SPECTRAL ESTIMATES FOR RIEMANNIAN AND ALMOST RIEMANNIAN SUBMERSIONS WITH FIBERS OF BASIC MEAN CURVATURE

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Abstract. We estimate the eigenvalues of the Laplace-Beltrami operator Δ of the total space M of a Riemannian submersion whose fibers have basic mean curvature vector field. The first Sobolev space of M splits in the L^2-orthogonal direct sum of the subspaces of the functions constant on each fiber, resp. of zero integral on the fibers. As these subspaces are invariant for Δ, its spectrum is the union of the spectra of Δ itself restricted to them. The small eigenvalues are related to eigenfunctions constant on the fibers. The first non-zero eigenvalue has a lower bound depending on the geometry of the basis of the submersion and on the volume of the fibers; when all the fibers are minimal submanifolds of M, the dependance on their (constant) volume disappears. For an almost Riemannian submersion, we modify the horizontal part of the metric of M to get a new metric which makes Riemannian the submersion. The min-max and max-min principles give a pinching of the eigenvalues of the almost Riemannian submersion by the eigenvalues of the Riemannian one.

1. The spectrum of a riemannian submersion with fibers of basic mean curvature vector.

Let $(M,g),(B,j)$ be two compact boundaryless Riemannian manifolds. A surjective C^∞ mapping $\pi : M \longrightarrow B$ is a *submersion* if its differential $(d\pi)_y : T_yM \longrightarrow T_{\pi(y)}B$ is a surjective mapping of maximal rank n =dim$B$ at any point $y \in M$. The fibers $F_x = \pi^{-1}(x),\ x \in B$, are regular p-dimensional manifolds of M $(p = m - n$, with m=dim$M > n)$, diffeomorphic to a model fiber F. A vector $X_y \in T_yM$ is *vertical* if it is tangent at y to the fiber $F_{\pi^{-1}(\pi(y))}$. The subspace $\mathcal{V}_y \subset T_yM$ of vertical vectors is the *vertical space*; the *horizontal space* $\mathcal{H}_y$ is the orthogonal complement of $\mathcal{V}_y$ in T_yM with respect to the metric g:

$$\mathcal{H}_y = \mathcal{V}_y^\perp \quad , \quad T_yM = \mathcal{V}_y \oplus \mathcal{H}_y \quad , \quad g(\mathcal{V}_y, \mathcal{H}_y) = 0.$$

The space $\mathcal{H}_y$ is naturally isomorphic to $T_{\pi(y)}$. The submersion $\pi : (M,g) \longrightarrow (B,j)$ is *Riemannian* if the restriction of its differential to horizontal vectors, $(d\pi)_y \restriction_{\mathcal{H}_y} : \mathcal{H}_y \longrightarrow T_{\pi(y)}B$, is an isometry:

$$j\left((d\pi)_y(X),(d\pi)_y(X)\right) = g(X,X)$$

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for any horizontal vector $X \in \mathcal{H}_y$ (see B. O'Neill [6]).

A horizontal vector field $X \in \Gamma(TM)$ is *basic* if it is projectable by π, i.e. if its image by the differential $(d\pi)_y$ is the same vector $\overline{X}_x \in T_xB$ for all points y in the fiber $F_x = \pi^{-1}(x)$, $x \in B$. In other words, X is basic if and only if it is the lift of a vector field $\overline{X} \in \Gamma(TB)$. The inner product of two basic vector fields X, Y is constant along any fiber F_x:

$$\langle X_y, Y_y \rangle = \langle (d\pi)_y(X), (d\pi)_y(Y) \rangle = \langle \overline{X}_x, \overline{Y}_x \rangle \tag{1.1}$$

(from now on, we shall write briefly $\langle\ ,\ \rangle$ to denote inner products). For any fixed $x \in B$, let us denote by g_x the restriction of the metric g to the fiber F_x, by v_{g_x} the induced canonical measure on F_x, and by $V(x)$ the corresponding volume of F_x, $V(x) = \int_{F_x} dv_{g_x}$. When X is the (basic) lift of $\overline{X} \in \Gamma(B)$, then for any function $f \in C^\infty(M)$ one has:

$$\overline{X}\left(\int_{F_x} f(y)dv_{g_x}(y)\right) = \int_{F_x} (Xf)_y dv_{g_x}(y) - \int_{F_x} f(y)\langle H_y, X_y\rangle dv_{g_x}(y) \tag{1.2}$$

where H_y is the *mean curvature vector* at $y \in F_x$ of the fiber F_x, i.e. the trace of the vectorial second fundamental form of F_x at y (cf. G. Besson [1]).

We shall assume in the sequel that H is basic, and denote $\overline{H}$ its projection.

(1.3) Lemma. *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with fibers of basic mean curvature vector field. Then the measure $\frac{v_{g_x}}{V(x)}$, $x \in B$, is invariant by the holonomy of the fibration.*

Proof. Notice that, when H is basic, 1.2 gives for the volume of the fibers:

$$\overline{X}\,(V(x)) = -\int_{F_x} \langle H_y, X_y \rangle dv_{g_x}(y) = -\langle \overline{H}_x, \overline{X}_x \rangle \cdot V(x). \tag{1.4}$$

From 1.4,1.2 and 1.1 we get, for any $f \in C^\infty(M)$:

$$\begin{aligned}
\overline{X}\left(\int_{F_x} f(y)\frac{dv_{g_x}(y)}{V(x)}\right) &= -\frac{\overline{X}(V(x))}{(V(x))^2}\int_{F_x} f(y)dv_{g_x}(y) + \frac{1}{V(x)}\overline{X}\left(\int_{F_x} f(y)dv_{g_x}(y)\right) \\
&= \frac{\langle \overline{H}_x, \overline{X}_x\rangle V(x)}{(V(x))^2} + \frac{1}{V(x)}\left(\int_{F_x} (Xf)_y dv_{g_x}(y) - \int_{F_x} f(y)\langle H_y, X_y\rangle dv_{g_x}(y)\right) \\
&= \int_{F_x} (Xf)_y \frac{dv_{g_x}(y)}{V(x)}. \qquad \square
\end{aligned}$$

Let us define $\mathcal{E}_c, \mathcal{E}_0$ to be the subspaces of the Sobeolev space $H^1(M)$ consisting of the functions $f \in H^1(M)$ which are constant on the fibers, respectively of zero average on the fibers:

$$\begin{aligned}
&\mathcal{E}_c = \{f : M \longrightarrow \mathbb{R},\ f \in H^1(M)|\ f = u \circ \pi \text{ with } u : B \longrightarrow \mathbb{R}\}, \\
&\mathcal{E}_0 = \{h : M \longrightarrow \mathbb{R},\ h \in H^1(M)|\ \int_{F_x} h(y)dv_{g_x}(y) = 0\}.
\end{aligned} \tag{1.5}$$

(1.6) Theorem. *Let $\pi : (M, g) \longrightarrow (B, j)$ be a Riemannian submersion with fibers of basic mean curvature vector field. Then:*

(1) *the space $H^1(M)$ splits into the direct sum $H^1(M) = \mathcal{E}_c \oplus \mathcal{E}_0$;*
(2) *the decomposition in (1) is simultaneously L^2-orthogonal and q-orthogonal, where q is the quadratic form*

$$q(f) := \int_M |\nabla f|_g^2 dv_g;$$

(∇f denotes the gradient of f);
(3) *the spaces $\mathcal{E}_c$ and $\mathcal{E}_0$ are stable under the action of the Laplace-Beltrami operator Δ_M,*

$$\Delta_c := \Delta_M|_{\mathcal{E}_c} : \mathcal{E}_c \longrightarrow \mathcal{E}_c \quad \textit{and} \quad \Delta_0 := \Delta_M|_{\mathcal{E}_0} : \mathcal{E}_0 \longrightarrow \mathcal{E}_0,$$

hence

$$Spec(\Delta_M) = Spec(\Delta_c) \cup Spec(\Delta_0);$$

(4) *the heat operator on M splits: $\exp^{-t\Delta_M} = \exp^{-t\Delta_c} \oplus \exp^{-t\Delta_0}$ (in the sense that, if $f = u \circ \pi + h \in \mathcal{E}_c \oplus \mathcal{E}_0$, then $\exp^{-t\Delta_M}(f) = \exp^{-t\Delta_c}(u \circ \pi) + \exp^{-t\Delta_0}(h)$, and thus*

$$Trace\left(\exp^{-t\Delta_M}\right) = Trace\left(\exp^{-t\Delta_c}\right) + Trace\left(\exp^{-t\Delta_0}\right).$$

Proof of (1). For any $f \in H^1(M)$, define $u : B \longrightarrow \mathbb{R}$ by

$$u(x) = \frac{1}{V(x)} \int_{F_x} f(y) dv_{g_x}(y).$$

Then $u \circ \pi : M \longrightarrow \mathbb{R}$ is constant along the fibers by definition. We are going to show that $u \circ \pi$ has H^1-norm bounded, so its belongs to $\mathcal{E}_c$.

Fubini's property and Cauchy-Schwarz inequality give:

$$\begin{aligned}
\|u \circ \pi\|_{L^2(M)}^2 &= \int_B \left(\int_{F_x} (u \circ \pi)^2(y) dv_{g_x}(y) \right) dv_j(x) \\
&= \int_B \left(u^2(x) \int_{F_x} dv_{g_x}(y) \right) dv_j(x) \\
&= \int_B \frac{1}{V(x)} \left(\int_{F_x} f(y) dv_{g_x}(y) \right)^2 dv_j(x) \\
&\leq \int_B \frac{1}{V(x)} \left(\int_{F_x} f^2(y) dv_{g_x}(y) \cdot \int_{F_x} dv_{g_x}(y) \right) dv_j(x) \\
&= \int_B \left(\int_{F_x} f^2(y) dv_{g_x}(y) \right) dv_j(x) \\
&= \|f\|_{L^2(M)}^2.
\end{aligned}$$

To compute the differential of $u \circ \pi$, let $\overline{e}_1, \dots, \overline{e}_n$ be a local orthonormal basis of TB and let $e_1, \dots, e_n$ be their horizontal lifts. Consider also a local orthonormal basis $v_1, \dots, v_p$, $p = m - n$, of vertical vectors (i.e. tangent to the fibers). As $u \circ \pi$ is constant along the fibers, all the derivatives $v_h(u \circ \pi)$ vanish. For any basic vector field X with projection $\pi_* X = \overline{X}$, one has by 1.1,1.2,1.4:

$$
\begin{aligned}
X(u \circ \pi) = \overline{X}(u) &= \frac{\left(\overline{X}\left(\int_{F_x} f(y) dv_{g_x}(y)\right)\right) \cdot V(x) - \left(\int_{F_x} f(y) dv_{g_x}(y)\right) \cdot \overline{X}(V(x))}{(V(x))^2} \\
&= \frac{1}{(V(x))^2} \cdot \left(\int_{F_x} (Xf)_y dv_{g_x}(y) - \int_{F_x} f(y) \langle H_y, X_y \rangle dv_{g_x}(y)\right) \cdot V(x) \\
&+ \frac{1}{(V(x))^2} \cdot \left(\int_{F_x} f(y) dv_{g_x}(y) \cdot \int_{F_x} \langle H_y, X_y \rangle dv_{g_x}(y) \cdot V(x)\right) \\
&= \frac{1}{V(x)} \cdot \int_{F_x} (Xf)_y dv_{g_x}(y).
\end{aligned}
$$

Then, for any $i = 1, \dots, n$, one has

$$
(d(u \circ \pi))^i = e_i(u \circ \pi) = \overline{e}_i(u) = (du)^i = \frac{1}{V(x)} \cdot \int_{F_x} (df)^i(y) dv_{g_x}(y),
$$

thus

$$
\begin{aligned}
|d(u \circ \pi)|^2 &= \sum_{i=1}^n \left((d(u \circ \pi))^i\right)^2 = \sum_{i=1}^n \left((du)^i\right)^2 = |du|^2 \\
&= \frac{1}{(V(x))^2} \sum_{i=1}^n \left(\int_{F_x} (df)^i(y) dv_{g_x}(y)\right)^2 \\
&\leq \frac{1}{(V(x))^2} \sum_{i=1}^n \int_{F_x} \left((df)^i(y)\right)^2 dv_{g_x}(y) \cdot \int_{F_x} dv_{g_x}(y) \\
&= \frac{1}{V(x)} \sum_{i=1}^n \int_{F_x} \left((df)^i(y)\right)^2 dv_{g_x}(y) \\
&\leq \frac{1}{V(x)} \left(\sum_{i=1}^n \int_{F_x} \left((e_i f)_y\right)^2 dv_{g_x}(y) + \sum_{j=1}^p \int_{F_x} \left((v_j f)_y\right)^2 dv_{g_x}(y)\right) \\
&= \frac{1}{V(x)} \int_{F_x} |(df)(y)|^2 dv_{g_x}(y).
\end{aligned}
$$

An integration on M gives, again via Fubini's property and Cauchy-Schwarz inequality:

$$
\|d(u \circ \pi)\|^2_{L^2(M)} \leq \|df\|^2_{L^2(M)}.
$$

After all, we have shown that

$$
\|u \circ \pi\|^2_{H^1(M)} \leq \|f\|^2_{H^1(M)}
$$

and so that $u \circ \pi \in H^1(M)$. As $u \circ \pi$ is constant along the fibers, it belongs to $\mathcal{E}_c$.

A straightforward computation shows that $f - u \circ \pi$ belongs to $\mathcal{E}_0$, i.e. that its integral on each fiber is equal to 0.

It remains to show that the sum $H^1(M) = \mathcal{E}_c + \mathcal{E}_0$ is a direct sum: this shall follows from (2) or also from the fact that the space $\mathcal{E}_c$, restricted to each fiber F_x, is $L^2(F_x)$-orthogonal to the space of the functions with integral on F_x equal to 0.

Proof of (2). $\mathcal{E}_c$ and $\mathcal{E}_0$ are L^2-orthogonal:

$$\begin{aligned}
\langle\langle u \circ \pi, h\rangle\rangle_{L^2(M)} &= \int_M (u \circ \pi)(y) h(y) dv_g(y) \\
&= \int_B \left(\int_{F_x} (u \circ \pi)(y) h(y) dv_{g_x}(y) \right) dv_j(x) \\
&= \int_B \left(u(x) \cdot \int_{F_x} h(y) dv_{g_x}(y) \right) dv_j(x) \\
&= 0.
\end{aligned}$$

$\mathcal{E}_c$ and $\mathcal{E}_0$ are q-orthogonal: let us call q also the symetric bilinear form associated to q,

$$q(f_1, f_2) = \int_M \langle \nabla f_1, \nabla f_2 \rangle = \int_M \langle df_1, df_2 \rangle.$$

Notice first of all that the vector field $\nabla(u \circ \pi)$ is basic and its projection is ∇u:

$$\nabla(u \circ \pi) = \sum_{i=1}^n (e_i(u \circ \pi)) e_i + \sum_{h=1}^p (v_h(u \circ \pi)) v_h = \sum_{i=1}^n (\overline{e}_i(u)) e_i,$$

hence

$$\pi_*(\nabla(u \circ \pi)) = \sum_{i=1}^n (\overline{e}_i(u \circ \pi)) \pi_*(e_i) = \sum_{i=1}^n (\overline{e}_i(u \circ \pi)) \overline{e}_i = \nabla u.$$

Therefore one has by 1.2 and 1.1 (D_X denotes here the directional derivative with respect to X):

$$\begin{aligned}
0 = D_{\nabla u} \left(\int_{F_x} h \right) &= \int_{F_x} D_{\nabla(u\circ\pi)} h - \int_{F_x} \langle D_{\nabla(u\circ\pi)}, H \rangle h \\
&= \int_{F_x} D_{\nabla(u\circ\pi)} h - \langle D_{\nabla u}, \overline{H} \rangle \int_{F_x} h \\
&= \int_{F_x} D_{\nabla(u\circ\pi)} h.
\end{aligned}$$

On the other hand it is

$$\begin{aligned}
\langle \nabla(u \circ \pi), \nabla h \rangle &= \sum_{i=1}^n (\nabla(u \circ \pi))^i (\nabla h)^i = \sum_{i=1}^n e_i(u \circ \pi) \cdot e_i(h) \\
&= \sum_{i=1}^n \left(e_i(u \circ \pi) \cdot e_i \right)(h) \\
&= D_{\nabla(u\circ\pi)} h.
\end{aligned}$$

After all, we have

$$q(u\circ\pi,h)=\int_M\langle\nabla(u\circ\pi),\nabla h\rangle=\int_B\left(\int_{F_x}D_{\nabla(u\circ\pi)}h\right)=0.$$

Proof of (3). This is a consequence of (2) and of the fact that q is the quadratic form associated to the symetric operator Δ_M, thus any orthonormal Hilbertian complete basis diagonalizing q diagonalizes also Δ_M. Namely, if

$\{\varphi_i\}_{i\in I}$ is a q-orthonormal Hilbertian basis of $\mathcal{E}_c$,

$\{\psi_j\}_{j\in J}$ is a q-orthonormal Hilbertian basis of $\mathcal{E}_0$,

then $\{\varphi_i,\psi_j\}_{i\in I,j\in J}$ is a q-orthonormal Hilbertian basis of $H^1(M)$ consisting of eigenfunctions of Δ_M:

$$\Delta_M\varphi_i=\lambda_i\varphi_i\quad,\quad\Delta_M\psi_j=\mu_j\psi_j.$$

Therefore, for $u\circ\pi\in\mathcal{E}_c$ it is $u\circ\pi=\sum_{i\in I}a_i\varphi_i$, that implies $\Delta_M(u\circ\pi)=\sum_{i\in I}a_i\lambda_i\varphi_i\in\mathcal{E}_c$, i.e. $\Delta_c=\Delta_M|_{\mathcal{E}_c}$ acts stably on $\mathcal{E}_c$. In the same way, one sees that $\Delta_0=\Delta_M|_{\mathcal{E}_0}$ acts stably on $\mathcal{E}_0$.

Proof of (4). The splitting of the heat operator is an immediate consequence of (3). The result on the trace is also trivial, it is only to write

$$Trace(e^{-t\Delta_M})=\sum_{i\in I}e^{-\lambda_i t}+\sum_{j\in J}e^{-\mu_j t}.\qquad\square$$

For any fixed $x\in B$, denote $\lambda_1(F_x)=\lambda_1(F_x,g_x)$ the first non-zero eigenvalue of the Laplace-Beltrami operator Δ_{F_x} of the fiber F_x.

(1.7) Corollary. *Any eigenvalue $\lambda_j\in$Spec(Δ_M) belonging to Spec(Δ_0) satisfies*

$$\lambda_j\geq\Lambda:=\inf_{x\in B}\lambda_1(F_x).$$

Proof. For any $f\in C^\infty(M)$, the gradient $\nabla^{F_x}f$ of the restriction of f to the fiber F_x (the vertical part of ∇f) verifies

$$|\nabla f|_y^2\geq|\nabla^{F_x}f|_y^2$$

at any $y\in F_x$.

Suppose now that $f\in\mathcal{E}_0$, i.e. that f is orthogonal to the functions constant on the fibers, then

$$\begin{aligned}\int_{F_x}|\nabla^{F_x}f|_y^2dv_{g_x}(y)&=\int_{F_x}\langle\Delta_{F_x}f,f\rangle_y dv_{g_x}(y)\geq\lambda_1(F_x,g_x)\int_{F_x}f^2(y)dv_{g_x}(y)\\&\geq\Lambda\int_{F_x}f^2(y)dv_{g_x}(y)\end{aligned}$$

by the first variational principle.

When $f \in \mathcal{E}_0$ is an eigenfunction of Δ_M related to the eigenvalue $\lambda_j \in$Spec(Δ_0), $\Delta_M f = \Delta_0 f = \lambda_j f$, the previous inequalities and Fubini's property give

$$\begin{aligned}\lambda_j \int_M f^2(y) dv_g(y) &= \int_M \langle \Delta_M f, f\rangle_y dv_g(y) = \int_M |\nabla f|_y^2 dv_g(y)\\ &= \int_B \left(\int_{F_x} |\nabla f|_y^2 dv_{g_x}(y) \right) dv_j(x)\\ &\geq \int_B \left(\int_{F_x} |\nabla^{F_x} f|_y^2 dv_{g_x}(y) \right) dv_j(x)\\ &\geq \Lambda \int_B \left(\int_{F_x} f^2(y) dv_{g_x}(y) \right) dv_j(x)\\ &= \Lambda \int_M f^2(y) dv_g(y). \qquad \square\end{aligned}$$

It follows that if λ_j is an eigenvalue of (Δ_M) such that $\lambda_j < \Lambda$, then $\lambda_j \in$Spec(Δ_c). Thus, in order to estimate "small" eigenvalues of Δ_M, it suffices to estimate the eigenvalues of Δ_c, where "small" means less than a lower bound of Λ.

In some cases, one can find such a lower bound. For instance, assume that the fibers F_x are diffeomorphic to a fiber-type F, endowed with a reference metric g_F, and that the metric on F_x is $g_x = (b(x))^2 g_F$ (typical example: manifolds of revolution). Then the min-max principle gives

$$\lambda_i(F_x, g_x) = \frac{1}{(b(x))^2} \lambda_i(F, g_F),$$

in particular $\lambda_1(F_x, g_x) = \frac{1}{(b(x))^2}\lambda_1(F, g_F) \geq \frac{1}{\sup_{x\in B}(b(x))^2}\lambda_1(F, g_F)$.

(1.8)Example. Assume that F_x is a closed curve, i.e. F_x diffeomorphic to S^1, and that the metric on F_x is $g_x = (b(x))^2 d\theta^2$, where $d\theta^2$ is the canonical metric of S^1. As the length of F_x is $\ell(x) = \ell(F_x) = \int_0^{2\pi} b(x) d\theta = 2\pi b(x)$, one has $b(x) = \frac{\ell(x)}{2\pi}$ and thus

$$\lambda_1(F_x, g_x) = \frac{4\pi^2}{(\ell(x))^2} \geq \frac{4\pi^2}{\sup_{x\in B}(\ell(x))^2}$$

(recall that $\lambda_1(S^1, d\theta^2) = 1$).

Another situation in which it is possible to find a lower bound of Λ is when the fibers F_x have Ricci curvature bounded from below by $-(p-1)k^2$, where p is the dimension of the fibers, and diameter bounded from above by D: in this case one get

$$\lambda_1(F_x) \geq \Gamma(p, k, D)$$

where Γ is an explicit constant, see P. Li and S.T. Yau [5] and S. Gallot [3].

2. Estimates of the spectrum of Δ_c.

Let us consider, as before, a Riemannian submersion $\pi : (M, g) \longrightarrow (B, j)$ with basic mean curvature, and the subspace $\mathcal{E}_c \subset H^1(M)$ of the functions $f = u \circ \pi$ constant on the fibers. A staightforward computation gives:

(2.1) Proposition. *The mapping* $\mathcal{E}_c \longrightarrow H^1(B)$ *which maps the function* $f = u \circ \pi$ *onto the function* u *is bijective and maps:*

(1) *the* L^2*-norm* $\|f\|^2_{L^2(M)} = \int_M f^2(y) dv_g(y)$ *onto the quadratic form*

$$\|u\|_0^2 = \int_B V(x) u(x)^2 dv_j(x);$$

(2) *the quadratic form* $q(f) = \int_M |\nabla f|^2_y dv_g(y)$ *onto the quadratic form*

$$q_0(u) = \int_B V(x) |\nabla u|^2_x dv_j(x).$$

Proof. The bijectivity of the linear mapping $u \circ \pi \longmapsto u$ is obvious. The claim (i) is an immediate consequence of Fubini's property:

$$\begin{aligned}\|u \circ \pi\|^2_{L^2(M)} &= \int_M (u \circ \pi)^2(y) dv_g(y) = \int_B \left(\int_{F_x} (u \circ \pi)^2(y) dv_{g_x}(y) \right) dv_j(x) \\ &= \int_B \left(u^2(x) \int_{F_x} dv_{g_x}(y) \right) dv_j(x) = \int_B u^2(x) V(x) dv_j(x) = \|u\|_0^2.\end{aligned}$$

For (ii), we have shown in the proof of Theorem 1.6,(2), that $\nabla(u \circ \pi)$ is basic and that its projection by π_* is ∇u. As π_* restricted to horizontal vectors is an isometry, we have $|\nabla(u \circ \pi)|_y = |\nabla u|_x$ for any $y \in M$ and $x = \pi(y)$. Then, the claim follows again by Fubini's property. □

Let us call $R(u)$ the Rayleigh quotient of a function u with respect to the canonical L^2-norm and $R_0(u)$ the Rayleigh quotient of u of the quadratic form q_0 with respect to the L^2_0-norm:

$$R_0(u) = \frac{q_0(u)}{\|u\|_0^2} = \frac{\int_B V(x) |\nabla u|^2_x dv_j(x)}{\int_B V(x) u(x)^2 dv_j(x)}.$$

Assume that the volume $V(x) = Vol(F_x)$ of the fibers is bounded when x ranges over B: $0 < V_0 \leq V(x) \leq V_1 < +\infty$. As $q(f) = q_0(u) \geq V_0 q(u)$ and $\|f\|^2_{L^2(M)} = \|u\|_0^2 \leq V_1 \|u\|^2_{L^2(B)}$, one has

$$R(f) = R_0(u) \geq \frac{V_0}{V_1} R(u) \quad \text{and} \quad R(f) = R_0(u) \leq \frac{V_1}{V_0} R(u).$$

Therefore, $\lambda_i(\Delta_c)$ is equal to the i-th eigenvalue of the diagonalization of q_0 with respect to the L^2_0-norm. Moreover, the min-max and max-min principle give:

$$\frac{V_0}{V_1} \lambda_i(B) \leq \lambda_i(\Delta_c) \leq \frac{V_1}{V_0} \lambda_i(B). \tag{2.2}$$

In a similar way, from $R(f) = R_0(u) \geq \frac{q_0(u)}{V_1 \|u\|^2_{L^2(B)}}$, it follows

$$\lambda_i(\Delta_c) \geq \frac{1}{V_1} \lambda_i(q_0) \tag{2.3}$$

where now q_0 is diagonalized with respect to the canonical L^2-norm and no more with respect to the L^2_0-norm.

(2.4) Theorem. *Let $\pi : (M,g) \longrightarrow (B,j)$ be a Riemannian submersion with fibers of basic mean curvature vector field. The first non-zero eigenvalue of Δ_M verifies:*

$$\lambda_1(\Delta_M) \geq \frac{\Gamma^{-1}}{V_1}\left(\int_B V(x)^{-\frac{n}{2}} dv_j(x)\right)^{-\frac{2}{n}}$$

where $\Gamma = \Gamma(n,k,D,V)$ is a positive constant depending on the geometry of B: n is the dimension, k is a lower bound of the Ricci curvature, $Ric \geq -(n-1)k$, D is the diameter, and V is the volume; $V(x)$ is the volume of the fiber F_x, and V_1 is its lower bound.

Notice that this estimate is optimal by its dependance on the power of $\frac{1}{V}$, cf. S. Gallot and D. Meyer [4].

Proof. S. Gallot showed ([2], Corollary 2.6) that for any $u \in H^1(B)$ orthogonal to the constant functions (the eigenvalue 0 corresponds to the constant functions), it holds the following Sobolev inequality:

$$\int_B u(x)^2 dv_j(x) \leq \Gamma\left(\int_B |\nabla u|_x^{\frac{2n}{n+2}} dv_j(x)\right)^{\frac{n+2}{n}} ;$$

the explicit value of Γ is also given in [2]. Hölder inequality gives (remember that $V(x)$ is the volume of the fiber F_x):

$$\begin{aligned}\int_B |\nabla u|_x^{\frac{2n}{n+2}} dv_j(x) &= \int_B |\nabla u|_x^{\frac{2n}{n+2}} V(x)^{\frac{n}{n+2}} \frac{1}{V(x)^{\frac{n}{n+2}}} dv_j(x)\\ &\leq \left(\int_B |\nabla u|_x^2 V(x) dv_j(x)\right)^{\frac{n}{n+2}} \cdot \left(\int_B \frac{1}{V(x)^{\frac{n}{2}}} dv_j(x)\right)^{\frac{2}{n+2}} .\end{aligned}$$

Injecting this in the Sobolev inequality we get

$$\begin{aligned}\int_B u(x)^2 dv_j(x) &\leq \Gamma \cdot \int_B |\nabla u|_x^2 V(x) dv_j(x) \cdot \left(\int_B V(x)^{-\frac{n}{2}} dv_j(x)\right)^{\frac{2}{n}}\\ &= \Gamma q_0(u) \cdot \left(\int_B V(x)^{-\frac{n}{2}} dv_j(x)\right)^{\frac{2}{n}} .\end{aligned}$$

Hence

$$\lambda_1(q_0) = \inf_u \frac{q_0(u)}{\|u\|_{L^2}^2} \geq \Gamma^{-1} \cdot \left(\int_B V(x)^{-\frac{n}{2}} dv_j(x)\right)^{-\frac{2}{n}}$$

(u orthogonal to constants) which, together with 2.3, gives the claim:

$$\lambda_1(\Delta_M) = \lambda_1(\Delta_c) \geq \frac{1}{V_1}\Gamma^{-1} \cdot \left(\int_B V(x)^{-\frac{n}{2}} dv_j(x)\right)^{-\frac{2}{n}} . \qquad \square$$

(2.5)Remark. When all the fibers of the submersions are *minimal submanifolds* of M, which means $H \equiv 0$, (1.4) shows that they have the same volume: $V(x) =$ *constant*. Then the inequality of Theorem 2.4 takes the simplified form:

(2.6) $$\lambda_1(\Delta_M) \geq C$$

where the constant $C = C(n,k,D,V)$ is equal to $\Gamma^{-1}V^{-\frac{2}{n}}$.

3. Approximating an almost riemannian submersion by a riemannian one.

Let $(M, g'), (B, j)$ be two compact boundaryless Riemannian manifolds and let $\pi : M \longrightarrow B$ be a submersion. We shall say that $\pi : (M, g') \longrightarrow (B, j)$ is an *almost-Riemannian submersion* when the restriction of g' to horizontal vectors is an almost isometry, i.e. there exist two real constants a and b (not depending on the point $y \in M$) such that

$$a^2 j\left((d\pi)_y(X), (d\pi)_y(X)\right) \leq g'(X, X) \leq b^2 j\left((d\pi)_y(X), (d\pi)_y(X)\right) \tag{3.1}$$

for any horizontal vector $X \in H_y$.

Define on M the Riemannian metric g at any $y \in M$ by $g_y|_{V_y} = g'_y|_{V_y}$, by $g_y(X, V) = 0$ for any horizontal X and vertical V, and by $g_y(X, Y) = (\pi^* j_x)(X, Y) = j_x\left((d\pi)_y(X), (d\pi)_y(Y)\right)$ for any horizontal X and Y. In substance, the metric g preserves the orthogonality between vertical and horizontal vectors, it reduces to the old metric g' for vertical vectors, and it is the metric of the basis B for horizontal vectors. Then $\pi : (M, g) \longrightarrow (B, j)$ is a Riemannian submersion and

$$a^2 g(X, X) \leq g'(X, X) \leq b^2 g(X, X) \tag{3.2}$$

for any horizontal X.

Let us denote by $\{\lambda_i(M, g)\}_{i=0,1,2,\dots}$ the spectrum of the Laplace-Beltrami operator $\Delta_{(M,g)}$ and so on (each eigenvalue is repeated according to its multiplicity).

(3.3) Proposition. *The eigenvalues of $\Delta_{(M,g)}$ and of $\Delta_{(M,g')}$ satisfy, for any $i = 0, 1, 2, \dots$:*

$$\frac{b^m}{a^{m+2}} \lambda_i(M, g) \geq \lambda_i(M, g') \geq \frac{a^m}{b^{m+2}} \lambda_i(M, g)$$

where m =dim(M).

Proof. Let $g^*, g^{'*}$ be the dual metrics of g, g' resp. and let $v_g, v_{g'}$ be the canonical measures induced by g, g' resp. We have:

$$\frac{1}{b^2} g^* \leq g^{'*} \leq \frac{1}{a^2} g^* \quad \text{and} \quad a^m v_g \leq v_{g'} \leq b^m v_g,$$

hence

$$\frac{a^m}{b^{m+2}} \frac{\int_M |du|^2_{g^*} dv_g}{\int_M u^2 dv_g} \leq \frac{\int_M |du|^2_{g'^*} dv_{g'}}{\int_M u^2 dv_{g'}} \leq \frac{b^m}{a^{m+2}} \frac{\int_M |du|^2_{g^*} dv_g}{\int_M u^2 dv_g}$$

for any function $u \in H^1(M) \setminus \{0\}$ (notice that the Sobolev spaces $H^1(M, g)$ and $H^1(M, g')$ concide). Then the claim follows by min-max and max-min principles. $\Box$

Proposition 3.3 implies for the traces of the heat kernels the following inequality:

$$Z_{(M,g')}(t) \leq Z_{(M,g)}\left(\frac{a^m}{b^{m+2}} t\right) \tag{3.4}$$

for any positive t, where $Z_{(M,g')}(t) = \sum_{i=0}^{+\infty} \exp -\lambda_i(M, g')t$ and so on.

We should also consider the case when π is an almost-Riemannian submersion only outside a subset A of zero capacity in B. It is not hard to see that cap(A)=0 implies cap($\pi^{-1}(A)$)=0. *A priori* the capacity of a subset A' in M depends on the metric, but when g and g' are almost isometric according to 3.2 the capacity of A' with respect to g is equal to zero if and only if the capacity of A' with respect to g' is equal to zero. Then, we could settle and prove the analogous of 3.3 and 3.4 for the Neumann problem on $M \setminus \pi^{-1}(A)$.

References

[1]. G. Besson, *A Kato type inequality for Riemannian submersions with totally geodesic fibers*, Annals of Global Analysis and Geometry **4, n. 3** (1986), 273–289.

[2]. S. Gallot, *Inégalités isopérimétriques, courbure de Ricci et invariants géométriques I*, C.R.Acad.Sc. Paris **Série I, t.296** (1983), 333–336.

[3]. S. Gallot, *Inégalités isopérimétriques at analytiques sur les variétés riemanniennes*, Astérisque **163–164** (1988), 33–91.

[4]. S. Gallot, D. Meyer, *D'un résultat hilbertien à un principe de comparaison entre spectre. Applications*, Ann. Sci. Éc. Norm. Supér. **IV. Sér. 21** (1988), 561–591.

[5]. P. Li, S.T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Sympos. Pure Math. **36** (1980), 205–239.

[6]. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.

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