An "eulerian" approach to a class of matching problems

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Abstract. We study a card game called *He Loves Me, He Loves Me Not* ($(HLM)^2N$), which can be considered as a generalization of the classical games *Treize* and *Mousetrap*. We give some results by a theoretical point of view and by a numerical one, by means of Monte Carlo simulations. Furthermore, we introduce a new technique which allows us to obtain the best result at least for French card decks (52 cards with 4 seeds). This technique allows us to answer to some open questions related to the game *Mousetrap*.

1. Introduction

In some recent papers **[3], [18], [19], [28], [29], [30], [31], [43], [46]** new interest has been given to a class of *solitaires*, related to the classical game called *Treize*, introduced by Rémond de Montmort in 1708 **[16, pp. 130 − 131 , 142 − 143]**. The original game, known as Problem I, as translated from French on **[47]**, is the following:

"Pierre, having an entire deck composed of fifty-two cards shuffled at discretion, draws them one after the other, naming and pronouncing one when he draws the first card, two when he draws the second, three when he draws the third and thus in sequence, up to the thirteenth, which is a King. Now if in all this sequence of cards he has drawn none of them according to the rank that he has named them, he pays that which each of the players has wagered in the game and gives the hand to the one who follows him at the right. But if it happens to him, in the sequence of thirteen cards, to draw the card which he names, for example, to draw one ace at the time which he names one, or a two at the time which he names two, or a three at the time which he names three etc., he takes all that which is in the game and restarts as before, naming one, two, three etc. It is able to happen that Pierre, having won many times and restarting with one, has not enough cards in his hand in order to go up to thirteen, now he must, when the deck falls short to him, to shuffle the deck, to give to cut and next to draw from the entire deck the number of cards which is necessary to him in order to continue the game, by commencing with the one where he is stopped in the preceding hand. For example, if drawing the last card from them he has named seven, he must, in drawing the first card from the entire deck, after one has cut, to name eight and next nine etc. up to thirteen, unless he rather not win, in which case he would restart, naming first one, next two and the rest as it happens in

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the explanation. Whence it seems that Pierre is able to make many hands in sequence and likewise he is able to continue the game indefinitely."

The above described game (which I will quote as **T0**, in order to distinguish it from the successive versions) has not been studied for a long time, due to its difficulty. At page 298 of his book, de Montmort reports a remark by Jean Bernoulli:

"The four problems that you propose at the end of your treatise are interesting: but the first seems to me insoluble because of the length of the calculation that it would demand and that the human lifespan would not suffice to accomplish it".

Only in 1994 Doyle et al. [**19**], using Markovian chains and numerical tools, have found the answer to Problem I by de Montmort. Even if their paper does not clarify completely the mathematical tools to arrive at the solution, the answer seems correct, compared with Monte Carlo simulations [**3**].

A simpler version, which I will call **T1**, was introduced by de Montmort himself [**16, pp. 131 – 141**]:

"Pierre has a certain number of different cards which are not repeated at all and which are shuffled at discretion: he bets against Paul that if he draws them in sequence and if he names them according to the order of the cards, beginning of them either with the highest, or with the lowest, there will happen to him at least one time to draw the one that he will name. For example, Pierre having in his hand four cards, namely an ace, a deuce, a three and a four shuffled at discretion, bets that drawing them in sequence and naming one when he will draw the first, two when he will draw the second, three when he will draw the third, there will happen to him either to draw one ace when he will name one, or to draw a deuce when he will name two, or to draw a three when he will name three, or to draw a four when he will name four. Let be imagined the same thing for all other number of cards. One asks what is the strength or the expectation of Pierre for whatever number of cards that this may be from two up to thirteen" (from now on, we will indicate with "eleven" the "knave", with "twelve" the "queen" and with "thirteen" the "king").

This problem gave raise to the theory of *rencontres* (matchings), a classical part of Probability Theory and of Combinatorial Analysis (see for example [**14**], [**21**], [**48**]), which has been deeply studied by many authors. Just to recall the most important ones, let me cite Jean and Nicolas Bernoulli (see, for example, [**16, pp. 301 – 324**] and [**35, pp. 457 – 506**]), de Moivre [**15**], Euler [**20**], Borel [**6**], Fréchet [**24**].

As observed by Guy and Nowakowski (see [**29**] and references therein), these games belong to a more general theory on permutations with restricted position, which includes, besides the matching problems, the *problème des ménages* and the latin rectangles.

For a mathematical and historical survey of the progresses obtained on *Treize* up to the end of the XIX Century, see, for example, [**3**], [**13**], [**34**], [**47**], [**55**], [**56**] and references therein.

The case in which no coincidences occur belongs to the theory of *derangements*, with different and even curious names: *hat check*, *secretary's packet problem*, *misaddressed letters problem* etc. (see, for example, [**21**], [**48**]).

The problem $T1$ has been more and more complicated by several authors. In particular, Fréchet [**24**] studied in a very deep way the versions **T2**, already discussed by de Montmort and Bernoulli in [**16, p. 308**], where the deck is formed by $m \cdot s$ cards and s is the number of seeds:

"Any given number m of cards, being each repeated a certain number s of times, and taken promiscuously as it happens, draw m of them one after the other; to find the probability that of some of those sorts, some one card of each may be found in its place, and at the same time, that of some other sorts, no one card be found in its place" (see [**56, p. 156**]) and **T3**

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as particular cases of a much more general problem:

"There are two card decks of several suits (or ranks) (respectively $\nu_1, \nu_2, ..., \nu_m$ and $\rho_1, \rho_2, ..., \rho_m$), the decks not necessarily having the same number of cards in each suit, nor need the decks be of identical composition. One card is dealt from each deck, and if the cards at a drawing are of the same suit, a matching is said to occur".

Let us observe that the case of two decks of unequal numbers of cards n' and n'' (for example, $n' < n''$) is readily handled by substituting, for the smaller deck, one obtained by adding $n'' - n'$ "blank" cards, that is, cards of any type not already appearing in either deck and regarding these as an additional suit. Consequently, in this game it is possible to consider decks containing cards without any value. This last problem was already studied in special cases by Catalan [**9, p. 477 − 482**] as an extension of $T1$ and by Haag [**32, p. 670**] as an extension of $T2$.

Though other authors studied the problem before (see [**25**] for the first results and [**2**], [**3**], [**24**] for a review), Fréchet's monograph [**24**] - apart from all the original and brilliant contributions contained in it - is fundamental because the author perfectly systematizes all the results given by the previous authors and by himself, arriving at very elegant, general, formulas. In spite of the very fine results obtained, his monograph seems not so much known and many recent authors seem to ignore it.

The most relevant results obtained by Fréchet will be recalled in the next Section.

In some books on card *solitaires* (see, for example, [**8, p. 221**] and [**42, p. 54**]) it is proposed an advanced version of *Treize*, called in different ways (*Roll Call*, *Hit or Miss*, *Talkative*, *Harvest* etc.) and which we will call **T4**, where the player discards every matching card and, when he has drawn all the cards, he turns off the deck and continues the game, without shuffling:

"Deal cards one at time into one pile, called 'ace' for the first card, 'two' for the second and so on. After 'King', call the next card 'ace' and so through the pack. When a card proves to be of the rank you call, it is *hit*. Discard all hit cards from the pack. The object of play is to hit every card in the pack - eventually. Whenever the stock is exhausted, turn over, without shuffling, the waste-pile to form a new stock and continue counting from where you left off."

The game can be stopped following different rules: in [**42, p. 54**] we read:

"The game is construed lost if you go through the stock twice in succession without a hit. The point of this rule is that unlimited redealing would surely win the game unless a no-hit were encountered when the stock comprises 52, 39, 26 or 13 cards".

On the other hand, Carobene [**8, p. 221**] proposes the following counting: "ace", "king", "queen", "knave", down to "two" and allows the player to go through the stock up to 6 times without a hit, in order to give to the player appropriate chances to win.

Other possibilities consist in counting from "king" to "ace" and/or allowing the player to turn the deck as many times as he wants, concluding the game when he has stored all the cards or when in the residual deck the cards are in such a position that the player cannot store any card more and the game enters into a loop without hits.

Curiously, Morehead and Mott-Smith state that, following their rules, the probability of winning is more or less equal to 0.02, without any mathematical support. Actually, in **[3]** numerical results are given, showing that the probability of winning, in this case, is more or less equal to 0.002.

Let us remark that the game allows the player to store the matching cards and to turn over the deck without shuffling it.

This version of the game has not yet been studied by a mathematical point of view, apart from **[3]**, where mainly numerical results are attained. For a deeper insight to this game, see **[3]**.

On the other hand, in 1857 Cayley **[10]** proposed a game similar to *Treize*, called *Mousetrap*, played with a deck containing only one seed; here we report the description given in **[28, p. 237]**:

"Suppose that the numbers $1, 2, ..., n$ are written on cards, one to a card. After shuffling (permuting) the cards, start counting the deck from the top card down. If the number on the card does not equal the count, transfer the card to the bottom of the deck and continue counting. If the two are equal then set the card aside and start counting again from "one". The game is won if all the cards have been set aside, but lost if the count reaches $n + 1$."

The fundamental question posed by Cayley is the following **[11]**: "Find all the different arrangements of the cards, and inquire how many of these there are in which all or any given smaller number of the cards will be thrown out; and (in the several cases) in what orders the cards are thrown out."

Relatively few authors (in chronological order: **[11]**, **[54]**, **[28]**, **[30]**, **[43]**, **[29]**, **[31]**) have studied the problem, arriving, only recently **[29, 43]**, at partial results.

In **[28, p. 238]**, **[29]** and **[30]** Guy and Nowakowski consider another version of the game, called *Modular Mousetrap*, where, instead of stopping the game when no matching happens counting up to n, we start our counting again from "one", arriving to set aside every card or at a loop where no cards can be set aside anymore. Obviously, in this game, if n is prime, we have only two possibilities: *derangement* or winning deck. Guy and Nowakowski study only few cases ($n \leq 5$), while in the case $n = 6$ they consider only decks starting with an "ace".

The games are studied in the case of only one seed. Both versions of *Mousetrap* will be here generalized to the case of several decks (*"multiseed" Mousetrap*: $n = m \cdot s$).

Let us remark that the *"multiseed" Mousetrap* is introduced here for the first time. *Mousetrap* rules could be generalized at least in two different ways: when the player has counted up to m, without coming to a card which ought to be thrown out, he can

a) either stop the game (*Mousetrap-like rule*)

b) or eliminate the last m cards and continue his counting, restarting from "one", up to the exhaustion of the deck, when all the cards have been eliminated or stored.

We will choose the second option, that recalls a different *solitaire*, which will be considered in Section 3. It is not known in the mathematical literature, but, as told in **[3]** and in **[46]**, it has been studied for a relatively long time. It is commonly called *He Loves Me, He Loves Me Not* ($(HLM)^2N$) or *Montecarlo*:

"From a deck with s seeds and m ranks, deal one at a time all the cards into a pile, counting "one", "two", "three" etc. When a card whose value is k proves to be of the rank you call, it is *hit*. The card is thrown out and stored in another pile, the record is increased by k, the preceding $k - 1$ cards are put at the end of the deck, in the same order in which they were dealt and you start again to count "one", "two", "three" etc. If you have counted up to m without coming to a card has been thrown out, the last m cards are "burned", i.e. definitively discarded and you begin the count afresh, counting "one", "two", "three" etc. with the residual cards. When the number n_c of cards in the residual deck is less than m, the count can arrive, at most, at the value n_c. The game ends when you have stored and/or "burned" all the cards and there are no more cards in the deck. The record of the game is given by the sum of the face values of all the stored cards."

Aim of the game is to yield as the greatest record as possible.

It can be considered an intermediate game between $T0$, $T4$ and a *"multiseed" Mousetrap*: in fact,
a) the storage of every matching card in a record deck recalls $T4$ and *Mousetrap*;
b) differently from $T0$, when the deck falls short, it must be turned off without any shuffling, like in $T4$ and in *Mousetrap*;
c) differently from $T4$, after every matching, the player does not continue the counting but starts again from "one", i.e. with the same rules as in $T0$ and in *Mousetrap*;
d) if the number of residual cards n_c is less than m, the count cannot arrive at m, as in $T0$, $T4$ and *Mousetrap*, but at most at n_c.

Up to now, this game has been studied only by a numerical point of view, by means of Monte Carlo simulations, separately by Andrea Pompili **[46]** and by the author.

In this paper we introduce a new technique, which allows us to obtain the number of winning decks for many values of m and s, without any need of simulations, not only for $(HLM)^2N$, but also for *Mousetrap* and *Modular Mousetrap*, in their "multiseed" version, too. The technique has been implemented in a computer program. New results have been obtained in a very efficient way and many others could be reached, if the algorithm could be implemented in a parallel calculus framework.

The paper is divided into seven Sections.

In Section 2 we recall the most important results related to the games $T3$ and *Mousetrap*, quoting Fréchet's formulas and some approximation theorems for the probability distributions.

In Section 3 we consider the introductory notions of $(HLM)^2N$ and state two conjectures: a stronger one (SC) and a weaker one (WC), concerning the possibility to find at least one winning deck; other remarks and questions are posed.

In Section 4 some numerical results, concerning some questions posed in Section 3, are shown. These numerical results, based on Monte Carlo simulations, have given few results and, up to now, they were the unique method used to validate the two conjectures. In fact, only few theoretical results were, up to now, available.

In Section 5 a completely new method is shown, which is highly performing and which allows us not only to give a positive answer to (SC) at least up to a deck of French cards ($m = 13; s = 4$), but, for a large range of m and s, gives the exact number of winning decks, i.e. of deck giving the best reachable record. Thanks to the new method, an answer to the question of the number of winning decks at $(HLM)^2N$ is given, up to $s = 2$, $m = 14$; $s = 3$, $m = 10$; $s = 4$, $m = 7$.

In Section 6, adapting the new technique to the games *Mousetrap* and *Modular Mousetrap* and to their "multiseed" versions, the results are compared with those ones known in the literature. Thanks to the new method introduced in Section 5, an answer to the question of the number of winning decks is given, up to $s = 1$, $m = 15$; $s = 2$, $m = 8$; $s = 3$, $m = 6$; $s = 4$, $m = 5$ for *Mousetrap*; $s = 1$, $m = 13$; $s = 2$, $m = 7$; $s = 3$, $m = 5$; $s = 4$, $m = 4$ for *Modular Mousetrap*.

In Section 7 a short review of open problems and perspectives is shown.

2. Preliminary results on *Treize* and *Mousetrap*

At p. 82 of their monograph **[1]**, Blom, Holst and Jansen describe the game $T3$, where all the $n = m \cdot s$ cards are dealt, counting s times "one", "two", ... "m", without starting the count afresh, from "one", after every matching, but continuing the count. Aim of the game is to obtain as many matchings as possible.

As far as the author knows, the game appeared for the first time in a mathematical journal in 1844, when Fodot proposed it on *Nouvelle Annales de Mathematique* **[22]**.

Studying a more general problem, Lindelöf **[39], [40]**, yielded the formula for the probability of *derangements* $P_{T3}(0)$ in a form which is not so easily readable.

Let us remark that, in the case of several seeds, we have to speak about *rank-derangement*, as suggested by Doyle, Grinstead e Laurie Snell **[18]**, because a deck obtained from another one only exchanging the position between cards of the same rank is, playing $T3$, identical to the original.

Consequently, the number of different decks, in $T3$ as in all the "multiseed" games we will consider in this paper, is given by

$$N_{m \cdot s} = \frac{(m \cdot s)!}{(s!)^m} \ . \tag{2.1}$$

$T3$ and the general problem considered by Lindelöf are today commonly known as *frustration solitaires*, because the winning probability - where the player wins if he obtains a *derangement* - is very low. The reader can find on **[18]** the curious story of this game.

After a long sequence of papers in the first half of the XX Century, considering more and more general games (see **[2]**, **[3]** and **[24]** for a review), Fréchet published the monograph **[24]** where at pp. 173 – 176 he obtained many interesting and fine results for $T1$, $T2$, $T3$, *frustration solitaire* and many other problems as particular cases of a very general theory (see at pp. 162 – 176).

Curiously, while Riordan **[48]** cites Fréchet's monograph, neither **[1]**, nor **[18]**, where the games are introduced and discussed, cite it.

In his monograph, the author prefers to speak about labelled balls and compartments **[24, pp. 162 – 163]**. Here we adapt the problem to a card game.

"Let us consider a deck with n' cards, ν_1 assuming value 1, ν_2 assuming value 2, ... ν_m assuming value m $\left(\sum_{k=1}^{m} \nu_k = n'\right)$ and n'' compartments, ρ_1 assuming value 1, ρ_2 assuming value 2, ... , ρ_m assuming value m $\left(\sum_{k=1}^{m} \rho_k = n''\right)$. In the most general version of the game, $\rho_i \neq \rho_j$ and $\nu_i \neq \nu_j$ if $i \neq j$; $\rho_i \neq \nu_i$ and $n' \neq n''$.

Let us deal p cards ($p \leq n = \min\{n' \ , \ n''\}$) and arrange them in a p-ple $(j_1, j_2, ..., j_p)$ of compartments. We have a matching when a card is arranged in a compartment having its same value."

Actually, the problem studied by Fréchet is even more general, including the possibility to have "blank" cards or compartment, i.e. cards or compartments without any value. As already remarked, in this case we can considered the blank as another value.

For the sake of simplicity and since the games we are considering in this paper do not provide for blank cards, we will ignore this possibility. However, the reader can examine the monograph **[24]** for a deeper insight.

As already underlined in the Introduction, the game can be described in terms of comparison of two decks. Let us observe that, in both the descriptions, the role of deck and of compartment (or, analogously, the role between the two different decks) is exchangeable. In order to reproduce Fréchet's formulas, from now on we will suppose $n' \geq n''$. Consequently, we will always have $n = n''$.

Let us indicate with $a_k \geq 0$ the number of cards arranged in a compartment with value k. Since the number of these compartments is ρ_k, we have, obviously, $a_k \leq \rho_k \quad \forall k = 1, ..., m$. Clearly, if we arrange $p = n$ cards, we have $a_k = \rho_k \quad \forall k$. Moreover, $\sum_{i=0}^{m} a_i = p \leq n$. When exactly p matchings occur, we have $a_i \leq \nu_i \quad \forall i = 1, ..., m$. In this case we have $a_i \leq \delta_i \ := \min\{\nu_i, \rho_i\}$, for every $i = 1, ..., m$. Consequently

$$p \leq \sum_{k=1}^{m} \delta_k =: \Delta \leq n \ .$$

In other words, Δ represents the maximal number of matchings.

We will indicate with W the number of matchings; with $P(W = r) =: P(r)$ the probability distribution; with $\mathbb{E}(W) = \mu$ its mean value or expectation and with $\mu_2 = \sigma^2 = \mathbb{E}((W - \mu)^2)$ its variance.

We now quote the most important results, where the formulas are surprisingly very elegant.

Theorem 2.1. ([24, pp. 168 – 169])

$$P(r) = \sum_{k=r}^{\Delta} (-1)^{k-r} \binom{k}{r} \frac{(n'-k)!}{n'!} \left\{ \sum_{\sum a_j = k} \left[\prod_{i=1}^{m} a_i! \binom{\nu_i}{a_i} \binom{\rho_i}{a_i} \right] \right\} \tag{2.2}$$

$$P(W \geq r) = \sum_{k=r}^{\Delta} (-1)^{k-r} \binom{k-1}{r-1} \frac{(n'-k)!}{n'!} \left\{ \sum_{\sum a_j = k} \left[\prod_{i=1}^{m} a_i! \binom{\nu_i}{a_i} \binom{\rho_i}{a_i} \right] \right\} \tag{2.3}$$

$$\max\{\min \rho_k, \min \nu_k\} \leq \mathbb{E}(W) =: \mu = \frac{\sum \rho_k \cdot \nu_k}{n'} \leq \min\{\max \rho_k, \max \nu_k\}$$

$$\begin{aligned} \sigma^2 &= \frac{1}{n'-1} \left\{ \sum_{t=1}^{m} \rho_t \nu_t + \frac{1}{(n')^2} \left(\sum_{t=1}^{m} \rho_t \nu_t \right)^2 - \frac{1}{n'} \sum_{t=1}^{m} \rho_t \nu_t (\rho_t + \nu_t) \right\} = \\ &= \mu - \frac{1}{(n'-1)} \left[\frac{1}{n'} \sum_{t=1}^{m} \rho_t \cdot \nu_t \cdot (\rho_t + \nu_t) - \mu^2 - \mu \right] \ . \end{aligned} \tag{2.4}$$

Corollary 2.2.([5, pp. 90 – 91], [15, pp. 109 – 117, Problème XXXVI], [16, pp. 315, 323], [18], [24, pp. 174 – 176], [38, p. 222], [55, p. 233])

a) $T2 : \rho_i = 1 \ ; \ \nu_i = s \quad \forall i = 1, ..., m \ ; \ p = \Delta = m \ ; \ n' = n'' = n = m \cdot s.$

$$\begin{aligned} P_{m \cdot s}(r) &= \sum_{k=r}^{m} (-1)^{k-r} \binom{k}{r} \frac{(ms-k)!}{(ms)!} \binom{m}{k} s^k = \\ &\frac{1}{r!} \sum_{k=r}^{m} (-1)^{k-r} \frac{1}{(k-r)!} \cdot \frac{(ms-k)!}{(ms)!} \cdot \frac{m!}{(m-k)!} \ s^k \end{aligned} \tag{2.5}$$

$$\mathbb{E}(W) = \frac{ms}{ms} = 1 \quad ; \quad \sigma^2 = 1 - \frac{s-1}{n-1} = \frac{s(m-1)}{ms-1}$$

b) $T3 : \rho_i = \nu_i = s \quad \forall i = 1, ..., m \; ; \; p = \Delta = ms = n' = n'' = n.$

$$P_{m\cdot s}(r) = \sum_{k=r}^{ms} (-1)^{k-r} \binom{k}{r} \frac{(ms-k)!}{(ms)!} \sum_{\sum a_j = k} \left[\prod_{j=1}^{m} a_j! \binom{s}{a_j}^2 \right] \; ; \tag{2.6}$$

$$P_{m\cdot s}(0) = \sum_{k=0}^{ms} (-1)^{k} \frac{(ms-k)!}{(ms)!} \sum_{\sum a_j = k} \left[\prod_{j=1}^{m} a_j! \binom{s}{a_j}^2 \right] \; ; \tag{2.7}$$

$$\mathbb{E}(W) = s \quad ; \quad \sigma^2 = \frac{(m-1)s^2}{ms-1} = \frac{n-s}{n-1}\, s \; . \tag{2.8}$$

Formula (2.7) represents the probability of *derangement* in the game $T3$. This formula has been reobtained by Doyle et al. in **[18]**, using different techniques.

In **[44]**, Olds shows a method, based on *permanents* of a matrix, to compute the central moments of the probability distribution in $T3$. He computes directly only the first four; the first two coincide with Fréchet's results, while

$$\begin{aligned} \mu_3 &= \frac{s(n-s)(n-2s)}{(n-1)(n-2)} \\ \mu_4 &= \frac{s(n-s)}{(n-1)(n-2)(n-3)}\{(n-2s)(n-3s)(3s+1) + (s-1)(12ns - n - 18s^2 - 6s)\} \end{aligned} \; . \tag{2.9}$$

Let us observe that, for $s = 1$ ($T1$), formulas (2.8) and (2.9) yield

$$\mathbb{E}(W) = 1 \quad ; \quad \mu_2 = 1 \quad ; \quad \mu_3 = 1 \quad ; \quad \mu_4 = 4 \; ,$$

similarly to **[24, p. 141]**.

Corollary 2.3. *The probabilities of the maximal number of matchings are*

$$\textbf{a)}\; T2 \; : \quad P_{m\cdot s}(m) = \frac{s^m}{\frac{1}{m!}\cdot \binom{n}{m}} \; ;$$

$$\textbf{b)}\; T3 \; : \quad P_{m\cdot s}(ms) = \frac{1}{N_{m\cdot s}} = \frac{(s!)^m}{(m\cdot s)!} \; . \tag{2.10}$$

Just as an example, let us compute the probabilities of *derangement* and of maximal number of matchings for $T2$ and $T3$ in the case of French cards ($m = 13 \; , \; s = 4$).

$T2: \; P_{13\cdot 4}(0) \sim 0.356935 \quad ; \quad P_{13\cdot 4}(13) \sim 1.697 \cdot 10^{-14}$;
$T3: \; P_{13\cdot 4}(0) \sim 0.016233 \quad ; \quad P_{13\cdot 4}(52) \sim 1.087 \cdot 10^{-50}$;

Remark 2.4. Recalling that $T1$ is a particular case of $T3$, where $s = 1$, we can obtain $P_{T1}(0)$ and $P_{T1}(r)$ from formulas (2.7) and (2.6). The first was already obtained by Jean Bernoulli in 1710 **[16, p. 290]**, who yielded his very famous formula

$$P_{T1,m}(0) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \; . \tag{2.11}$$

The second coincides with the formula yielded in 1837 by Catalan **[9, p. 474]**

$$P_{T1,m}(r) = \frac{1}{r!} \sum_{j=0}^{m-r} \frac{(-1)^j}{j!} \qquad\qquad r = 0, ..., m \ . \tag{2.12}$$

It is noteworthy the fact that, already in 1711, Nicolas Bernoulli **[16, pp. 323 – 324]** obtained for the first time the value $P_{13\cdot 4}(0)$ for $T2$. The technique used by the Bernoullis, today known as *inclusion-exclusion principle* (see, for example, **[7], [21], [48]**), was really innovative in his time.

Clearly, the formulas obtained by Fréchet, though very elegant, are not so simple to be handled, first of all if we are interested only on the order of the values of the probabilities.

Thus it is natural to ask for an approximation of the probability distributions (2.5) and (2.6) with more easily readable distributions. Already Laplace **[38, pp. 222 - 223]** observed, for example, that, in the game $T2$, when m (and, consequently, n) grows, the *derangement* probability is asymptotically equivalent to $\left(1 - \frac{1}{m}\right)^m$; thus

$$\lim_{m \to \infty} P_{T2}(0) = e^{-1} \ .$$

Other authors (see, for example, **[6, pp. 61 – 68]**, **[24, pp. 143 – 144]** and **[37]**) have dealt with the problem of the best approximation, comparing in particular the probability distribution with the poissonian and the binomial. For a review of the most important results, see, for example, **[1]**, **[33]**, **[58]**.

It is well known that the probability to yield k successes in n independent, equiprobable trials, with winning probability p, is given by the binomial distribution (by Bernoulli) Bi:

$$P_n(W = k) = Bi(n, p)(k) = \binom{n}{k} p^k (1 - p)^{n-k} \ .$$

The mean value and the variance of this distribution are respectively

$$\mathbb{E}(W) = \sum_{k=1}^{n} k \cdot Bi(n, p)(k) = np \quad ; \quad \sigma^2 = np(1 - p) \ .$$

On the other hand, the *poissonian distribution Po* is defined by the formula

$$P_n(W = k) = Po_\lambda(k) = \frac{(\lambda)^k}{k!} \ e^{-\lambda}$$

and represents the probability that a random variable W takes on the value k. The letter λ is the unique characteristic parameter of the distribution. The mean value and the variance of this distribution are

$$\mathbb{E}(W) = \sigma^2 = \lambda \ .$$

If in the binomial distribution the mean value np is fixed and $n \to \infty$, we obtain the well-known *Rare Events (or Poisson's) Law* (see, for example, **[21, pp. 142 – 143]**):

Theorem 2.5. *(Poisson's Theorem) If k and $\lambda = n \cdot p$ are fixed,*

$$\lim_{n \to \infty} Bi(n, p)(k) = \frac{(np)^k \cdot e^{-np}}{k!} = Po_{np}(k) \ .$$

More precisely

Theorem 2.6. ([1, p.1])

$$Bi(n,p)(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{(np)^k}{k!} e^{-np} \left[1 + \mathcal{O}\left(np^2, \frac{k^2}{n} \right) \right] \qquad \forall k \in I\!N \ .$$

From 1837, when the paper by Poisson **[45]** appeared, the *Rare Events Law* has been proved to be valid in much more general situations, where the trials are neither independent nor equiprobable, provided the number n is "sufficiently" large and p "sufficiently" small. In **[1], [45], [57], [58]** it is possible to find the history of the *poissonian approximation theory*, up to the most recent results, together with a remarkable bibliography.

The matching problems $T1$, $T2$ and $T3$ can be treated by means of the poissonian approximation, as shown in **[1, pp. 80 − 83]** and in the paper **[37]** by Lanke, where the author studies the following problem:

"Let A and B two decks of cards; let A have ν_1 cards of type "one", .., ν_m cards of type m and let $\rho_1, ..., \rho_m$ be the corresponding numbers for B. Further, let n the size of A which is supposed to equal that of B:

$$n = \sum_{i=1}^{m} \nu_i \ = \ \sum_{i=1}^{m} \rho_i \ .$$

Now the two decks are matched at random, i.e. each card from A is put into correspondence with one card from B. The n pairs so obtained are examined for matches where a pair is said to give a match if the two cards in the pair are of the same type."

Lanke studied the behavior of the probability distribution of matches when m tends to infinity.

Theorem 2.7. ([37]) *Suppose that* $\lim_{m\to\infty} \frac{1}{n} \sum_{i=1}^{m} \nu_i \cdot \rho_i = \lambda \in I\!R$*; then the following conditions are equivalent:*

1) *W asymptotically has a Poisson distribution with parameter λ;*

2)

$$\lim_{m\to\infty} \frac{1}{n^2} \sum_{i=1}^{m} \nu_i \cdot \rho_i \cdot (\nu_i + \rho_i) = 0 \ ; \tag{2.13}$$

3)

$$\lim_{m\to\infty} \frac{1}{n} \max_i (\nu_i + \rho_i) = 0 \quad \textit{provided} \quad \nu_i \cdot \rho_i > 0 \ ; \tag{2.14}$$

4)

$$\lim_{m\to\infty} \sigma^2 = \lambda \ . \tag{2.15}$$

Corollary 2.8.([1, p. 82], [37]) *For m tending to infinity, the probability distribution $P(W)$ in the game $T3$ asymptotically has a Poisson distribution with parameter $\lambda = s$.*

Proof It is sufficient to apply formulas (2.13), (2.14) and (2.15) to the case $\nu_i = \rho_i = s\ ,\ i = 1, ..., m\ ;\ n = ms$:

$$\sigma^2 = s - \frac{s(s-1)}{(n-1)} = \frac{s^2(m-1)}{sm-1} \qquad ; \qquad \lim_{m\to\infty} \sigma^2 = s\ ;$$

$$\frac{1}{n^2}\sum_{i=1}^{m} \nu_i \cdot \rho_i \cdot (\nu_i + \rho_i) = \frac{2s}{m} \quad ; \quad \lim_{m\to\infty} \frac{1}{n^2}\sum_{i=1}^{m} \nu_i \cdot \rho_i \cdot (\nu_i + \rho_i) = 0\ ;$$

$$\frac{1}{n}\max_i(\nu_i + \rho_i) = \frac{2}{m} \quad ; \quad \lim_{m\to\infty} \frac{1}{n}\max_i(\nu_i + \rho_i) = 0\ .$$

□

In **[57, pp. 192 − 197]** von Mises shows other necessary and sufficient conditions for the Poisson approximation, in terms of factorial moments and applies his theory to the game $T1$. We refer to von Mises' monograph for further information.

In order to give a quantitative idea of how much good an approximation of a discrete probability distribution is, we refer to the *total variation distance*.

Definition 2.9. ([1, pp. 1, 253 − 254]) Given any subset $A \subseteq I\!N$ and two discrete probability distributions λ and μ on A, the *total variation distance* between λ and μ is defined by

$$d_{TV}(\lambda, \mu) := \sup_{A\subseteq I\!N} |\lambda(A) - \mu(A)| \ = \ \frac{1}{2}\sum_{j\in A} |\lambda(j) - \mu(j)|\ .$$

In their book, Barbour et al. obtain the following estimates for the game studied by Lanke.

Proposition 2.10.([1, pp. 82 − 83])

$$\sigma^2 = \lambda - \frac{1}{(n-1)}\left[\frac{1}{n}\sum_{i=1}^{m} \nu_i \cdot \rho_i \cdot (\nu_i + \rho_i) - \lambda^2 - \lambda\right] \ \leq \ \lambda\ ;$$

$$d_{TV}(P, Po_\lambda) \ \leq \ \left(1 - e^{-\lambda}\right)\left[\frac{n-2}{\lambda n(n-1)}\left(\frac{1}{n}\sum_{i=1}^{m} \nu_i \cdot \rho_i \cdot (\nu_i + \rho_i) - \lambda^2 - \lambda\right) + \frac{2\lambda}{n}\right]\ . \tag{2.16}$$

Corollary 2.11.([1, pp. 81 − 82])

In the game $T2$

$$d_{TV}\ (P\ ,\ Po_1) \ \leq \ (1 - e^{-1}) \cdot \left[\frac{(s+1)n - 2s}{n(n-1)}\right] \ \leq \ (1 - e^{-1}) \cdot \frac{(s+1)}{n}\ .$$

In the game $T3$

$$d_{TV}\ (P\ ,\ Po_s) \ \leq \ (1 - e^{-s}) \cdot \left[\frac{3s-1}{n} - \frac{(s-1)(2s-1)}{n(n-1)}\right]\ .$$

As already remarked by Borel [**6, pp. 61 − 66**] and Fréchet [**24, pp. 142 − 144**] for $T1$ and by Barbour et al. [**1, pp. 80 − 83**] for $T2$ and $T3$, the binomial distribution seems to approximate the probability distribution even better than the poissonian. Borel interpreted this fact observing that, for $n = m$ tending to infinity, the number of cards is so high that every card drawing can be considered approximately an event independent from the other ones and, consequently, the binomial distribution can be considered a good approximation for large decks.

Fréchet has proved this fact for the game $T1$. Extending Borel's ideas to $T2$ and $T3$, we can expect that $Bi\left(m \cdot s, \frac{1}{m}\right)$ approximates well P_{T3}. Numerical computations confirm that

$$d_{TV}\left(P_{T3,13\cdot 4}, Bi\left(52, \frac{1}{13}\right)\right) \leq d_{TV}(P_{T3,13\cdot 4}, Po_4)\ .$$

In Tables 1 and 2 it is possible to compare the values of the three distributions respectively for the games $T2$ and $T3$ for a French card deck ($m = 13\ ,\ s = 4$). It is evident that, already for $m = 13$, the two approximations are sufficiently good.

Curiously, in their monograph [**1, p.81**], Barbour et al. state that, by means of numerical computations, it is possible to show that the binomial distribution giving the best approximation in the game $T2$ in the case ($m = 13\ ,\ s = 4$) is $Bi\left(17, \frac{1}{17}\right)$. Actually, our computations show not only that

$$d_{TV}\left(P, Bi\left(13, \frac{1}{13}\right)\right)\ \sim\ 0.00564\ <\ d_{TV}\left(P, Bi\left(17, \frac{1}{17}\right)\right)\ \sim\ 0.137\ ,$$

but also that $Bi\left(\frac{1}{13}, 13\right)$ is the unique binomial distribution having a total variation distance from the probability distribution less than the Po_1.

In general, numerical results seem to confirm Borel's idea, because the best approximation in terms of binomial distribution are obtained with $Bi\left(m, \frac{1}{m}\right)$ for $T2$ and with $Bi\left(m \cdot s, \frac{1}{m}\right)$ for $T3$, for every m and s we have examined.

Concerning *Mousetrap*, there are few results, obtained, in particular, by Steen [**54**], already in 1878 and, much more recently, by Guy and Nowakowski [**29**] and Mundfrom [**43**]. Steen calculated, for any n, the number $a_{n,i}$ of permutations that have $i\ ,\ 1 \leq i \leq n$, as the first card set aside; the numbers $b_{n,i}$ and $c_{n,i}$ of permutations that have "one" (respectively "two") as the first hit and i as the second. He obtained the following recurrence relations:

$$\begin{aligned} &a_{n,i} = a_{n,i-1} - a_{n-1,i-1} \quad , \quad b_{n,i} = a_{n-1,i-1} \\ &c_{n,i} = c_{n,1} - (i-1)c_{n-1,1} + \sum_{k=2}^{i-2}(-1)^k \cdot \frac{i(i-1-k)}{2}\ c_{n-k,1} \qquad \forall n > i+1 \end{aligned} \tag{2.17}$$

Denoting with $a_{n,0}$ the number of *derangements*; $a_n = \sum_{k=1}^{n} a_{n,k}$ the total number of permutations which give hits; $b_{n,0}$ the number of permutations giving "one" as the only hit; $b_n = \sum_{k=2}^{n} b_{n,k}$ the total number of permutations giving a second hit, "one" being the first; $c_{n,0}$ the number of permutations giving "two" as the only hit; $c_n = \sum_{k=1}^{n} c_{n,k}$ $(k \neq 2)$ the total number of permutations giving a second hit, "two" being the first, Steen showed that, for $0 \leq i \leq n$ (putting $a_{0,0} = 1$),

$$a_{n,0} = n a_{n-1,0} + (-1)^n \quad , \quad a_{n,i+1} = \sum_{k=0}^{i} (-1)^k \binom{i}{k} (n-1-k)!$$

$$b_{n,i} = a_{n-1,i-1} = a_{n-2,i-2} - a_{n-3,i-2} \quad , \quad b_{n,0} = a_{n,1} - b_n = a_{n,1} - a_{n-1} = a_{n-1,0} \tag{2.18}$$

$$c_{n,i} = \left[\sum_{k=1}^{i-3} (-1)^{k+i-1} \frac{k(k+3)}{2} (n-i+k-1)! \right] - (i-1)(n-3)! + (n-2)! \ .$$

Guy and Nowakowski **[29]** and Mundfrom **[43]** showed separately that Steen's formula (2.18) is not valid for $i = n-1$ and $i = n$ and gave the exact relations. We quote the expressions, together with the equation for $c_n = \sum_{k=1}^{n} c_{n,k}$, $k \neq 2$, as shown in **[29]**, thanks to their compactness:

$$c_{n,n-1} = \sum_{k=0}^{n-3} (-1)^k \binom{n-3}{k} (n-k-2)!$$

$$c_{n,n} = (n-2)! + \left[\sum_{k=0}^{n-5} (-1)^{k+1} \left(\binom{n-4}{k} + \binom{n-3}{k+1} \right) (n-k-3)! \right] + 2(-1)^{n-3} \tag{2.19}$$

$$c_n = (n-2)(n-2)! - \left[\left[\frac{1}{e}((n-1)! - (n-2)! - 2(n-3)! \right]\right] \ ,$$

where $[[x]]$ is the nearest integer to x.

In **[29]** we can see a table, giving the numbers of permutations eliminating just i cards ($1 \leq i \leq 9$); the diagonal represents the numbers of winning permutations, i.e. permutations setting aside all the n cards. Since the table does not derive from any closed formula, it was probably obtained by means of direct computations, considering that, for $n = 9$, all the permutations, whose number is equal to $9! = 362880$, could be easily checked by means of a computer.

Furthermore, the authors give the probability that only the card k is set aside from a deck of $n > 2$ cards and show the related complete table of values, for $1 \leq k \leq n$, $1 \leq n \leq 10$, adding another table, for $11 \leq n \leq 17$, but $1 \leq k \leq 5$.

Knowing general formulas giving the numbers of permutations that have i as the k-th hit, given the previous $(k-1)$ hits, would be very useful to arrive at a closed formula for the probability distribution of the game. But, as remarked by Steen, already the computations to obtain $c_{n,i}$ are very difficult and it is hard to expect more advanced results in this direction.

Nevertheless, observing that a deck gives a *derangement* at $T1$ if and only if it gives a *derangement* at *Mousetrap*, from formula (2.11) we have the following

Corollary 2.12. *The probability of derangement for the games Mousetrap (M) and Modular Mousetrap (MM) is*

$$P_{M,m}(0) = P_{MM,m}(0) = P_{T1,m}(0) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \ .$$

and

$$\lim_{m \to \infty} P_{M,m}(0) = \lim_{m \to \infty} P_{MM,m}(0) = Po_1(0) \ = \ e^{-1} \ .$$

New results will be shown in Section 6, based not on closed formulas, but on computer tools, which can extend the results attained in **[29]** up to $m = 15$, $s = 1$ and to *"multiseed" Mousetrap.*

Finally, only in **[29]** some results are yielded for the game *Modular Mousetrap.* Guy and Nowakowski's paper is mostly focussed on the problem of reformed permutations, i.e. decks obtained as lists of cards in the order they were set aside in a winning deck. This problem will be taken into consideration in another paper **[4]**.

3. $(HLM)^2N$: definitions and first combinatorial results

The game I am going to describe is known as *He Loves Me, He Loves Me Not* ($(HLM)^2N$), or, as I discovered very recently, *Montecarlo* **[46]**:

"From a deck with s seeds and m ranks, deal one at a time all the cards into a pile, counting "one", "two", "three" etc. When a card whose value is k proves to be of the rank you call, it is *hit.* The card is thrown out and stored in another pile, the record is increased by k, the preceding $k-1$ cards are put at the end of the deck, in the same order in which they were dealt and you start again to count "one", "two", "three" etc. If you have counted up to m without coming to a card has been thrown out, the last m cards are "burned", i.e. definitively discarded and you start again to count "one", "two", "three" etc. with the residual cards. When the number n_c of cards in the residual deck is less than m, the count can arrive, at most, at the value n_c. The game ends when you have stored and/or "burned" all the cards and there are no more cards in the deck. The record of the game is given by the sum of the face values of all the stored cards."

The game can be played with arbitrary values of m and s.

Since, after every matching, we start counting again from "one", the game recalls $T0$ and *Mousetrap*, but it differs from $T0$ because, when the deck falls short, it is turned without reshuffling. Furthermore, the game differs from all the others mentioned in Section 1 for the following reasons:

a) we record the sum of the values of the cards, not their number; obviously, in a deck of $m \cdot s$ cards, we can, at most, obtain

$$s \cdot \sum_{k+1}^{m} k \ = \ \frac{s}{2} m(m+1) \quad \text{points} \ ;$$

b) we "burn", i.e., we eliminate m cards, if no coincidences occur counting from 1 to m, but we do not stop the game and we continue our counting starting again from "one".

The game, as the other ones, can have some variants:

a) the counting can be performed in different ways, according to the literature **[8, p. 221]**, **[24, p. 161]**, **[42, p. 54]**; differently from *Treize* ($T1, T2, T3$ and $T4$) and *Mousetrap*, where the counting order does not affect the results, because a different counting can be considered just as a permutation of the other ones, in this game, where it is important to obtain the best sum of values, the order in which the values are pronounced affects the final result in a fundamental way, as we will see more clearly in the next Section.

b) the game could be stopped either when, remaining in the deck a number $n_c < m$ of cards, we don't obtain any matching counting up to n_c, or, following $T4$ and *Mousetrap* rules, when the deck falls short, we turn the deck of n_c cards and we continue our counting up to m; in this second case, if no matching happens counting up to m, the game stops; otherwise the counting can be restarted, after having stored the last matching card. In the first case, we play $(HLM)^2N$; in the second we play the *"multiseed" Mousetrap.*

As already remarked, differently from *Mousetrap* and $(HLM)^2N$, in the games $T3$ and $T4$, after every matching, we continue our counting where we stopped, instead of starting again from "one". Consequently, it is very simple to find at least a winning deck: it is sufficient to take a deck, ordered (independently from the seeds) by the values of the cards i.e. a deck where the cards are put in the following way:

$$1\ 2\ 3\ ...\ (m-1)\ m\ 1\ 2\ 3\ ...\ (m-1)\ m\ ...$$

repeated s times. In this way, all the cards are stored in the first turn. While many other decks win at $T4$, this is the only one storing all the cards playing $T3$, as already shown in formula (2.10). On the other hand, *Mousetrap* and $(HLM)^2N$ are more intriguing, because there is no *a priori* information on any potential winning deck.

Moreover, the rule followed by *Mousetrap* allows the player to store all the $m \cdot s$ cards (in particular, at *Mousetrap*, if we remain with only one card in the deck, we know that we will store it, because we will count up to m visiting always the same card, whose values is, obviously, less or equal to m). Instead, in the following Proposition we show that in $(HLM)^2N$ we can store at most $ms-1$ cards. In other words, when we consider *Mousetrap* with more than one seed, this game is easier than $(HLM)^2N$ and every deck winning at $(HLM)^2N$ wins at *Mousetrap*.

Proposition 3.1. *In $(HLM)^2N$, for every s, m we can at most store $ms-1$ cards and the record cannot exceed*

$$C_{max} := \frac{s}{2}[m(m+1)] - 2\ . \tag{3.1}$$

Proof The proof is based on contradiction.

Let us suppose that we can store all the $n = m \cdot s$ cards. Since the storage mechanism implies that, once a card is stored, the number of residual cards in the deck is lowered by one, the last stored card lowers the residual deck from one card to no cards. Consequently, the only card storable as the last one is an "ace". Proceeding backward in the storage mechanism, when we store the last but one card, the deck passes from two cards to one. One of these two cards, as already observed, is an "ace". The second one, that must be stored, can be only an "ace" or a "two". But if we want to store the "two", the other card, which precedes it, cannot be an "ace" (otherwise, counting the two cards, we should have stored firstly the "ace"!). Thus the last two cards must be two "aces". Continuing our process backward and reasoning in the same way as before, since we want to store all the last three cards, the last but two card must be an "ace", a "two" or a "three". But if the last but two card is a "two" or a "three", if we want to store it we should not have an "ace" as the first of the three cards, in contradiction with the fact that the other two cards are two "aces". Consequently, the last three cards must be three "aces". The backward reasoning can be iterated, arriving at the conclusion that, for every k, the last k cards must be "aces". But the number of "aces" is equal to s, so, when $k > s$, we arrive at a contradiction. Formula (3.1) immediately follows from the first thesis. □

The crucial question is if it is always possible to find a deck from which we can store all the decks but a "two" and, consequently, we can obtain C_{max}.

In **[3]** we stated the following two conjectures:

Strong Conjecture (SC) For every $s \geq 3$; $m \geq 2$, there exists at least one deck from which we store $sm - 1 = n - 1$ cards, obtaining the best record, i.e.

$$C_{max} \ = \ \frac{s}{2}m(m+1) - 2\ .$$

Weak Conjecture (WC) For every $s \geq 2$; $m \geq 2$, there exists at least one deck from which we store $sm - 1 = n - 1$ cards.

For $s = 1$ (SC) is clearly false.

Proposition 3.2. *Formula (3.1) is not valid for $s = 1$.*

Proof Let us observe that, for $s = 1$, the only way to store the card with value m consists in putting it in the m-th place, without having any other coincidences in the previous $(m - 1)$ places. Let us indicate with $X_1\ X_2\ X_3\ ...\ X_{m-2}\ X_{m-1}$ an arbitrary *derangement* of the first $(m - 1)$ cards; thus the m cards have the following sequence in the deck:

$$X_1\ X_2\ X_3\ ...\ X_{m-2}\ X_{m-1}\ m\ .$$

But in the turn following the matching of the card m, the residual deck is formed by $(m - 1)$ cards, placed in a *derangement*; consequently we cannot have any other coincidences. □

Remark 3.3. For $s = 2$ there exist cases for which it is not possible to obtain the best record given by (3.1). The case $s = 2$, $m = 3$ (90 different decks) can be verified directly, "by hand". The best record reachable, in this case, is 9, instead of 10. In the other cases, with an increasing value of m, we need numerical simulations: for $s = 2$, $m = 4$ (2530 different decks) and for $s = 2$, $m = 5$ (113400 different decks), we obtain, respectively, 17 points, instead of 18 and 27 points, instead of 28. for $s = 2$, $6 \leq m \leq 13$ the best record, given by (3.1), has been obtained. In Section 5 we will prove this fact.

$(HLM)^2N$ implies several questions:

1) what is the maximum record available, depending on m and s? what is its probability or even only its estimate?

2) what is the most probable record?

3) what is the probability distribution of records or even only its asymptotic estimate?

4) what is the probability of *derangement*, that is of no coincidences?

5) what is the record average?

6) what is the average of the number of stored cards?

7) what is the probability distribution of the values of stored cards? what is the average of the values of stored cards?

In general, we can give only numerical answers, by means of Monte Carlo simulations; some of them will be examined in Section 4. Questions **2)**, **5)**, **6)** and **7)** have been discussed in [**3**].

Due to the possibility to turn the deck, and, consequently, to the fact that every matching is strongly dependent on the previous events (a matching occurring not at the first turn is affected by the event that, in the previous turns, the card has not matched any counting), the usual theory of probability distributions for matching problems, as the classical approximation theory, based on binomial or poissonian distributions, does not hold anymore.

However, we can give an exact answer to question **4)** for the *derangement* probability, by means of the general results by Fréchet, recalled in Section 2.

Proposition 3.4. *The probability $P_{(HLM)^2N,ms}(0)$ of derangement in the game $(HLM)^2N$ coincides with the probability $P_{T3,ms}(0)$ of derangement in $T3$, i.e.*

$$P_{(HLM)^2N,ms}(0) = P_{T3,ms}(0) = \sum_{k=0}^{ms} (-1)^k \ \frac{(ms-k)!}{(ms)!} \ \sum_{\sum a_j = k} \left[\prod_{j=1}^{m} a_j! \ \binom{s}{a_j}^2 \right] .$$

Consequently

$$\lim_{m\to\infty} P_{(HLM)^2N,ms}(0) = Po_s(0) \ = \ e^{-s} .$$

Proof It is evident that a deck gives a *derangement* in $(HLM)^2N$ if and only if it gives a *derangement* in $T3$. In fact, in both cases the player arrives s times at m without any matching and the deck is "burned" in the first turn.

The asymptotic value is related to the fact that, as already shown in Section 2, for $m \to \infty$, the probability distribution of $T3$ is satisfactorily approximated by the Poisson distribution. □

Remark 3.5. Note that the values of $P(0)$ obtained in Proposition (3.4) hold for *Mousetrap* and *Modular Mousetrap*, too. The choice of the *Mousetrap-like* option **a)**, described in the Introduction, for the "multiseed" game should imply a different value for $P_{M,m\cdot s}(0)$. In fact, in this case, a deck gives a *derangement* if and only if it gives a *derangement* in $T2$. Consequently, we should have

$$P_{M,m\cdot s}(0) = P_{T2,m\cdot s}(0) = \sum_{k=0}^{m} (-1)^k \ \frac{(ms-k)!}{(ms)!} \ \binom{m}{k} s^k$$

and

$$\lim_{m\to\infty} P_{M,m\cdot s}(0) = e^{-1} .$$

In Table 3 we compare the values of $P_{(HLM)^2N,4m}(0)$, and $Bi\left(4m \ , \ \frac{1}{m}\right)(0)$, where we have fixed the value of $s = 4$ and we vary m. It is evident that both the sequences of $P_{(HLM)^2N,ms}(0)$ and $Bi\left(4m \ , \ \frac{1}{m}\right)(0)$ approach the value $Po_4(0)$ when $m \to \infty$.

Let us remark that the approximation by means of Poisson and binomial distributions holds only for the *derangement*, as already known for the game $T4$, since the option of turning the deck highly complicates the computation of probabilities, because, for example, the probability to have a matching in the following turns is affected by the probability not to have matchings for the same card in the previous turns. Moreover, since in the games $T1$, $T2$ and $T3$ the player continues his counting also after a matching, without restarting from "one", all the cards have en equal probability to match. Instead, in $(HLM)^2N$ and in *Mousetrap*, since the player starts his counting afresh after every matching, the cards with lower values have greater probability to match, because they are pronounced more times than the cards with higher values.

For every value of s, when m grows, $P_{(HLM)^2N}(0)$ tends, increasing, to the value e^{-s}. It is possible to estimate numerically that, when $m = s+1$, $P_{(HLM)^2N}(0) = P_{(HLM)^2N}(1)$ and $P_{(HLM)^2N}(0) > P_{(HLM)^2N}(1)$ for every $m > s+1$. It is not possible to obtain this results starting from $P_{T3}(1)$, not only for the considerations made just above, but also because in $T3$ we can obtain exactly one matching with whatever value k , $1 \leq k \leq m$. If the unique card giving the matching in $(HLM)^2N$ is an "ace", the deck does not need to be turned and the number of decks giving the matching with an "ace" at $(HLM)^2N$ coincide with the decks giving one matching with an "ace" at $T3$. On the other hand, every deck giving only one matching in $(HLM)^2N$ with a card with value $k > 1$ needs to be turned; consequently decks giving only one matching at $T3$, in general, cannot be related to the decks giving a record equal to 1 at $(HLM)^2N$.

However, roughly speaking, we are considering decks that, playing at $(HLM)^2N$, turn "not so many" cards, so that we could expect that the probability $P_{(HLM)^2N}(1)$ could roughly be obtained starting from $P_{T3}(1)$.

It is noteworthy the fact that, though without any theoretical proof, but, probably, thanks to the fact that we are speaking about approximating distributions, the binomial distribution gives the expected result.

In fact, putting the approximation $P_{(HLM)^2N}(1) \sim \dfrac{P_{T3}(1)}{m}$ and approximating the probability distribution of $T3$ with the binomial, we have

$$Bi\left(ms, \frac{1}{m}\right)(0) = \frac{Bi\left(ms, \frac{1}{m}\right)(1)}{m} \iff \left(1 - \frac{1}{m}\right)^n = \frac{n\frac{1}{m}\left(1 - \frac{1}{m}\right)^{n-1}}{m}$$
$$\iff \left(1 - \frac{1}{m}\right) = \frac{s}{m} \iff s = m - 1 \ .$$

Incidentally, this result cannot be obtained starting from the poissonian approximation. In fact

$$Po_s(0) = \frac{Po_s(1)}{m} \iff e^{-s} = \frac{s\ e^{-s}}{m} \iff s = m \ .$$

Let us observe that, from formulas (2.17), (2.18) and (2.19), when $s = 1$, we have $P_{(HLM)^2N}(1) = b_{n,0}$ and $P_{(HLM)^2N}(2) = c_{n,0}$. For greater records the rules become much more complicated. Since we are interested on the cases $s > 1$, we should have a great achievement if we could obtain formulas similar to (2.17), (2.18) and (2.19) for $s > 1$.

Obviously, our aim should be to find (at least an approximation of) the probability distribution $P(k)$ for $(HLM)^2N$, for every value of k and, in particular, for $P_{max} := P(C_{max})$. In fact, knowing it would give a definitive answer to (SC). Unfortunately, as far as I know, classical literature gives us only the following estimates **[21, pp. 219 – 220]**, **[49, p. 48]**, **[57, pp. 112 – 118, 125 – 126]**

Theorem 3.6. *Čebishev inequality. Given any random variable ξ having finite variance and any number $\epsilon > 0$,*

$$P(|\xi| \geq \epsilon) \ \leq \ \frac{1}{\epsilon^2}\mathbb{E}(\xi^2) \ ;$$

Theorem 3.7. *Markov inequality. Given any random variable $\xi > 0$, having finite mean $\mathbb{E}(W) = \mu$ and any number $\epsilon > \mu$,*

$$P(\xi \geq \epsilon) \ \leq \ \frac{\mu}{\epsilon} \ ;$$

or related inequalities (see, for example, **[27, p. 12]**, **[59, p. 93]**).

Another estimate of $P(C_{max})$ will be given in Section 5 and compared with the above inequalities. In any case an appropriate tool to give a satisfactory approximation of $P(C_{max})$ has not yet been established.

In the next Section we will show all the numerical results, obtained by Monte Carlo simulations, which can give some answers to the questions mentioned at the beginning of this Section, though only by a numerical point of view.

4. Numerical results

In order to obtain at least numerical answers to (SC) and (WC), a computer software has been built up, based on Monte Carlo simulations (which allow us to approximate the probability distribution by means of the frequency distribution of a sufficiently high number of experiments), according to the following, simple steps:

a) deck "shuffling", by means of random permutations of an initial deck;

b) playing the game: in a vector **C**, with $\frac{s}{2}[m(m+1)]$ components, the first sm components are filled with the shuffled deck. A cursor passes through all the ordered components. When the first matching happens at a card, whose value is k_1, the preceding $(k_1 - 1)$ cards are put in the same order just after the last nonzero component of **C**, filling the vector components from the $(ms+1)$-th position to the $(ms+k_1-1)$-th one. The cursor restarts from the (k_1+1)-th position, counting from "one". The card k_1 is stored and the actual record is increased by k_1 points. Subsequently, at the r-th matching, corresponding to the card k_r, the preceding $(k_r - 1)$ cards are shifted, in the same order, just after the last nonzero component of the vector and so on.

Calling n_c the minimum value between the number of residual cards in the deck and m, when no coincidences happen after n_c cards, they are eliminated and if $n_c \leq m$ the game stops because there are no more cards to be "visited".

c) data storage: at the end of every game, if the record exceeds a determined threshold (for example, the previous best record), the record, the number of stored cards, the number of values of the cards and the winning deck are stored in a data file. If we are interested on the statistics, all the information for every deck is stored in frequency distribution vectors and, consequently, the averages of records, of the number of stored cards and of the values of stored cards are computed. If we are interested only on the best record, when in a deck m consecutive cards are eliminated, due to no coincidences, the deck is discarded, because no more able to obtain the record and the game restarts with a new deck.

Remark 4.1. In general, we cannot relate the final number of nonzero components of the card vector **C** and the record obtained with the corresponding deck. For example, if we obtain only 2 points by means of two "aces", the length of the card vector remains ms. If we obtain 2 points by means of a "two", we have to shift the card preceding the "two" in position $(ms + 1)$ and the card vector length increases.

However, if we are in presence of a *derangement*, the vector **C** will have only ms nonzero components. On the other hand, it is easy to show that to a winning deck, which stores all the $(ms - 1)$ cards but a "two", corresponds a card vector whose length is $\frac{sm(m+1)}{2} - 1$.

The method is very efficient, considering the speed of execution and, in particular, the disk usage for the data storage; in fact, after the game, it is always possible to obtain back the deck we have examined, considering the first $m \cdot s$ components of the card vector **C**.

The software has been written in FORTRAN code and implemented in a PC, equipped with a Pentium IV. On the other hand, Andrea Pompili, in [**46**], used a Borland C language.

We can count in three different ways (they are the three counting ways I know, from direct experience and from literature on *solitaires*, but many other ways could be chosen!):

a) ace (1), m, m-1, m-2, ..., 4, 3 , 2 ([**8, p. 221**], [**24, p. 161**])

b) m, m-1, m-2, ..., 4, 3 , 2, ace (1)

c) ace (1), 2, 3, ..., m-1, m .

As already remarked, while the frequency distribution of the number of stored cards in each game does not depend on the counting method chosen, the record depends on the counting method chosen. In fact, because of the fact that the numbers which are firstly pronounced have a greater probability to match and to be stored, it is simpler, in cases **a)** and **b)**, to obtain the best result reachable, equal to C_{max} in case **a)** and equal to $C_{max} + 1 \ = \ \frac{s}{2} m(m+1) - 1$ in case **b)**.

Moreover, the three cases show some peaks in the frequency distribution, which apparently seem not to follow any rule in cases **a)** and **b)**, while in case **c)** peaks corresponding to the first multiples of m (including $k = 0$, when $m > s + 1$) are evident, besides other secondary peaks.

Just as an example, in Fig. 1 we reproduce the frequency distribution of the records, obtained by means of numerical simulations, for $m = 5$, $s = 4$, for the three counting ways. In the case **c)**, the presence of some principal peaks (corresponding to the first three multiples of m) is evident. There are also secondary peaks, corresponding to values which seem not to follow any particular rule.

The presence of the principal peaks corresponding to the first multiples of m has been noted only when we use the counting method **c)**. This is another reason to prefer it.

Consequently, to answer to the some of the questions posed in Section 3, we will focus on the classical counting method, i.e., method **c)**.

Remark 4.2. Let us observe that the principal peaks appear in *Mousetrap*, too, when we compute the sum of the stored cards values. This seems to suggest that the presence of peaks is related firstly to the fact that after every matching we restart our counting from "one". However, the mathematical reason why these peaks appear is still under investigation.

In Table 4 we show the best records, obtained with Monte Carlo methods, in order to check the validity of (SC) or, at least, of (WC). We have studied the game, varying both m and s. For the sake of simplicity, we report here only the first 13 values of m and the first 4 values of s. The possibility to validate the conjectures becomes very hard when m and s increase too much. In fact, the higher m and s are, the higher the number of different decks is, as shown in the formula (2.1), which gives the number of different decks for every fixed couple (s, m).

In order to give an idea of the computational complexity of the problem, let us observe that a French card deck has $\dfrac{52!}{(4!)^{13}} \sim 9.2 \cdot 10^{49}$ permutations (without considering the *rank derangements*). Supposing that each one of the over 6 billions Earth inhabitants could examine every day 20 billions decks, **each one different from the others and from the decks examined by the other people**, with a computer (this is the actual capacity of my FORTRAN program, implemented in a PC equipped with a Pentium IV), we should need more than $2 \cdot 10^{27}$ years to test all the different decks!

The threshold for the number of decks to be checked beyond which the numerical simulations seem to become inadequate is around 10^{20}. Nevertheless, it is noteworthy the case $m = 10$, $s = 2$. In fact, even after more than 600 billions simulations, no evidence of a winning deck appeared, though the number of different decks is "only" almost $2.58 \cdot 10^{15}$. In this case, the Monte Carlo method has only given a positive answer to (WC), obtaining, at most, 106 points, instead of $C_{max} = 108$, as predicted in (3.1). This situation could be *a priori* related either to an effective negative answer to (SC) for $m = 10, s = 2$ or to the high number of different decks, in front of a too low number of winning decks. Actually, the question will be answered in a surprisingly easy way in the next Section: there are only 656 winning decks and, consequently, the probability of finding one of them is $P_{10\cdot 2}(108) \sim 2.76 \cdot 10^{-13}$. Then it seems that we should have needed $\mathcal{O}(10^{12} \div 10^{13})$ simulations to expect to find a winning deck.

5. The "eulerian" approach.

In the previous Section we have shown a table, where we have reported the best records obtained with Monte Carlo simulations and we have remarked the difficulties related to the high computational complexity of the problem.

Here we introduce a new technique, which can give much more satisfactory answers than the numerical simulations, in a very efficient way, yielding not only a positive answer to (SC) at least up to $m = 13$, $s = 4$, i.e. for the classical deck of French cards (though it can be used to explore much larger decks), but also the exact number of winning decks, and consequently, the exact probability of winning, for a large number of cases, as shown in Table 5.

Let us firstly explain the method.

Trying an analogy with Rational Mechanics, we can interpret the way to deal the cards one after the other as an *eulerian* description of the game, since the observer is fixed and all the elements of the system, i.e. the cards, pass in front of it, while in the numerical schemes another description, consisting in arranging sequentially all the cards and visiting them by means of a cursor, can be interpreted as a *lagrangian* description, because in this case we follow the evolution in time of the whole system, starting from its initial configuration.

The lagrangian description is much more useful in the numerical simulations: in fact, as already observed, after having assigned the first $m \cdot s$ components of the vector $\mathbf{C}$ (which can have, at most, $\frac{s}{2}m(m+1) - 1$ components), after every matching, the card with value k_1 giving this matching is stored and the preceding $k_1 - 1$ cards are put just after the last nonzero component of $\mathbf{C}$, ready to be visited again by the cursor, which, in this way, never comes back, but continues forward, up to the end of the game.

In other words, the lagrangian description allows us to generate, from every deck, a string, called *lagrangian string*, whose length is, at most, $C_{max} = \frac{s}{2}m(m+1) - 1$ (in this case, we played with a winning deck and the last component in the string is a "two"), whose first $m{\cdot}s$ components give the initial configuration of the system, i.e. the original deck. A *derangement* corresponds to a string with length $m \cdot s$, coinciding with the original deck.

In every lagrangian string the number of times a card whose value is k is repeated varies from deck to deck. Let us consider, just as an example, the only two winning decks, found by means of the Monte Carlo method, in the case $m = 7, s = 4$, in at least 60 billions simulations:

$$\begin{array}{ccccccc} 4 & 3 & 1 & 4 & 7 & 7 & 2 \\ 6 & 5 & 4 & 7 & 3 & 1 & 3 \\ 3 & 6 & 2 & 2 & 1 & 7 & 6 \\ 6 & 5 & 1 & 5 & 4 & 5 & 2 \end{array} \quad (\mathbf{a}) \qquad \begin{array}{ccccccc} 4 & 3 & 7 & 3 & 1 & 6 & 7 \\ 2 & 7 & 5 & 3 & 5 & 2 & 2 \\ 4 & 4 & 6 & 2 & 1 & 7 & 5 \\ 6 & 3 & 1 & 6 & 5 & 1 & 4 \end{array} \quad (\mathbf{b}) \qquad (5.1)$$

They generate, respectively, the following lagrangian strings, S_1 and S_2:

$$\begin{array}{cccccccccccccccccccccccccccc} 4 & 3 & 1 & \mathbf{4} & 7 & 7 & 2 & 6 & \mathbf{5} & 4 & 7 & \mathbf{3} & \mathbf{1} & 3 & 3 & 6 & 2 & 2 & 1 & \mathbf{7} & 6 & 6 & 5 & 1 & \mathbf{5} & 4 & 5 & 2 \\ \mathbf{4} & 3 & 1 & 7 & 7 & 2 & \mathbf{6} & 4 & 7 & \mathbf{3} & 3 & 6 & 2 & 2 & 1 & \mathbf{6} & 6 & 5 & 1 & \mathbf{4} & 5 & \mathbf{2} & 3 & 1 & 7 & 7 & 2 & 4 \\ \mathbf{7} & 3 & 6 & 2 & 2 & 1 & \mathbf{6} & 5 & 1 & 5 & 3 & 1 & 7 & \mathbf{7} & 2 & 4 & \mathbf{3} & 6 & \mathbf{2} & 2 & 1 & 5 & 1 & \mathbf{5} & 3 & 1 & 7 & 2 \\ 4 & \mathbf{6} & 2 & 1 & 5 & 1 & 3 & 1 & \mathbf{7} & 2 & 4 & 2 & 1 & \mathbf{5} & \mathbf{1} & 3 & 1 & 2 & \mathbf{4} & 2 & 1 & \mathbf{3} & \mathbf{1} & 2 & \mathbf{2} & \mathbf{1} & 2 & \end{array}$$

4	3	7	3	1	**6**	7	**2**	7	5	**3**	5	**2**	2	4	4	6	2	1	**7**	5	6	**3**	**1**	6	5	1	**4**
4	3	7	3	1	7	**7**	5	5	2	**4**	4	6	2	1	**5**	6	6	5	1	4	3	**7**	3	1	7	5	**5**
2	4	6	2	1	**6**	6	5	1	**4**	3	3	1	7	**5**	2	4	6	2	1	**6**	5	1	**3**	3	1	7	2
4	**6**	2	1	5	1	3	1	**7**	2	4	2	1	**5**	**1**	3	1	2	**4**	2	1	**3**	**1**	2	**2**	**1**	2	

where the cards yielding matchings are written in boldface characters. In these strings, the number of times that the different cards are "visited" are respectively:
in S_1: ace: 23 times; 2: 23 times; 3: 15 times; 4: 12 times; 5: 12 times; 6: 12 times; 7: 14 times;
in S_2: ace: 23 times; 2: 19 times; 3: 15 times; 4: 14 times; 5: 15 times; 6: 13 times; 7: 12 times.

We can also consider another string, with the same length of the lagrangian one, formed by the values pronounced by the player, instead of the values "visited". In this case, all the strings of maximal length contain s times the value m, $2s$ times the value $m-1$, $3s$ times the value $m-2$, $s(m-k+1)$ times the value k $(k \neq 2)$, down to $s(m-2)$ times the value 3, $sm = n$ times the value 1 e $s(m-1)-1$ times the value 2.

By construction, these new strings, which we will call *eulerian*, have exactly $(n-1)$ matchings with the corresponding lagrangian strings, when they are generated by a winning deck.

Since our main goal is to study the winning decks, for the sake of simplicity in our presentation, let us focus only on strings generated by winning decks, if not differently indicated.

It is immediate to note that, since in the eulerian strings every value k is preceded by the first $k-1$ values, we can indicate only with $[k]$, or, better, with k, the substring $123...k$. Consequently, we can represent every eulerian string by means of a reduced string, which is formed by the cards which have given a matching, put in the same order in which they were stored, and, at the end, an "ace". Thus, the length of the reduced strings generated by winning decks is $n = m \cdot s$. In this case, if we substitute the final "ace" with the residual "two" of the winning deck, we form a new deck, i.e. a permutation of the original deck. In other words, the so modified winning reduced strings correspond to the reformed decks (or permutations) introduced by Guy and Nowakowski **[28,** §**E37]**, **[29]**, **[30]** in the game *Mousetrap*. We will call them *reduced (eulerian) string.*

In the above mentioned example, to the winning decks for $m = 7, s = 4$, we can associate the following eulerian strings

12341234512311234567123451234123456123123456123412123456
7123456123456712312123451234561234567123451123412311211 ,

which generates the reduced eulerian string

4531754636427673256751431212

and

12345612123121234567123112341234567123412345123456712345
1234561234123451234561231234561234567123451123412311211 ,

which generates the reduced eulerian string

6232731474575645636751431212 .

In the winning reduced strings the value 2 is always the final component. Consequently, the number of all the potential winning reduced strings is given by

$$S_{m\cdot s} = \frac{(n-1)!}{(s!)^{m-1}\cdot(s-1)!} \; . \tag{5.2}$$

If we had a bijective correspondence among the winning decks and the potential winning strings, we should have answered to question 1) in Section 3, dividing $S_{m\cdot s}$ by the number of all the possible decks:

$$\frac{(n-1)!}{(s!)^{m-1}\cdot(s-1)!}\cdot\frac{(s!)^m}{n!} = \frac{s}{n} = \frac{1}{m} \; . \tag{5.3}$$

Unfortunately, we cannot have bijection. For the sake of simplicity, let us consider the case $m = 2$, $s = 3$. Among the $\frac{6!}{(3!)^2} = 20$ decks, only four of them win. Here we show the winning decks and the associated reduced strings (or reformed decks):

the string 111222 is generated by the deck 111222;
the string 112122 is generated by the deck 112212;
the string 121122 is generated by the deck 122112;
the string 211122 is generated by the deck 221112.

The potential winning reduced strings are, in this case, $\frac{5!}{(3!)\cdot(2!)} = 10$:

$$\{221112\} \; ; \; \{212112\} \; ; \; \{211212\} \; ; \; \{211122\} \; ; \; \{122112\} \; ;$$
$$\{121212\} \; ; \; \{121122\} \; ; \; \{112212\} \; ; \; \{112122\} \; ; \; \{111222\} \; .$$

Actually, only the fourth, the seventh, the ninth and the tenth are generated by winning decks. However, even if we cannot have bijection between winning decks and potential winning reduced strings, we can use formula (5.3) as a rough upper bound for $P(C_{max})$. At the end of this Section we will improve this bound, comparing it with Čebishev and Markov inequalities.

When we associate to a winning deck a reduced string we have a very deep information, related to the fact that the procedure of string generation is (maintaining a physical language) reversible: knowing the generated string, we can rebuild the original deck. Considering example (5.1 **a)**) ($m = 7$, $s = 4$), let us consider a vector with 28 components. Let us put in the fourth component the first element of the string, i.e. the first stored card, which is clearly a "four". Then we will put in the ($4 + 5 =$) ninth component the second stored card, i.e. a "five" and so on. When the counting arrives at 28, or, in general, at $m \cdot s$, we restart our counting from the first component, taking into account only the zero components, inserting the first 27 stored cards. The last card, i.e. a "two", will be put in correspondence with the last zero component. In this way we have rebuilt the original winning deck from the winning reduced string.

The "eulerian" approach can thus provide a very efficient method for the study of the winning decks, highly more efficient than the Monte Carlo simulations.

The technique, implemented in a computer program, rebuilds strings of more and more increasing length (up to the winning strings of length n, or n-strings), storing in data files only those ones such that the sub-decks, rebuilt from them, win playing $(HLM)^2N$, i.e. store all the cards but the final "two". The program, starting from a k-string, read in a data file, builds all the $(k+1)$-strings, obtained adding at the beginning of the actual k-string all the allowed values from 1 to m; rebuilds the corresponding sub-decks; plays with the sub-decks; if a sub-deck sets aside all the cards, but a "two" and generates the original string, the program stores the corresponding winning $(k + 1)$-string.

In order to save disk usage, the strings are stored as "characters" in the FORTRAN data files. Every idea about any further memory saving improvement would be welcome.

More precisely, the algorithm is the following: starting from the last "two", we proceed backward, building all the sub-strings, of increasing length, that can guarantee the storing of all the cards, apart from the last "two". Obviously, the last stored card can be only an "ace" or a "two"; similarly, the last but one can be only an "ace" or a "two": the drawing of a "three" as the last but one stored card is excluded by Prop. (3.2). Continuing our reasoning, the last but two stored card can be only an "ace", a "two" or a "three", the last but three an "ace", a "two", a "three" or a "four" and so on, up to the last but $(m-1)$ stored card, which cannot assume a value greater than $(m-1)$. From the last but m stored card on, every card value is admitted.

Practically, let us recall that in the winning reduced n-strings the last position must be occupied by a card whose value is "two" and that, in order to have winning strings (since the strings of length $k \leq m$ (or k-strings), cannot be occupied by a card whose value is greater or equal to k), the position just before the last "two" can be occupied only by an "ace" or a "two"; thus we have only two winning final strings of length two: 12 and 22, which are respectively generated by the sub-decks 12 and 22.

The final strings of length three can be four: 112 ; 212 ; 122 ; 222. Clearly, the choice of these strings is related to s. If, for example, $s = 2$, the fourth string must be excluded, because it contains three identical cards.

Each one of these strings is in a one-to-one correspondence with a sub-deck generating it. In fact

from the string 112 we build the sub-deck 112, which generates the string 112 ;
from the string 212 we build the sub-deck 221, which generates the string 212 ;
from the string 122 we build the sub-deck 122, which generates the string 122 ;
from the string 222 we build the sub-deck 222, which generates the string 222 .

When we pass to the final strings of length four we have 12 possibilities:

$$1112 \; ; \; 2112 \; ; \; 3112 \; ; \; 1212 \; ; \; 2212 \; ; \; 3212 \; ; \; 1122 \; ; \; 2122 \; ; \; 3122 \; ; \; 1222 \; ; \; 2222 \; ; \; 3222 \; .$$

While we can associate to eight of them the corresponding generating winning deck, according to the following list:

$$\begin{aligned}
&\text{the sub}-\text{deck } 1112 \text{ generates the string } 1112 \; ; \\
&\text{the sub}-\text{deck } 2211 \text{ generates the string } 2112 \; ; \\
&\text{the sub}-\text{deck } 1221 \text{ generates the string } 1212 \; ; \\
&\text{the sub}-\text{deck } 2132 \text{ generates the string } 3212 \; ; \\
&\text{the sub}-\text{deck } 1122 \text{ generates the string } 1122 \; ; \\
&\text{the sub}-\text{deck } 2212 \text{ generates the string } 2122 \; ; \\
&\text{the sub}-\text{deck } 1222 \text{ generates the string } 1222 \; ; \\
&\text{the sub}-\text{deck } 2222 \text{ generates the string } 2222 \; ;
\end{aligned} \tag{5.4}$$

we realize that the strings 3112 ; 2212 ; 3122 ; 3222 have no corresponding winning deck. In fact, considering, for example, the string 3122, the deck generating it must have in the third position the card "three"; in the fourth position the card "ace" and, having no other components after, the second "ace" must be put in first position. Consequently, the card "two" must me put in the unique place remained, that is in the second position. So the generating deck should be 1231. But it is evident that this deck, instead of the considered string, generates the loosing string formed only by an "ace", without any other coincidences.

Moreover, to the string 2212 corresponds the deck 1222, which generates the string 1222, which is still a winning string, but different from the original one. This last consideration shows that there is no bijective correspondence between decks and reduced strings: if every deck generates only one string, the reverse is in general not true: the same deck can be rebuilt from different strings!

In order to avoid these situations, the algorithm we have implemented contains a test, where we check if the original string coincides with the reformed string, obtained from the deck given back by the original string. Otherwise the string must be discarded.

Continuing the procedure, we select winning strings of more and more increasing length, with the fundamental restriction that they must be generated by a deck, following the rules of $(HLM)^2N$.

By virtue of this technique we have been able to show that (SC) is true at least up to the case of French cards ($m = 13$, $s = 4$), finding, in less than one second, four winning decks. The first winning deck of French cards found by the computer is the following:

7	9	5	9	7	3	8	6	6	2	5	12	11
4	12	9	7	7	10	2	4	5	3	11	13	2
4	4	11	13	3	6	10	10	10	3	5	12	2
1	1	1	1	12	9	11	13	8	8	6	8	13

,

while the first deck of Italian cards ($m = 10$, $s = 4$) is

6	8	9	7	5	5	3	6	6	10
2	7	4	7	4	10	2	8	5	3
9	2	4	4	3	6	10	7	10	3
5	2	1	1	1	1	9	9	8	8

.

The search for **at least** one winning deck is, in general, very fast. But, as shown in Table 5, we have, in many cases, found also the exact number of winning decks. Let us remark the fact that for the case $m = 10$, $s = 2$ (in comparison with un unsuccessful research of winning decks with Monte Carlo methods, after more than $6 \cdot 10^{11}$ simulations) we yielded all the 656 winning decks, by virtue of the "eulerian" technique, in less than one second.

Let us apply this method to prove the following

Proposition 5.1. *For $s = 2$, $m = 3, 4, 5$ there are no winning decks. For $s = 2$, $m = 6$ there exists only one winning deck.*

Proof Let us consider firstly strings with an arbitrary m and $s = 2$. Following the above described procedure, we must build all the winning strings of length, respectively, $3 \times 2 = 6$; $4 \times 2 = 8$; $5 \times 2 = 10$; $6 \times 2 = 12$, where, as already remarked, the last position must be occupied by a "two". According to the list (5.4) and recalling that $s = 2$, the 4-strings we are interested on are 2112 ; 1212 ; 3212 ; 1122. These 4-strings generate only the following 5-strings: 32112 ; 42112 ; 31212 ; 41212 ; 13212 ; 33212 ; 43212 ; 31122 ; 41122. Among them, only 42112 ; 31212 ; 13212 are generated by decks (respectively 21142 ; 21312 ; 12132). Continuing backward, we arrive at nine strings of length 6:

342112 ; 442112 ; 542112 ; 331212 ; 431212 ; 531212 ; 313212 ; 413212 ; 513212 ;

among them, only one (431212) is generated by a deck: 312421, which contains a "four". Thus, there are no winning 6-strings (and, consequently, winning decks) for $s = 2$, $m = 3$. Let us now build the four final 7-strings: 3431212 ; 4431212 ; 5431212 ; 6431212. Among them, only two are generated by decks:

3431212 is generated by 2133124;
5431212 is generated by 2421531.

Continuing: among the 9 strings of length 8, only four are generated by decks:

63431212 is generated by 33124621;

35431212 is generated by 31324215;

45431212 is generated by 53142421;

65431212 is generated by 21531624.

All of them contain cards whose value is greater than 4. Consequently, there are no winning decks for $m = 4$, $s = 2$.

The nine 9-strings generated by decks are:

563431212 is generated by 462153312;

663431212 is generated by 246216331;

435431212 is generated by 215431324;

345431212 is generated by 213531424;

545431212 is generated by 242155314;

845431212 is generated by 314242185;

365431212 is generated by 243215316;

665431212 is generated by 316246215;

765431212 is generated by 531624721.

In order to conclude the proof, let us consider only strings where the cards assume at most value "six". Among all the 51 10-strings, only 17 are formed with cards whose value is at most "six". The strings generated by decks are 21. Among them, only 7 are formed with cards whose value is at most "six":

4563431212 is generated by 3124462153;

5563431212 is generated by 3312546215;

4663431212 is generated by 3314246216;

6345431212 is generated by 3142462135;

4365431212 is generated by 3164243215;

5365431212 is generated by 5316524321;

4665431212 is generated by 2154316246.

All of them contain at least one "six". Consequently, there are no winning decks for $m = 5$, $s = 2$. Finally, iterating the procedure only for cards whose value is at most "six", we arrive at 13 12-strings. Among them, only one, 534665431212, is generated by a deck: 316254632154. Then, for $m = 6$, $s = 2$, there is only one winning deck. □

In Table 5 we report the number of winning decks for $s = 2, 3, 4$.

Finally, recalling that, in Proposition (3.2), we have already shown that, for $s = 1$, it is not possible to yield C_{max}, we can, however, determine the best record reachable. Table 6 shows the best results obtained by virtue of a modified version of the computer program explained in this Section. The results we have yielded following this method coincide with the best records obtained with Monte Carlo simulations when the number m is sufficiently small. For larger m, the simulations need too much time to reach the best record, while the eulerian method arrives at the correct answer very quickly.

The "eulerian" method for the reconstruction of winning strings of increasing length allows us to give a rough estimate of the probability P_{max}. In fact, as already remarked, since the last card in every winning string must be a "two", the number N_{max} of winning strings cannot exceed the value (5.2). Continuing backward our reasoning, since there are only two winning 2-strings, i.e. (1) : {12} and (2) : {22}, the total number of winning strings cannot exceed the sum of all the n-strings (no matter if winning or not) rebuilt starting from (1) and (2). The total numbers of strings rebuilt from (1) and (2) are respectively

$$N_1 = \frac{(ms-2)!}{[(s-1)!]^2\ (s!)^{m-2}} \quad ; \quad N_2 = \frac{(ms-2)!}{(s-2)!\ (s!)^{m-1}}$$

Thus

$$N_{max} \leq N_1 + N_2 \leq 2\max\{N_1\ ,\ N_2\} = 2\ \frac{(ms-2)!}{\min\{[(s-1)!]^2\ (s!)^{m-2}\ ,\ (s-2)!\ (s!)^{m-1}\}} =$$
$$= \frac{2\ (ms-2)!}{(s-2)!(s-1)!(s!)^{m-2}\cdot\min\{(s-1)\ ,\ s\}} = \frac{2\ (ms-2)!}{[(s-1)!]^2(s!)^{m-2}} \leq S_{m\cdot s} \qquad \forall m > 2$$

and, consequently,

$$P_{max} \leq \frac{2\ (ms-2)!}{[(s-1)!]^2(s!)^{m-2}}\cdot\frac{(s!)^m}{(ms)!} = \frac{2s}{m(ms-1)}\ .$$

Similarly, we can show that, when we consider the four winning 3-strings {112} , {122} , {212} , {222} and all the n-strings (no matter if winning or not) rebuilt starting from them,

$$P_{max} \leq \frac{4(ms-3)!}{(s-3)!(s-1)!(s!)^{m-2}\cdot\min\{(s-2)\ , s\}}\cdot\frac{(s!)^m}{(ms)!} =$$
$$= \frac{4(ms-3)!}{(s-2)!(s-1)!(s!)^{m-2}}\cdot\frac{(s!)^m}{(ms)!} = \frac{4s^2(s-1)}{ms(ms-1)(ms-2)}\ .$$

The reasoning can be repeated up to the m-strings. However, increasing the length of the reduced k-strings implies an increasing difficulty to establish which of them are winning strings. Thus, in order to yield only a rough estimate of P_{max}, it is sufficient here to consider all the k-strings rebuilt from the $(k-1)$-strings, no matter if they win or not. Let us firstly observe that, among all the k-strings, $\{123.....k\}$ gives the minimal number of n-strings reconstructed from a $(k-1)$-string and consequently, calling $C_{k,ms}$ the number of winning k-strings $(k = 3, ..., m)$,

$$N_{max} \leq \frac{C_{k,ms}(ms-k)!}{(s!)^{m-k+1}(s-2)![(s-1)!]^{k-2}}\ . \tag{5.5}$$

It is immediate to show that the deck rebuilt starting from the string $\{123.....k\}$ can never win, thus we could discard it and improve the estimate (5.5). But, as already stressed, we are just looking for a rough estimate.

Thus

$$P_{max} = N_{max}\cdot\frac{(s!)^m}{(ms)!} \leq \frac{C_{k,ms}s^{k-1}(s-1)}{\binom{ms}{k}k!} \quad , \quad \forall k = 3, ..., m\ . \tag{5.6}$$

The greater k is, the more restrictive (5.6) is. Consequently

$$P_{max} \leq \frac{C_{m,ms}s^{m-1}(s-1)}{\binom{ms}{m}m!}\ .$$

From what remarked above, it is very difficult to establish the exact value of $C_{k,ms}$, when k grows. However, this number cannot exceed the total number of m-strings that can be rebuilt with the rules of

$(HLM)^2N$, in the extremal case for which $s = m$, which is given by the product of the first two 2-strings times the number of cards which can be put in the last but two position (i.e. 2) times the number of cards which can be put in the last but three position (i.e. 3) .... times the number of cards which can be put in the last but $(m-1)$ position (i.e. $m-1$). Thus $C_{m,ms} < 2(m-1)!$ and, finally,

$$P_{max} \leq \frac{2s^{m-1}(s-1)}{\binom{ms}{m} m} \ . \tag{5.7}$$

Clearly, estimate (5.7) can be highly improved, considering separately, for example, the different values of s, or checking what is the highest value of k, for s fixed, for which the number $C_{k,ms}$ does not depend on m.

However, it is evident that (5.7), even if improved, represents an estimate exceeding too much the real value of P_{max}. For example, for $m = 7$, $s = 4$, we know that $P_{max} \sim 6.19 \cdot 10^{-12}$, but the three estimates, given respectively by Čebishev inequality, Markov inequality and (5.7), provide

Čebishev inequality: $P_{(HLM)^2N,7\cdot4}(C_{max}) \leq 0.0162$;

Markov inequality: $P_{(HLM)^2N,7\cdot4}(C_{max}) \leq 0.1741$;

from formula (5.7): $P_{(HLM)^2N,7\cdot4}(C_{max}) \leq 0.002965$,

where the values of the mean and of the variance of the probability distribution have been approximated with the corresponding values of the frequency distribution of the Monte Carlo simulations.

Considering that the number of winning decks respectively at *Modular Mousetrap* ($N_{max}(MM)$), at *Mousetrap* ($N_{max}(M)$) and at $(HLM)^2N$ ($N_{max}((HLM)^2N)$) satisfy the following inequality

$$N_{max}(MM) \ > \ N_{max}(M) \ > \ N_{max}((HLM)^2N) \ ,$$

(because every deck winning at $(HLM)^2N$ wins at *Mousetrap* and every deck winning at *Mousetrap* wins at *Modular Mousetrap*) another possibility should consist in to estimate from above P_{max} for the game $(HLM)^2N$ with the winning probability $P_{M,m\cdot s}(m\cdot s)$ obtained, again by means of Monte Carlo simulations, for the game *Mousetrap* or - if this probability is still too much small to be estimated with simulations - the probability $P_{MM,m\cdot s}(m\cdot s)$ for the game *Modular Mousetrap*, where the winning decks are obtained much more easily that in the previous games.

For example,

Čebishev inequality: $P_{(HLM)^2N,8\cdot4}(C_{max}) \leq 0.0125$;
Markov inequality: $P_{(HLM)^2N,8\cdot4}(C_{max}) \leq 0.1536$;
from formula (5.7): $P_{(HLM)^2N,8\cdot4}(C_{max}) \leq 0.001168$;
$P_{(HLM)^2N,8\cdot4}(C_{max}) \leq P_{M,8\cdot4}(32) \sim 0.0000014$.

Čebishev inequality: $P_{(HLM)^2N,10\cdot4}(C_{max}) \leq 0.0081$;
Markov inequality: $P_{(HLM)^2N,10\cdot4}(C_{max}) \leq 0.1244$;
from formula (5.7): $P_{(HLM)^2N,10\cdot4}(C_{max}) \leq 0.0001856$;
$P_{(HLM)^2N,10\cdot4}(C_{max}) \leq P_{M,10\cdot4}(40) \sim 3 \cdot 10^{-8}$;
$P_{(HLM)^2N,10\cdot4}(C_{max}) \leq P_{MM,10\cdot4}(40) \sim 0.1759$.

Čebishev inequality: $P_{(HLM)^2N,13\cdot4}(C_{max}) \leq 0.0048$;
Markov inequality: $P_{(HLM)^2N,13\cdot4}(C_{max}) \leq 0.0968$;

from formula (5.7): $P_{(HLM)^2N,13\cdot 4}(C_{max}) \leq 0.00001219$;
$P_{(HLM)^2N,13\cdot 4}(C_{max}) \leq P_{MM,13\cdot 4}(52) \sim 0.4508$.

6. Applications to the game *Mousetrap*

As already remarked in the Introduction, there are few results related to the game *Mousetrap*. In particular, there are no (even approximated) formulas giving the probability of winning decks. The technique introduced in the previous Section, adequately adapted to this game, allows us to obtain not a closed formula, but a sequence of values, giving the number $N_{max,m\cdot s}$ of winning decks and, consequently, the probability $P_{max,m\cdot s}$ for different values of m and s. Up to now, according to **[12]**, **[50]**, **[51]**, **[52]**, the sequence of values of P_{max} was obtained only for $s = 1$ and up to $m = n = 13$. In **[52]** this sequence can be read not in A007709, but can be easily obtained from A007711, from the sequence of non-winning decks (or unreformed decks), because their number is, obviously, equal to $n! - N_{max,n}$.

According to Kok Seng Chua **[12]**, this sequence has been obtained playing with all the $n! = m!$ decks, by means of a computer program generating all the permutations of a set of n elements and of parallel calculus, due to the very high number of trials to be performed.

Our new technique allows us to obtain the same results very quickly (my PC yielded the exact number of winning 13-decks in less than 2 hours, in comparison with one week job of parallel calculus used by K.S. Chua **[12]**) and to extend the sequence, for $s = 1$, up to $m = 15$, without any aid of parallel calculus.

The new sequence of reformed decks (starting from $n = 1$), quoted in **[52]** as A007709, is thus

$$\mathbf{1}\ ;\ \mathbf{1}\ ;\ \mathbf{2}\ ;\ \mathbf{6}\ ;\ \mathbf{15}\ ;\ \mathbf{84}\ ;\ \mathbf{330}\ ;\ \mathbf{1,812}\ ;\ \mathbf{9,978}\ ;\ \mathbf{65,503}\ ;\ \mathbf{449,719}\ ;\ \mathbf{3,674,670}\ ;\ \mathbf{28,886,593}\ ;$$
$$266,242,729\ ;\ 2,527,701,273$$

while the sequence of unreformed decks (i.e. the total number of non winning decks), quoted as A007711, is now

$$\mathbf{0}\ ;\ \mathbf{1}\ ;\ \mathbf{4}\ ;\ \mathbf{18}\ ;\ \mathbf{105}\ ;\ \mathbf{636}\ ;\ \mathbf{4,710}\ ;\ \mathbf{38,508}\ ;\ \mathbf{352,902}\ ;\ \mathbf{3,563,297}\ ;\ \mathbf{39,467,081}\ ;$$
$$\mathbf{475,326,930}\ ;\mathbf{6,198,134,207}\ ;\ 86,912,048,471\ ;\ 1,305,146,666,727$$

(the values in boldface were already quoted in **[52]** or in **[12]**).

The "eulerian" technique has been adapted to the game *Modular Mousetrap*, too. Though experimentally $N_{max}(MM)$ grows much faster than $N_{max}(M)$, the new technique has proved to be very powerful, as shown in the Tables 7 and 8.

Let us remark that, for example, in **[29]** the authors study *Modular Mousetrap* only in the cases $s = 1$, $m \leq 5$, while in the case $s = 1$, $m = 6$ they study only the decks where the first card is an "ace".

Furthermore, we have obtained a huge amount of results in the *"multiseed" Mousetrap* ($s > 1$), arriving, just as a test of the efficiency of the new technique, to $s = 2$, $m = 8$; $s = 3$, $m = 6$; $s = 4$, $m = 5$ for *Mousetrap* and to $s = 1$, $m = 13$; $s = 2$, $m = 7$; $s = 3$, $m = 5$; $s = 4$, $m = 4$ for *Modular Mousetrap*.

The cases $s = 2$, $m = 9$; $s = 3$, $m = 7$; $s = 4$, $m = 6$ for *Mousetrap* are still under investigation.

These results, shown in Tables 7 and 8, can be extended to the cases $s > 4$ and, by means of parallel calculus, to higher values of m.

Remark 6.1. As already observed in **[29]**, at *Modular Mousetrap*, when $s = 1$ and m is prime, every deck which is not a *derangement* is a winning deck, because the cards have no possibilities to end in a loop.

Consequently, while, if m is not prime, there is no *a priori* rule showing what is the winning probability, Table 8 shows that, when $s = 1$ and m is prime, it is very easy to know the **exact** winning probability:

$$P_{MM,m}(m) = 1 - P_{MM,m}(0) \qquad \forall m \quad prime \ .$$

Thus, knowing the sequence **[52]** A002467 of permutations with at least one fixed point, we immediately obtain the sequence of numbers of winning decks, for n prime:

3,936,227,868 for $n = m = 13$; 224,837,335,816,336 for $n = m = 17$; 76,894,368,849,186,894 for $n = m = 19$ and so on.

For these cases the "eulerian" technique should have proved to be computationally too much expensive, for a single PC.

Furthermore, since $\lim_{m\to\infty} P_{MM,m}(0) = \frac{1}{e} \sim 0.367879441$, it follows that, at *Modular Mousetrap*, $\lim_{m\to\infty} P_{MM,m}(m) = 1 - \frac{1}{e} \sim 0.632120559$, if we consider only the sequence of prime numbers m.

For the other numbers, the winning probability seems to oscillate and tend to zero very slowly, when $m \to \infty$.

Finally, the new technique has proved to be extremely efficient when applied to the study of reformed decks (or reformed permutations): as already recalled, when a deck wins at *Mousetrap*, *Modular Mousetrap* or $(HLM)^2N$, it generates a new deck, which corresponds to a reduced eulerian string or, equivalently, to a reformed deck. We can play again with this new deck, in order to check if it will win again.

Guy and Nowakowski firstly proposed the study of reformed decks in **[29]** and obtained only partial results, for *Mousetrap* and *Modular Mousetrap*.

K. S. Chua **[12]** achieved a substantial improvement only for the game *Mousetrap*. His results are quoted in **[52]** in the sequences A007711, A007712, A055459, A067950.

We have further improved their results, extending the sequences A007711, A007712, A055459, A067950 up to $m = 15$ for *Mousetrap*, but, first of all, obtaining very long sequences of reformed decks in the game *Modular Mousetrap* (for example, the deck

$$11 \ , \ 5 \ , \ 4 \ , \ 19 \ , \ 14 \ , \ 18 \ , \ 16 \ , \ 3 \ , \ 1 \ , \ 9 \ , \ 7 \ , \ 10 \ , \ 13 \ , \ 17 \ , \ 12 \ , \ 15 \ , \ 8 \ , \ 6 \ , \ 2$$

is 45 times reformable!) and lots of nontrivial cycles, i.e. sequences of reformed decks where, starting from a deck, after a sequence of reformed permutations, we arrive again at the original deck; for example, the deck

$$1 \ , \ 11 \ , \ 5 \ , \ 8 \ , \ 2 \ , \ 6 \ , \ 9 \ , \ 4 \ , \ 7 \ , \ 10 \ , \ 3 \ ,$$

after 137 reformations, enters in a cycle, because the 203-th reformed deck gives back the 137-th deck.

Moreover, the case $m = 7 \ , \ s = 2$ shows seven different 1-cycles, besides the trivial one (i.e. $\{12345671234567\}$).

The reason why we can obtain very long sequences and cycles in *Modular Mousetrap* must be found in the fact that, as already remarked, in this game, when n is prime, we have always a winning deck or a *derangement* and that the probability to yield a winning deck is very high (see Table 8). Consequently, in this case it is very easy to obtain, from a deck, a reformed one.

The complete list of the most important results on reformed decks, for *Mousetrap*, *Modular Mousetrap* and $(HLM)^2N$ will be published in another paper [**4**].

7. Conclusions and further developments

The ”eulerian” technique here introduced has proved to be very powerful for the study of the games *He Loves Me, He loves Me Not, Mousetrap* and *Modular Mousetrap.* Clearly, it can give only the number of winning decks, without any possibility to arrive at a closed formula. But the complexity of the game studied is so high that it is very hard to expect to find general closed formulas. In fact, as already remarked, only partial results have been obtained in the previous literature.

The contraindication of this ”eulerian” methods (which consists in rebuilding the winning decks starting from strings, of increasing length, formed by the last stored cards in the decks) is related to disk usage problems: in order to build all the strings of length $(k+1)$, the program needs to store all the strings of length k.

Even if we should not be interested on the storage of all the winning decks, but only on their number, it is however necessary to store all the winning $(n-1)$-strings. In the game *Mousetrap*, in the case $m = 15$, $s = 1$, the storage of all the winning n-strings needed a 42.3 GB memory, while the storage of all the winning $(n-1)$-strings needed a 70.6 GB memory.

Moreover, in the case of French cards, considering the growth rate of the number of winning cards at $(HLM)^2N$ when m grows, for $s = 4$, we should expect, in the most cautious estimate, at least 10^{24} winning French decks. Actually, a number absolutely unreasonable, for an actual PC.

Certainly, the usage of parallel computers or (as actually done playing $(HLM)^2N$ in the cases $m = 9$, $s = 3$ and $m = 7$, $s = 4$) the storage of all the k-strings in several data subfiles, which could be processed separately, could help the search of all the winning decks for higher and higher values of m and/or s.

Anyway, the importance of the technique consists firstly in having shown that (SC) is true at least for $m = 13$, $s = 4$ (but the test can be performed for much larger decks). The growth rate of the number of winning decks allows us to suppose that (SC) is true for every value of m and s, though the winning probability is decreasing with m. However this technique cannot give a definitive positive answer to (SC) for every value of m and s.

The results here obtained can be, obviously, extended, in particular by means of more powerful computers or by means of parallel calculus, to which the technique is absolutely adaptable. Less disk usage expensive computer methods or other improvements could help, too.

Many questions still need an answer. In particular, is it possible to find an estimate for the probability of winning decks in the three games and, in particular, in $(HLM)^2N$, which is the most difficult? We are able to find very rough upper bounds, by means of Čebishev and Markov inequalities, of the inequality (5.7), which can be improved and of the $P_{m\cdot s}(m\cdot s)$ values yielded for *Mousetrap* or *Modular Mousetrap.* A positive answer to the Strong Conjecture could be obtained finding a lower bound to P_{max}.

Moreover, is it possible to explain by a theoretical point of view the principal peaks in the frequency distribution of records in *Mousetrap* and $(HLM)^2N$?

Is it possible to find a distribution which can approximate the probability distribution of records and of stored cards in the three games here studied?

Some, though interlocutory, answers to this questions will be published in a paper in preparation.

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I will highly appreciate every information on improvements of the results reported in this paper and/or of the computer programs to obtain them.

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	$P(k)$	$Po_1(k)$	$Bi\left(13, \frac{1}{13}\right)(k)$
$k = 0$	0.3569	0.3679	0.3533
$k = 1$	0.3788	0.3679	0.3827
$k = 2$	0.1896	0.1839	0.1913
$k = 3$	$5.934 \cdot 10^{-2}$	$6.131 \cdot 10^{-2}$	$5.847 \cdot 10^{-2}$
$k = 4$	$1.295 \cdot 10^{-2}$	$1.533 \cdot 10^{-2}$	$1.218 \cdot 10^{-2}$
$k = 5$	$2.085 \cdot 10^{-3}$	$3.066 \cdot 10^{-3}$	$1.827 \cdot 10^{-3}$
$k = 6$	$2.547 \cdot 10^{-4}$	$5.109 \cdot 10^{-4}$	$2.030 \cdot 10^{-4}$
$k = 7$	$2.393 \cdot 10^{-5}$	$7.299 \cdot 10^{-5}$	$1.692 \cdot 10^{-5}$
$k = 8$	$1.730 \cdot 10^{-6}$	$9.124 \cdot 10^{-6}$	$1.057 \cdot 10^{-6}$
$k = 9$	$9.514 \cdot 10^{-8}$	$1.014 \cdot 10^{-6}$	$4.895 \cdot 10^{-8}$
$k = 10$	$3.872 \cdot 10^{-9}$	$1.014 \cdot 10^{-7}$	$1.632 \cdot 10^{-9}$
$k = 11$	$1.105 \cdot 10^{-10}$	$9.216 \cdot 10^{-9}$	$3.708 \cdot 10^{-11}$
$k = 12$	$1.986 \cdot 10^{-12}$	$7.680 \cdot 10^{-10}$	$5.151 \cdot 10^{-13}$
$k = 13$	$1.697 \cdot 10^{-14}$	$5.908 \cdot 10^{-11}$	$3.302 \cdot 10^{-15}$

Table 1

In this table we report the values of the probability distribution for the game $T2$ with $m = 13$, $s = 4$, given by formula (2.5), of the poissonian distribution Po_1 and of the binomial distribution $Bi\left(13, \frac{1}{13}\right)$. The corresponding values of the total variation distances are

$$d_{TV}(P, Po) \sim 1.659 \cdot 10^{-2} \ ; \ d_{TV}(P, Bi) \sim 5.635 \cdot 10^{-3}$$

	$P(k)$	$Po_4(k)$	$Bi\left(52, \frac{1}{13}\right)(k)$
$k = 0$	$1.623 \cdot 10^{-2}$	$1.832 \cdot 10^{-2}$	$1.557 \cdot 10^{-2}$
$k = 1$	$6.890 \cdot 10^{-2}$	$7.326 \cdot 10^{-2}$	$6.748 \cdot 10^{-2}$
$k = 2$	0.1442	0.1465	0.1434
$k = 3$	0.1982	0.1954	0.1992
$k = 4$	0.2013	0.1954	0.2033
$k = 5$	0.1611	0.1563	0.1627
$k = 6$	0.1058	0.1042	0.1062
$k = 7$	$5.855 \cdot 10^{-2}$	$5.954 \cdot 10^{-2}$	$5.814 \cdot 10^{-2}$
$k = 8$	$2.790 \cdot 10^{-2}$	$2.977 \cdot 10^{-2}$	$2.726 \cdot 10^{-2}$
$k = 9$	$1.161 \cdot 10^{-2}$	$1.323 \cdot 10^{-2}$	$1.110 \cdot 10^{-2}$
$k = 10$	$4.276 \cdot 10^{-3}$	$5.292 \cdot 10^{-3}$	$3.979 \cdot 10^{-3}$
$k = 11$	$1.406 \cdot 10^{-3}$	$1.925 \cdot 10^{-3}$	$1.266 \cdot 10^{-3}$
$k = 12$	$4.159 \cdot 10^{-4}$	$6.415 \cdot 10^{-4}$	$3.605 \cdot 10^{-4}$
$k = 13$	$1.114 \cdot 10^{-4}$	$1.974 \cdot 10^{-4}$	$9.243 \cdot 10^{-5}$
....			
$k = 51$	0	$5.987 \cdot 10^{-38}$	$7.415 \cdot 10^{-56}$
$k = 52$	$1.087 \cdot 10^{-50}$	$4.606 \cdot 10^{-39}$	$1.188 \cdot 10^{-58}$

Table 2

In this table we report the values of the probability distribution for the game $T3$ with $m = 13$, $s = 4$, given by formula (2.6), of the poissonian distribution Po_4 and of the binomial distribution $Bi\left(52, \frac{1}{13}\right)$. The corresponding values of the total variation distance are

$$d_{TV}(P, Po) \sim 1.413 \cdot 10^{-2} \ ; \ d_{TV}(P, Bi) \sim 4.592 \cdot 10^{-3}$$

	$P_{4m}(0)$	$Bi\left(4m\ ,\ \frac{1}{m}\right)(0)$
$m\ =\ 3$	0.00999	0.00771
$m\ =\ 4$	0.01187	0.01002
$m\ =\ 5$	0.01307	0.01153
$m\ =\ 6$	0.01390	0.01258
$m\ =\ 7$	0.01451	0.01335
$m\ =\ 8$	0.01497	0.01394
$m\ =\ 9$	0.01533	0.01440
$m\ =\ 10$	0.01562	0.01478
$m\ =\ 11$	0.01586	0.01509
$m\ =\ 12$	0.01606	0.01535
$m\ =\ 13$	0.01623	0.01557
....		
$m = 100$	0.01804	0.01795
$m = 500$	0.01824	0.01824

Table 3

Values of $P_{4m}(0)$ and of $Bi\left(4m\ ,\ \frac{1}{m}\right)(0) = \left(1 - \frac{1}{m}\right)^{4m}$ in $(HLM)^2N$, for $s = 4$ and for different values of m. Both sequences tend, for $m \to \infty$, to the limiting value $Po_4(0) = e^{-4} \sim 0.01832$.

	$s=2$	$s=3$	$s=4$
$m=2$	6 4/4 (3/3) $-$ SC immediate	20 7/7 (5/5) $-$ SC immediate	70 10/10 (7/7) $-$ SC immediate
$m=3$	90 9/10 (5/5) $-$ WC immediate	1680 16/16 (8/8) $-$ SC immediate	34650 22/22 (11/11) $-$ SC immediate
$m=4$	2520 17/18 (7/7) $-$ WC immediate	369600 28/28 (11/11) $-$ SC immediate	63063000 38/38 (15/15) $-$ SC immediate
$m=5$	113400 27/28 (9/9) $-$ WC immediate	168168000 43/43 (14/14) $-$ SC 355,932	$\sim 3.06 \cdot 10^{11}$ 58/58 (19/19) $-$ SC 14,461,409
$m=6$	7484400 40/40 (11/11) $-$ SC 4,530,195	$\sim 1.37 \cdot 10^{11}$ 61/61 (17/17) $-$ SC 123,289,316	$\sim 3.25 \cdot 10^{15}$ 82/82 (23/23) $-$ SC 314,429,118
$m=7$	681080400 54/54 (13/13) $-$ SC 62,241,794	$\sim 1.83 \cdot 10^{14}$ 82/82 (20/20) $-$ SC 7,332,146,168	$\sim 6.65 \cdot 10^{19}$ 110/110 (27/27) $-$ SC 63,227,020,954
$m=8$	$\sim 8.17 \cdot 10^{10}$ 70/70 (15/15) $-$ SC 4,152,727,936	$\sim 3.69 \cdot 10^{17}$ 106/106 (23/23) $-$ SC $\sim$ 147,000,000,000	$\sim 2.39 \cdot 10^{24}$ 139/142 (31/31) $-$ WC $\sim$ 264,386,000,000
$m=9$	$\sim 1.25 \cdot 10^{13}$ 88/88 (17/17) $-$ SC $\sim$ 90,000,000,000	$\sim 1.08 \cdot 10^{21}$ 131/133 (26/26) $-$ WC $\sim$ 255,000,000,000	$\sim 1.41 \cdot 10^{29}$ 172/178 (34/35) $>$ 207,000,000,000
$m=10$	$\sim 2.38 \cdot 10^{15}$ 106/108 (19/19) $-$ WC $>$ 600,000,000,000	$\sim 4.39 \cdot 10^{24}$ 154/163 (28/29) $>$ 81,000,000,000	$\sim 1.29 \cdot 10^{34}$ 205/218 (37/39) $>$ 217,000,000,000
$m=11$	$\sim 5.49 \cdot 10^{17}$ 128/130 (21/21) $-$ WC 92,800,000,000	$\sim 2.39 \cdot 10^{28}$ 184/196 (31/32) 36,700,000,000	$\sim 1.75 \cdot 10^{39}$ 224/262 (39/43) 2,000,000,000
$m=12$	$\sim 1.51 \cdot 10^{20}$ 139/154 (22/23) 12,000,000,000	$\sim 1.71 \cdot 10^{32}$ 204/232 (33/35) 2,000,000,000	$\sim 3.40 \cdot 10^{44}$ 273/310 (43/47) 1,000,000,000
$m=13$	$\sim 4.92 \cdot 10^{22}$ 158/180 (22/25) 2,000,000,000	$\sim 1.56 \cdot 10^{36}$ 235/271 (34/38) 5,000,000,000	$\sim 9.20 \cdot 10^{49}$ 305/362 (45/51) 4,000,000,000

Table 4 - BEST RECORDS IN $(HLM)^2N$ OBTAINED WITH MONTE CARLO METHODS

In each box of this table we report the number $N_{m \cdot s}$ of different decks; the ratio between the best record and C_{max}; the ratio between the best number of stored cards and the number predicted by (WC); the number of simulations performed before yielding the first winning deck or performed without yielding any winning deck. The number of simulations is given by the sum of the trials done by F. Scigliano and by myself, while I have no information about the number of simulations done by A. Pompili. The symbols SC and WC indicate respectively if we proved the strong or the weak conjecture.

	$s=2$	$s=3$	$s=4$
$m = 2$	$3/6$; $P_{max} = 0.5$	$4/20$; $P_{max} = 0.2$	$15/70$; $P_{max} \sim 0.21$
$m = 3$	$0/90$; $P_{max} = 0$	$4/1680$; $P_{max} \sim 0.0024$	$5/34650$; $P_{max} \sim 0.00014$
$m = 4$	$0/2520$; $P_{max} = 0$	$9/369,600$ $P_{max} \sim 0.000024$	$229/63,063,000$ $P_{max} \sim 0.0000036$
$m = 5$	$0/113400$; $P_{max} = 0$	$63/168,168,000$ $P_{max} \sim 0.000000375$	$10568/3.06 \cdot 10^{11}$ $P_{max} \sim 0.000000035$
$m = 6$	$1/7,484,400$ $P_{max} \sim 1.34 \cdot 10^{-7}$	$1177/1.37 \cdot 10^{11}$ $P_{max} \sim 0.000000009$	$1,212,483/3.25 \cdot 10^{15}$ $P_{max} \sim 3.73 \cdot 10^{-10}$
$m = 7$	$7/681,080,400$ $P_{max} \sim 1.00 \cdot 10^{-8}$	$36144/1.83 \cdot 10^{14}$ $P_{max} \sim 1.98 \cdot 10^{-10}$	$411,488,689/6.65 \cdot 10^{19}$ $P_{max} \sim 6.19 \cdot 10^{-12}$
$m = 8$	$8/8.17 \cdot 10^{10}$ $P_{max} \sim 9.79 \cdot 10^{-11}$	$1,677,968/3.69 \cdot 10^{17}$ $P_{max} \sim 4.54 \cdot 10^{-12}$	
$m = 9$	$105/1.25 \cdot 10^{13}$ $P_{max} \sim 8.40 \cdot 10^{-12}$	$127,255,522/1.08 \cdot 10^{21}$ $P_{max} \sim 1.18 \cdot 10^{-13}$	
$m = 10$	$656/2.38 \cdot 10^{15}$ $P_{max} \sim 2.76 \cdot 10^{-13}$		
$m = 11$	$6745/5.49 \cdot 10^{17}$ $P_{max} \sim 1.23 \cdot 10^{-14}$		
$m = 12$	$76823/1.51 \cdot 10^{20}$ $P_{max} \sim 5.07 \cdot 10^{-16}$		
$m = 13$	$986,994/4.92 \cdot 10^{22}$ $P_{max} \sim 2.00 \cdot 10^{-17}$		
$m = 14$	$17,175,636/1.86 \cdot 10^{25}$ $P_{max} \sim 9.23 \cdot 10^{-19}$		
$m = 15$	*under investigation*		

Table 5 - WINNING DECKS AT *HE LOVES ME HE LOVES ME NOT*

In each box we report the ratio between the number of winning decks and the total number of decks and the winning probability $P_{max} = P(C_{max})$.

	$s = 1$	winning deck(s)
$m = 2$	1/1 (1/1)	1 , 2
$m = 3$	3/4 (1/2 *and* 2/2)	three decks
$m = 4$	6/8 (3/3)	2 , 1 , 3 , 4
$m = 5$	9/13 (3/4)	2 , 5 , 1 , 4 , 3
$m = 6$	14/19 (4/5)	6 , 1 , 4 , 3 , 5 , 2
$m = 7$	18/26 (4/6)	3 , 7 , 1 , 5 , 2 , 6 , 4
$m = 8$	25/34 (5/7)	8 , 1 , 5 , 2 , 6 , 4 , 7 , 3
$m = 9$	31/43 (7/8)	4 , 1 , 2 , 6 , 9 , 7 , 3 , 8 , 5
$m = 10$	39/53 (6/9)	10 , 1 , 6 , 2 , 7 , 3 , 8 , 5 , 9 , 4
$m = 11$	47/64 (8/10 *and* 9/10)	six decks
$m = 12$	56/76 (7/11 *and* 10/11)	three decks
$m = 13$	67/89 (11/12)	two decks
$m = 14$	79/103 (12/13)	two decks
$m = 15$	93/118 (13/14)	two decks
$m = 16$	108/134 (14/15)	two decks
.....		

Table 6

In this table we report the ratio between the best record at $(HLM)^2N$ with one seed and C_{max} and the ratio between the number of stored cards and the number of cards satisfying (WC). In some cases it is possible to obtain the same best record with a different number of cards. When there is only one winning deck, we report it, in the third column.

	$s=1$	$s=2$	$s=3$	$s=4$
$m=2$	$1/2$; $P=0.5$ $[\mathbf{G-N}]$	$3/6$; $P=0.5$	$4/20$; $P=0.2$	$15/70$; $P\sim 0.21$
$m=3$	$2/6$; $P\sim 0.33$ $[\mathbf{G-N}]$	$12/90$; $P\sim 0.13$	$90/1680$; $P\sim 0.054$	$675/34650$; $P\sim 0.019$
$m=4$	$6/24$; $P=0.25$ $[\mathbf{G-N}]$	$147/2520$; $P\sim 0.058$	$5232/369,600$ $P\sim 0.014$	$210,069/63,063,000$ $P\sim 0.0033$
$m=5$	$15/120$ $P=0.125$ $[\mathbf{G-N}]$	$2322/113,400$ $P\sim 0.020$	$476,042/168,168,000$ $P\sim 0.0028$	$119,375,881/3.06\cdot 10^{11}$ $P\sim 0.00039$
$m=6$	$84/720$ $P\sim 0.12$ $[\mathbf{G-N}]$	$71629/7,484,400$ $P\sim 0.0096$	$111,660,352/1.37\cdot 10^{11}$ $P\sim 0.00081$	
$m=7$	$330/5040$ $P\sim 0.065$ $[\mathbf{G-N}]$	$2,214,258/681,080,400$ $P\sim 0.0033$		
$m=8$	$1812/40320$ $P\sim 0.045$ $[\mathbf{G-N}]$	$118,228,868/8.17\cdot 10^{10}$ $P\sim 0.0014$		
$m=9$	$9978/362,880$ $P\sim 0.027$ $[\mathbf{G-N}]$			
$m=10$	$65503/3,628,800$ $P\sim 0.018$ $[\mathbf{C-S}]$			
$m=11$	$449,719/39,916,800$ $P\sim 0.011$ $[\mathbf{C-S}]$			
$m=12$	$3,674,670/479,001,600$ $P\sim 0.0077$ $[\mathbf{C-S}]$			
$m=13$	$28,886,593/6,227,020,800$ $P\sim 0.0046$ $[\mathbf{C-S}]$			
$m=14$	$266,242,729/8.72\cdot 10^{10}$ $P\sim 0.0031$			
$m=15$	$2,527,701,273/1.31\cdot 10^{12}$ $P\sim 0.0019$			

Table 7 - WINNING DECKS AT *MOUSETRAP*

In each box we report the ratio between the number of winning decks and $N_{m\cdot s}$ and the winning probability $P := P_{M,m\cdot s}(m\cdot s)$. We indicate with **[G-N]** and with **[C-S]** the results already quoted respectively in **[29]** and in **[12]**, **[52]**.

	$s=1$	$s=2$	$s=3$	$s=4$
$m=2$	$1/2$; $P=0.5$ **[G-N]**	$5/6$; $P\sim 0.83$	$19/20$; $P=0.95$	$69/70$; $P\sim 0.986$
$m=3$	$4/6$; $P\sim 0.67$ **[G-N]**	$60/90$; $P\sim 0.67$	$1081/1680$; $P\sim 0.64$	$22898/34650$ $P\sim 0.66$
$m=4$	$9/24$; $P=0.375$ **[G-N]**	$1182/2520$; $P\sim 0.47$	$173,053/369,600$ $P\sim 0.47$	$29,642,185/63,063,000$ $P\sim 0.47$
$m=5$	$76/120$; $P\sim 0.633$ **[G-N]**	$63063/113,400$ $P\sim 0.56$	$86,636,303/168,168,000$ $P\sim 0.52$	
$m=6$	$190/720$; $P\sim 0.26$	$1,797,350/7,484,400$ $P\sim 0.24$		
$m=7$	$3186/5040$; $P\sim 0.632143$	$364,572,156/681,080,400$ $P\sim 0.54$		
$m=8$	$11351/40320$ $P\sim 0.28$			
$m=9$	$132,684/362,880$ $P\sim 0.37$			
$m=10$	$884,371/3,628,800$ $P\sim 0.24$			
$m=11$	$25,232,230/39,916,800$ $P\sim 0.632120561$			
$m=12$	$50,436,488/479,001,600$ $P\sim 0.11$			
$m=13$	$3,936,227,868/6,227,020,800$ $P\sim 0.632120559$ [**A002467**]			

Table 8 - WINNING DECKS AT *MODULAR MOUSETRAP*

In each box we report the ratio between the number of winning decks and $N_{m\cdot s}$ and the winning probability $P := P_{MM,m\cdot s}(m\cdot s)$. We indicate with **[G-N]** the results already quoted in [**29**].

The result corresponding to $m = 13$, $s = 1$ has been obtained subtracting the total number of *derangements* to the total number of decks, $n! = m!$ (because m is prime). We indicate it with [**A002467**].

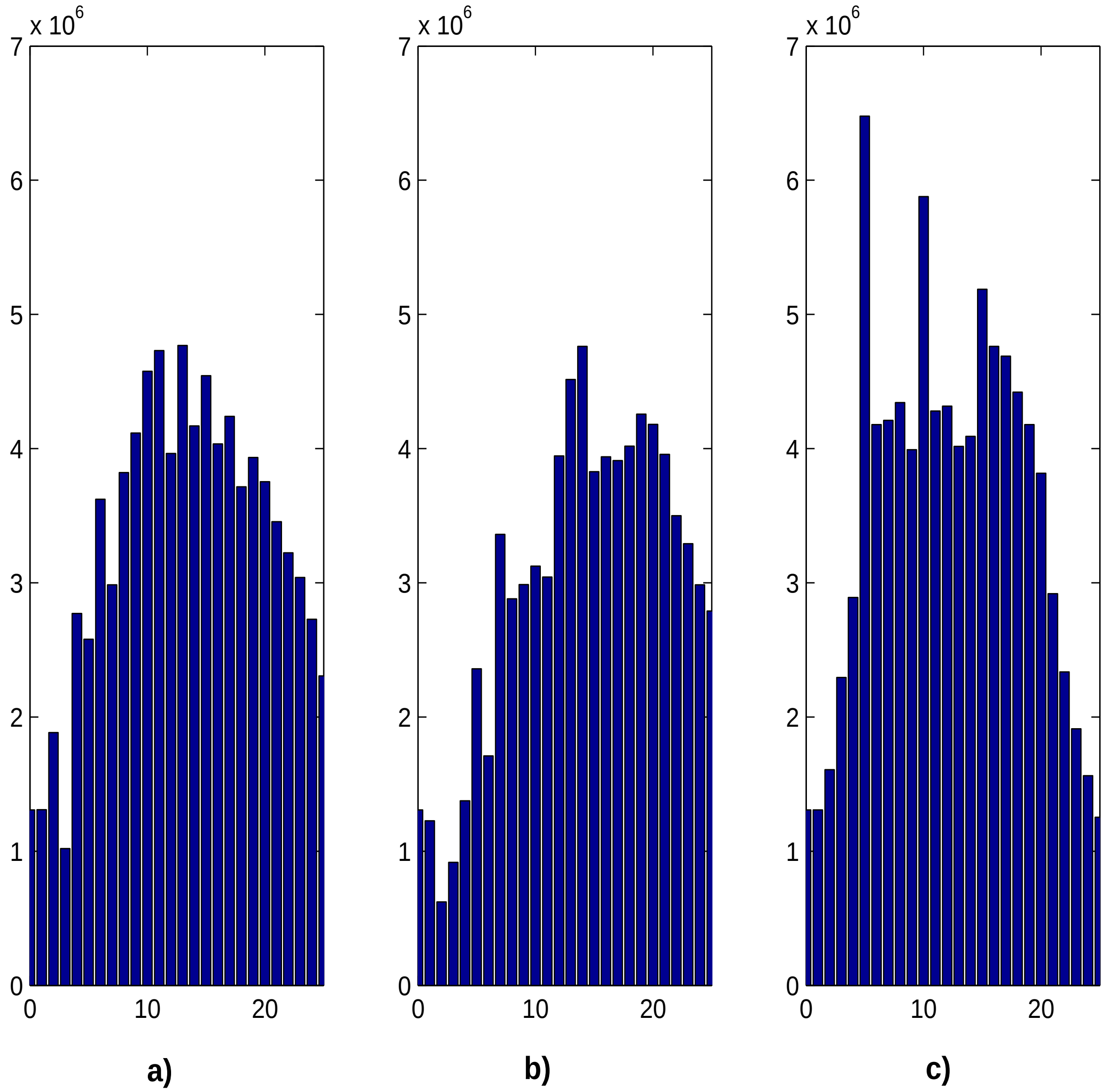

Fig. 1

In this figure we reproduce the frequency distribution of records in the game $(HLM)^2N$, obtained by means of numerical simulations, for $m = 5$, $s = 4$, for the three counting ways. In the case **c)**, the presence of some principal peaks (corresponding to the first three multiples of m) is evident. There are also secondary peaks, corresponding to values 8, 12, 46, which seem not to follow any particular rule. The presence of the principal peaks corresponding to the first multiples of m has been noted only when we use the counting method **c).**