilon_j,$$

and

$$\left\|\mathbf{P}\vec{\mathbf{u}}-(\operatorname{id}-\mathbf{P})\vec{\mathbf{v}}_i^{(j-1)}-\vec{\mathbf{v}}_i^{(j,K)}\right\|_{\ell_2(\mathcal{N})}\le\frac{\theta\epsilon_j}{6}+\frac{2\theta\epsilon_j}{3}, \tag{79}$$

which implies that

$$\left\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}-(\operatorname{id}-\mathbf{P})\vec{\mathbf{v}}_i^{(j-1)}-\vec{\mathbf{v}}_i^{(j,K)}\right\|_{\ell_2(\mathcal{N})}\le\theta\epsilon_j. \tag{80}$$

Due to (A1), (80), and $\|\cdot\|_{\mathcal{A}^{\tilde{s}}}$ being a quasi-norm, it follows that $\vec{\mathbf{v}}_i^{(j)} := \mathbf{COARSE}[(1-\theta)\epsilon_j, \vec{\mathbf{v}}_i^{(j,K)}]$, for i large enough, satisfies

$$\begin{aligned}\|\vec{\mathbf{v}}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} &\leq C_2(s)\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j} - (\mathrm{id}-\mathbf{P})\vec{\mathbf{v}}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}} \\ &\leq C_1(s)C_2(s)\left(\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}\|_{\mathcal{A}^{\tilde{s}}} + \|\mathrm{id}-\mathbf{P}\|\|\vec{\mathbf{v}}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}\right),\end{aligned}$$

so by (77) and $\epsilon_j = 3\rho^K/\theta\epsilon_{j-1}$ (see Algorithm 1),

$$\left(\epsilon_j^{\tilde{\tilde{s}}/s-1}\|\vec{\mathbf{v}}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}}\right) \leq C'\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s} + C_1(s)C_2(s)\|\mathrm{id}-\mathbf{P}\|(3\rho^K/\theta)^{\tilde{\tilde{s}}/s-1}\left(\epsilon_{j-1}^{\tilde{\tilde{s}}/s-1}\|\vec{\mathbf{v}}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}\right).$$

We can conclude that for $K > 0$ large enough, and by the assumption (75), the solutions of the homogeneous part of this recursion converge to zero, and so

$$\epsilon_j^{\tilde{\tilde{s}}/s-1}\|\vec{\mathbf{v}}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s}, \tag{81}$$

uniformly with respect to j. And, due to (A1) we also have

$$\begin{aligned}\#\,\mathrm{supp}(\vec{\mathbf{v}}_i^{(j)}) &\lesssim T(\#\,\mathrm{supp}(\vec{\mathbf{v}}_i^{(j)})) \\ &\lesssim \epsilon_j^{-1/\tilde{s}}\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j} - (\mathrm{id}-\mathbf{P})\vec{\mathbf{v}}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}} \\ &\lesssim \epsilon_j^{-\frac{\tilde{\tilde{s}}/\tilde{s}}{s}}\left(\epsilon_j^{\frac{\tilde{\tilde{s}}}{s}-1}\left[\|(\mathbf{P}\vec{\mathbf{u}})_{\epsilon_j}\|_{\mathcal{A}^{\tilde{s}}} + \|\mathrm{id}-\mathbf{P}\|\|\vec{\mathbf{v}}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}}\right]\right)^{1/\tilde{s}}.\end{aligned}$$

Therefore, by (77) and (81) we get with $\tilde{\tilde{s}}/\tilde{s} = 1+\delta$ that

$$\#\,\mathrm{supp}(\vec{\mathbf{v}}_i^{(j)}) \lesssim T(\#\,\mathrm{supp}(\vec{\mathbf{v}}_i^{(j)})) \lesssim \epsilon_j^{-\frac{\tilde{\tilde{s}}/\tilde{s}}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{\tilde{\tilde{s}}/\tilde{s}}{s}} = \epsilon_j^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}. \tag{82}$$

Next, recall that $0 < s < \tilde{\tilde{s}}$ and by Algorithm 1 we have $\varepsilon_i \lesssim \epsilon_j$ for all j. Then (81) implies that for all j and i large enough we have $\varepsilon_i^{\tilde{\tilde{s}}/s-1}\|\vec{\mathbf{v}}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s}$. In particular, for i large enough,

$$\varepsilon_i^{\tilde{\tilde{s}}/s-1}\|\vec{\mathbf{v}}_i\|_{\mathcal{A}^{\tilde{s}}} \lesssim \|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\tilde{\tilde{s}}/s}. \tag{83}$$

By the same argument, (82) yields for i large enough

$$\#\,\mathrm{supp}(\vec{\mathbf{v}}_i) \lesssim T(\#\,\mathrm{supp}(\vec{\mathbf{v}}_i)) \lesssim \varepsilon_i^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}. \tag{84}$$

Note that the stopping criterion in Algorithm 2 implies that for large enough i we have $\varepsilon \lesssim \varepsilon_i$. Therefore, from (84) we also get that

$$\#\,\mathrm{supp}(\vec{\mathbf{u}}_\varepsilon) \lesssim T(\#\,\mathrm{supp}(\vec{\mathbf{u}}_\varepsilon)) \lesssim \varepsilon^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}. \tag{85}$$

To prove part c) it is sufficient to show that the number of algebraic operations needed for each iterations, i.e., for the computation of $\vec{\mathbf{v}}_i := \mathbf{SOLVE}[\varepsilon_i, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A}_1(\vec{\mathbf{v}}_{i-1})]$, is $\lesssim \varepsilon_i^{-(1+\delta)/s}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{(1+\delta)/s}$ for all i large enough. Note that for small i the number of algebraic operations needed is bounded by $C\varepsilon^{-(1+\delta)/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s}$ for $\varepsilon \to 0$.

Due to $L < 1$ and by (67), there exists $N \in \mathbb{N}$ such that $L^N r^* \leq \varepsilon$. Therefore, the $N-$th, exit iteration, satisfies $N := \left\lceil \frac{\log_{\varepsilon_0^{-1}}(\varepsilon/r^*)}{\log_{\varepsilon_0^{-1}}(L)} \right\rceil < \left\lceil -\log_{\varepsilon_0^{-1}}(\varepsilon) - \frac{\log_{\varepsilon_0^{-1}}(r^*)}{\log_{\varepsilon_0^{-1}}(L)} \right\rceil$. This implies that we can estimate the total number of operations by

$$\begin{aligned}
&\lesssim \sum_{i=0}^{\left[-\log_{\varepsilon_0^{-1}}(\varepsilon)-\frac{\log_{\varepsilon_0^{-1}}(r^*)}{\log_{\varepsilon_0^{-1}}(L)}\right]} \varepsilon_i^{-(1+\delta)/s}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{(1+\delta)/s} &=& \|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s} \sum_{i=0}^{\left[-\log_{\varepsilon_0^{-1}}(\varepsilon)-\frac{\log_{\varepsilon_0^{-1}}(r^*)}{\log_{\varepsilon_0^{-1}}(L)}\right]} \left(\varepsilon_0^{-(1+\delta)/s}\right)^i \\
& &=& \|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s} \frac{1-\left(\varepsilon_0^{-(1+\delta)/s}\right)^{[-\log_{\varepsilon_0^{-1}}(\varepsilon)-\frac{\log_{\varepsilon_0^{-1}}(r^*)}{\log_{\varepsilon_0^{-1}}(L)}]+1}}{1-\varepsilon_0^{-(1+\delta)/s}} \\
& &\lesssim& \varepsilon^{-(1+\delta)/s}\|\mathbf{u}\|_{\mathcal{A}^s}^{(1+\delta)/s}.
\end{aligned}$$

The last inequality is due to the fact that here we consider the asymptotic behavior for $\varepsilon \to 0$.

Therefore to conclude the proof, it is sufficient to estimate the complexity of **SOLVE**. To do so one can follow the argument used in the proof of [38, Theorem 3.12 (II)] replacing [38, (3.19)] by (81) and using the assumptions (A1)-(A3) when relevant. Due to assumption (A3) one has $T(\#\operatorname{supp}(\vec{\mathbf{g}}_{i-1}^{(j)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$ and $\|\vec{\mathbf{g}}_{i-1}^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$. Therefore by assumption (A2) one has $T(\#\operatorname{supp}(\vec{\mathbf{f}}_{i-1}^{(j)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$ and $\|\vec{\mathbf{f}}_{i-1}^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$. By induction assumption we have that $\|\vec{\mathbf{v}}_i^{(j-1)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$ and $T(\#\operatorname{supp}(\vec{\mathbf{v}}_i^{(j-1)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$, that are trivially valid for $j=1$. Therefore, again by (A2) one has

$$\|\vec{\mathbf{v}}_i^{(j,k)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}), \tag{86}$$

and

$$T(\#\operatorname{supp}(\vec{\mathbf{v}}_i^{(j,k)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}, \tag{87}$$

for all $0 \le k \le K$. Therefore, by (A2) and (A3), the number of algebraic operations required to compute $\vec{\mathbf{v}}_i^{(j,K)}$ is $\lesssim \epsilon_j^{-1/\tilde{s}}(1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}} + T(\#\operatorname{supp}(\vec{\mathbf{v}}_{i-1}))$. Finally by assumption (A1) the application of $\vec{\mathbf{v}}_i^{(j)} := \mathbf{COARSE}[(1-\theta)\epsilon_j, \vec{\mathbf{v}}_i^{(j,K)}]$ costs $\lesssim T(\#\operatorname{supp}(\vec{\mathbf{v}}_i^{(j,K)})) + o(\epsilon_j^{-1/\tilde{s}}\|\vec{\mathbf{v}}_i^{(j,K)}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}) \lesssim \epsilon_j^{-1/\tilde{s}}(1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}}$. Thus, by (A1), (86) and (87) the induction hypothesis holds, i.e. $\|\vec{\mathbf{v}}_i^{(j)}\|_{\mathcal{A}^{\tilde{s}}} \lesssim (1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})$ and $T(\#\operatorname{supp}(\vec{\mathbf{v}}_i^{(j)})) \lesssim \epsilon_j^{-1/\tilde{s}}\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}}^{1/\tilde{s}}$ for any $j \in \mathbb{N}_0$. This implies by induction that computing $\vec{\mathbf{v}}_i^{(j)}$ from $\vec{\mathbf{v}}_i^{(j-1)}$ takes a number of operations $\lesssim \epsilon_j^{-1/\tilde{s}}(1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}} + T(\#\operatorname{supp}(\vec{\mathbf{v}}_{i-1}))$. Since ϵ_j decreases geometrically and by formulas (83) and (84), one can estimate, as done before, the cost of the computation of $\vec{\mathbf{v}}_i := \mathbf{SOLVE}[\varepsilon_i, \mathbf{A}, \vec{\mathbf{l}} - \mathbf{A}_1(\vec{\mathbf{v}}_{i-1})]$ as a multiple of

$$\begin{aligned}
& \varepsilon_i^{-1/\tilde{s}}(1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}} + \log(\varepsilon_i)T(\#\operatorname{supp}(\vec{\mathbf{v}}_{i-1})) \\
\lesssim\ & \varepsilon_i^{-1/\tilde{s}}(1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})^{1/\tilde{s}} + \log(\varepsilon_i)\varepsilon_{i-1}^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}} \\
\lesssim\ & \varepsilon_i^{-\frac{\tilde{s}/\tilde{s}}{s}}\left(\varepsilon_i^{\frac{\tilde{s}}{s}-1}(1+\|\vec{\mathbf{v}}_{i-1}\|_{\mathcal{A}^{\tilde{s}}})\right)^{1/\tilde{s}} + \log(\varepsilon_i)\varepsilon_{i-1}^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}} \\
\lesssim\ & \varepsilon_i^{-\frac{\tilde{s}/\tilde{s}}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{\tilde{s}/\tilde{s}}{s}} + \log(\varepsilon_i)\varepsilon_{i-1}^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}} \\
\lesssim\ & \varepsilon_i^{-\frac{1+\delta}{s}}\|\vec{\mathbf{u}}\|_{\mathcal{A}^s}^{\frac{1+\delta}{s}}.
\end{aligned}$$

In the last inequality we could ignore the log factor because of the arbitrary choice of $\delta > 0$, small as one wants. This concludes the proof. ∎

REMARKS: 1. Theorem 6.1 ensures that our scheme is arbitrarily close to optimality, i.e., it is quasi-optimal;
2. According to [38, Remark 3.13] the condition that $\mathbf{P}$ is bounded on $\mathcal{A}^t$ for all $t \in (0, \tilde{s})$ is almost a necessary requirement. This condition has been verified numerically in [46] in case of wavelet

frame discretization, by observing the optimal convergence of **SOLVE**. One could avoid requiring the boundedness of $\mathbf{P}$ and obtain a fully optimal scheme replacing **SOLVE** by its modified version **modSOLVE** (see [38]) and assuming, without loss of generality, that the number of algebraic operations needed to compute $\vec{\mathbf{g}}_i^{(j)}$ in assumption (A3) is bounded only by $\lesssim \epsilon_j^{-1/s} \|\vec{\mathbf{v}}_i\|_{\mathcal{A}^s}^{1/s}$. Unfortunately, **modSOLVE** requires the construction of an implementable alternative projector $\mathbf{P}$ onto $\operatorname{ran}(\mathbf{A})$ and it is not an easy task. Nevertheless, if $\mathcal{F}$ is a Riesz basis then the scheme is certainly optimal and our result confirms that appearing in [10, Theorem 7.5].

3. Due to Theorem 5.1, (76) holds if the physical parameters of the MHD problem allow for choosing L, ε_0 and r^* such that

$$0 < L =: r^*\gamma < \varepsilon_0 < r^* < \gamma^{-1}$$

for $\gamma = \gamma(\mathcal{F}, A, A_1) := 2\|\mathbf{A}^{-1}|_{\operatorname{ran}(\mathbf{A})}\| \|A_1\| \|F\|^3$. In particular, if $0 < \gamma < 1$ is small enough then (76) is satisfied. We show next that, for the particular MHD equations and in case $\mathcal{F}$ is a Riesz basis, the dependence of γ on the viscosity η and the electric resistivity σ^{-1} can be expressed explicitly. Observe that

$$\begin{aligned}
\langle \mathbf{A}\vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle &= \langle (D_{\mathbf{V}}^*)^{-1} F A F^* D_{\mathbf{V}}^{-1} \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle \\
&= \langle A F^* D_{\mathbf{V}}^{-1} \vec{\mathbf{u}}, F^* D_{\mathbf{V}}^{-1} \vec{\mathbf{u}} \rangle \\
&\geq \alpha \|F^* D_{\mathbf{V}}^{-1} \vec{\mathbf{u}}\|_{\mathbf{V}} \\
&= \alpha \|\sum_n (D_{\mathbf{V}}^{-1} \vec{\mathbf{u}})_n f_n\|_{\mathbf{V}} \sim \alpha \|D_{\mathbf{V}}^{-1} \vec{\mathbf{u}}\|_{\mathbf{V}_d} = \alpha \|\vec{\mathbf{u}}\|_{\ell_2(\mathcal{N})}.
\end{aligned}$$

Therefore, if $\mathcal{F}$ is a Riesz basis, then $\operatorname{ran}(\mathbf{A}) = \ell_2(\mathcal{N})$, and

$$\|\mathbf{A}|_{\operatorname{ran}(\mathbf{A})}^{-1}\| = \|\mathbf{A}^{-1}\| \lesssim \alpha^{-1}, \tag{88}$$

where from (11) we have $\alpha := \Big(\alpha_0 - 2\|a_1\| \ \|(\mathbf{u}_0, \boldsymbol{J}_0)\|_{\mathbf{X}_{(\mathbf{u},\boldsymbol{J})}}\Big)$ and $\alpha_0 = c(\Omega)\min\{\eta, \sigma^{-1}\}$. This implies that if η, σ^{-1} are large enough then $\gamma < 1$ can be made sufficiently small. The norm equivalence $\|\sum_{n\in\mathcal{N}} (D_{\mathbf{V}}^{-1}\vec{\mathbf{u}})_n f_n\|_{\mathbf{V}} \sim \|D_{\mathbf{V}}^{-1}\vec{\mathbf{u}}\|_{\mathbf{V}_d}$ is valid only if $\mathcal{F}$ is a Riesz basis, and it cannot be extended to pure frames. It is anyway possible to show a similar estimate as (88) under the additional assumption on the frame $\mathcal{F}$ that $D_{\mathbf{V}}^{-1} \operatorname{ran}(\mathbf{A}) \subseteq \operatorname{ran}(\mathbf{F})$, which should be specifically verified for any particular frame under consideration.

7. Construction of aggregated wavelet frames

In this section we discuss the existence of suitable multiscale bases on bounded domains for the MHD problem described in Sections 2-3. These bases are in fact Gelfand frames for $(\mathbf{V}, \mathcal{H}, \mathbf{V}')$, which ensures the correct application of Algorithms 1 and 2. In particular, we show that for the special case $\Gamma = \Gamma_1 = \Sigma_1$ we could use the wavelet bases constructed using [44, Chapter 2, Definition 5] from a biorthogonal system on $L_2(\Omega)$ satisfying [44, Chapter 2, Assumption 3], where Ω is some open and bounded domain, see for example [14, 15, 16].

We start by studying the properties of the functions in $\mathbf{V}$. Using

$$\mathbf{V} := \{(\mathbf{v}, \boldsymbol{K}) \in \mathbf{X}_{(\mathbf{v},\boldsymbol{K})} : \ b((\mathbf{v}, \boldsymbol{K}), (q, \psi)) = 0 \text{ for all } (q, \psi) \in M_{(q,\psi)}\},$$

and setting $\psi = 0$ in the definition of the form b we obtain

$$\int_\Omega (\nabla \cdot \mathbf{v}) q = 0 \ \text{ for all } \ q \in \dot{L}_2(\Omega).$$

By [7, Lemma 1.a)] (telling us that the divergence operator maps the functions in $\mathbf{H}_\Gamma^1(\Omega)$ onto $\dot{L}_2(\Omega)$) we get that $\nabla \cdot \mathbf{v} = 0$. Thus, $\mathbf{v} \in \mathbf{V}(\operatorname{div}; \Omega)$ satisfying $\mathbf{v}|_\Gamma = 0$, where

$$\mathbf{V}(\operatorname{div}; \Omega) := \{\mathbf{v} \in \boldsymbol{L}_2(\Omega) : \ \nabla \cdot \mathbf{v} = 0\}.$$

Therefore, $\mathbf{v}$ is in $\mathbf{V}(\operatorname{div}; \Omega) \cap \mathbf{H}_0^1(\Omega)$. A detailed discussion of construction of a wavelet basis for such a space is given, for example, in [42, 43, 44].

Note that the application of ∇ to ψ yields the same results regardless if $\psi \in \dot{H}^1(\Omega)$ or $\psi \in H^1(\Omega) \setminus \mathbb{R}$. Thus, substituting $q = 0$ into the definition of the form b we obtain

$$\int_\Omega \boldsymbol{K}(\nabla\psi) = 0 \quad \text{for all} \quad \psi \in H^1(\Omega) \setminus \mathbb{R}. \tag{89}$$

Corollary 3.4. in [24] yields, for simply-connected Ω, that $\boldsymbol{K} = \nabla \times \xi$ with $\xi \in \mathbf{H}(\mathrm{curl};\Omega)$, where $\mathbf{H}(\mathrm{curl};\Omega) := \{\xi \in \boldsymbol{L}_2,\ \nabla \times \xi \in \boldsymbol{L}_2\}$. There is also another way to describe the space, to which the functions $\boldsymbol{K}$ belong.

Proposition 7.1. *Let $\boldsymbol{K}$ and ψ be in $\boldsymbol{L}_2(\Omega)$ and $\dot{H}^1(\Omega)$, respectively. Then*

$$\int_\Omega \boldsymbol{K}(\nabla\psi) = 0 \quad \textit{for all} \quad \psi \in \dot{H}^1(\Omega)$$

if and only if $\nabla \cdot \boldsymbol{K} = 0$ and $\boldsymbol{K} \cdot \mathbf{n}|_\Gamma = 0$.

Proof. "$\Longrightarrow$" Integration by parts yields

$$\int_\Omega (\nabla \cdot \boldsymbol{K})\psi = -\int_\Omega \boldsymbol{K}(\nabla\psi) + \int_\Gamma \boldsymbol{K} \cdot \mathbf{n}\psi \quad \text{for all } \psi \in H^1(\Omega) \setminus \mathbb{R}. \tag{90}$$

And, by the assumption together with (89) we get that (90) reduces to

$$\int_\Omega (\nabla \cdot \boldsymbol{K})\psi = 0 \quad \text{for all } \psi \in H^1_0(\Omega) \subset H^1(\Omega) \setminus \mathbb{R}.$$

This implies that $\nabla \cdot \boldsymbol{K} = 0$ due to $H^{-1}(\Omega)$ being a dual for H^1_0. Thus, (90) becomes

$$\int_\Gamma \boldsymbol{K} \cdot \mathbf{n}\psi = 0 \quad \text{for all } \psi \in H^1(\Omega) \setminus \mathbb{R},$$

which implies $\boldsymbol{K} \cdot \mathbf{n}|_\Gamma = 0$ as $\boldsymbol{K} \cdot \mathbf{n}|_\Gamma$ is in $H^{1/2}(\Gamma)'$, a dual for $H^{1/2}(\Gamma)$, and $\psi \in H^{1/2}(\Gamma)$. The claim follows. "$\Longleftarrow$" follows directly from (90). ■

In the notation of [44, (2.26)], by Proposition 7.1 $\boldsymbol{K} \in \mathbf{V}_0(\mathrm{div};\Omega) := \mathbf{H}_0(\mathrm{div};\Omega) \cap \mathbf{V}(\mathrm{div};\Omega)$, where by [24, Theorem 2.6]

$$\mathbf{H}_0(\mathrm{div};\Omega) = \{\boldsymbol{K} \in \boldsymbol{L}_2(\Omega) :\ \nabla \cdot \boldsymbol{K} \in \boldsymbol{L}_2(\Omega),\ \boldsymbol{K} \cdot \mathbf{n}|_\Gamma = 0\}.$$

Thus, for example, in case $\Omega = [0,1]^n$, the existence of a Riesz basis for $\mathbf{V}_0(\mathrm{div};\Omega)$ follows from [44, Theorem 10 and Proposition 5, p. 100]. Therefore, we are guaranteed that a wavelet frame needed for discretizing **Problem 3** exists, at least, for any domain which is an affine image of $[0,1]^n$ [44, p. 103]. Restricting our problem to 2D, we could even work with divergence-free wavelets on domains $\Omega \subset \mathbb{R}^2$ which are conformal images of $[0,1]^2$ [44, p. 104].

In a forthcoming work we want to discuss in more details the methods for obtaining pure frames for $\mathbf{V}$ on more general domains. The following two approaches are of interest:

- Modify the construction of pure frames for $\mathbf{H}^s(\Omega)$ given in [13] and done using the ODD (Overlapping Domain Decomposition) technique. In particular, using ODD we assume that $\{\Omega_i\}_{i=1}^M$, $M \in \mathbb{N}$, are overlapping subdomains (see [13] for a more precise statement) such that $\cup_{i=1}^M \Omega_i = \Omega$. Of course, such domains could be assumed affine images of the reference domain $[0,1]^n$ (see Figure 2). Moreover, we assume that for each $1 \le i \le M$ a divergence-free wavelet basis $\Psi_i := \{\vec{\psi}^i_{j,k}\}_{j \ge -1, k \in \mathcal{J}^i_j}$ is given for $\mathbf{V}(\mathrm{div};\Omega_i) \cap \mathbf{H}^1_0(\Omega_i)$. Show that $\Psi := \cup_{i=1}^N \Psi_i = \{\vec{\psi}^i_{j,k}\}_{j \ge -1, k \in \mathcal{J}^i_j, i=1,\dots,N}$ is a Gelfand frame for $\mathbf{V}(\mathrm{div};\Omega) \cap \mathbf{H}^1_0(\Omega)$.
- Given a wavelet Gelfand frame for $H^1(\Omega)$, possibly constructed by ODD, follow a similar strategy as in [42, 43, 44] and construct a divergence-free Gelfand frame for $\mathbf{V}(\mathrm{div};\Omega) \cap \mathbf{H}^1_0(\Omega)$.

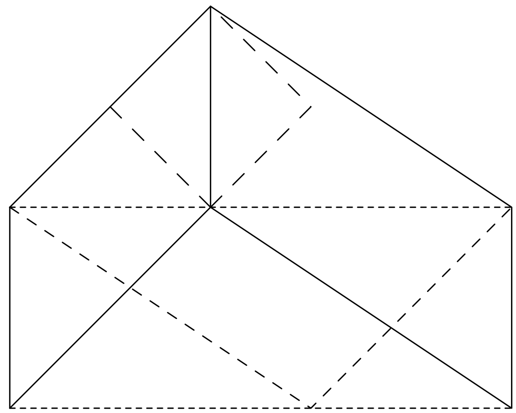

Figure 2. Example of Overlapping Domain Decomposition of a polygonal domain in 2D by means of patches which are affine images of squares.

These systems, called *aggregated divergence-free wavelet frames*, would allow us to avoid dealing with interfacing patches used in disjoint domain decompositions (DDD) (see [16, 44]). DDD are usually rather complicated to implement and can yield ill-conditioned systems (see [44, p. 104, sec. More General Domains]).

Moreover, in practice one never considers using the full frame $\Psi := \cup_{i=1}^M \Psi_i = \{\vec{\psi}^i_{j,k}\}_{j\geq -1, k\in\mathcal{J}^i_j, i=1,\dots,M}$, but rather $\Psi_J := \{\vec{\psi}^i_{j,k}\}_{-1\leq j\leq J, k\in\mathcal{J}^i_j, i=1,\dots,M}$ (for example, in the concrete implementation of the schemes discussed in [46], an upper bound in terms of concretely used scale levels j must be enforced for computational (memory/storage) limits, without spoiling the accuracy). Therefore, the first approach mentioned above is somewhat "ready-to-use". It is well-known that any finite system is already a (Gelfand) frame for its span. Using this fact one can consider solving **Problem 4** in $\mathbf{V}_J(\mathrm{div};\Omega) := \mathrm{span}(\Psi_J) \subseteq \mathbf{V}(\mathrm{div};\Omega) \cap \mathbf{H}^1_0(\Omega)$ for $J \geq 0$ large. The scheme we propose is well-defined in this case.

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