Arc-coloring of directed hypergraphs and chromatic number of walls

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Abstract

We define an arc-coloring for directed hypergraphs, such that any two arcs having either intersecting tails or the same head must be colored differently. We investigate the arc-coloring of those hypergraphs which can be represented by suitable adjacency matrices (*walls*), whereas a polynomial reduction is provided from the general arc-coloring problem to the *brick-coloring* of walls. An upper bound for the least number of required colors with fixed degree is established for a subclass of hypergraphs. Some particular walls are subsequently analyzed.

Key-words: arc-coloring, brick, chromatic index, directed hypergraph, wall.

2000 MSC: 05B99, 05C15, 05C20, 68Q17.

1 Introduction

There are several ways of introducing the notion of direction for the edges of a hypergraph. For example, in [8] a directed hypergraph is obtained from a hypergraph H, by partitioning every edge of H into two sets of vertices, namely the *tail* and the *head* of the edge. In some cases directed hypergraph have been given a different name, such as *labelled graphs* or *And-Or graphs* [9, 10]. In the present context we use the following notion [1, 2].

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Definition 1.1. A *directed hypergraph* $H = (V(H), E(H))$ consists of a set V, the *nodes*, together with a set $E \subseteq \mathcal{P}(V) \times V$. Every element $\varepsilon = (A, z) \in E$ is a *hyperarc* (or simply an *arc*). A and z are respectively the *tail* and the *head* of ε; we write $A = tail(\varepsilon)$, $z = head(\varepsilon)$. The *size* of H is $|H| = \sum_{\varepsilon \in E(H)} (|tail(\varepsilon)|+1)$. The *forward star* of a given node z is the set $fstar(z) = \{\varepsilon \in E(H) : z \in tail(\varepsilon)\}$, while the *backward star* is $bstar(z) = \{\varepsilon \in E(H) : z = head(\varepsilon)\}$. The *indegree* of z is $\delta^-(z) = |bstar(z)|$, its *outdegree* is $\delta^+(z) = |fstar(z)|$ and its *degree* is $\delta(z) = \max(\delta^-(z), \delta^+(z))$. Finally, the *degree* of H is the integer $\Delta(H) = \max_{z \in V}(\delta(z))$.

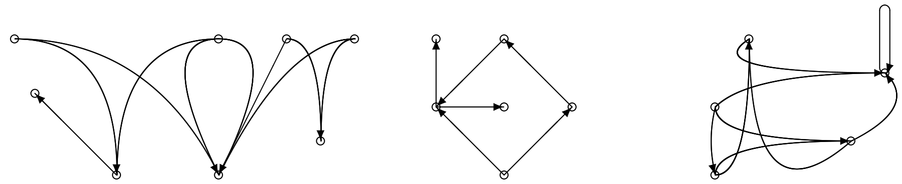


Figure 1: Directed hypergraphs

Directed hypergraphs play a basic role in many issues related to implicative structures. In particular, computer science has often availed of the support of such combinatorial structures, in connection with database theory and functional dependencies among attributes [1], as well as with artificial intelligence, deductive databases, diagnostic, logic. For further details, see for example [2].

In this paper we define and study a particular kind of arc-coloring, which takes into account the directed structure. In the case of digraphs, this notion has been introduced in [11]; several related results have also been presented in [5]. Thus, our definition extends the already existing notion of arc-coloring.

Definition 1.2. An *arc-coloring* of a directed hypergraph H is a map $\gamma : E(H) \to \mathbf{N}$ such that

i) $(\ (A,x) \in E, (B,y) \in E, A \cap B \neq \emptyset\ ) \Rightarrow \gamma(A,x) \neq \gamma(B,y)$

ii) $(\ (C,z) \in E, (D,z) \in E\ ) \Rightarrow \gamma(C,z) \neq \gamma(D,z)$

provided $(A,x) \neq (B,y)$ and $(C,z) \neq (D,z)$.

In simple words, we require that any two arcs having either intersecting tails or the same head be colored differently.

Definition 1.3. A directed hypergraph H is *k-colorable* if there exists some arc-coloring of H which requires at most k colors. The *(directed) chromatic index* of H, in symbols $q(H)$, is the least number k such that H is k-colorable.

Clearly, the trivial lower bound of $q(H)$ is $\Delta(H)$. The chromatic index may be interpreted, for example, as the minimal number of parallel processes, in some data exchange encoded by H, such that in each fixed process every node is not a target for more than one source, nor a (part of a) source for more than one target. In the case of digraphs many connections have been pointed out between this coloring and the design of permutation networks [5].

The present paper consists of two further sections besides the Introduction. In Section 2 we firstly show that the directed chromatic index of any digraph is not significant, for it is always equal to the degree. Then we consider the sub-class of *non-overlapping*, *interval* hypergraphs, proving that the elements of this class satisfy $q \leq 2\Delta - 1$ and that this bound cannot be lowered for any fixed degree (Proposition 2.4). The proof is given in terms of particular adjacency matrices (*walls*), which constitute a powerful means of rephrasing the hypergraph incidence. Using walls, the arc-coloring translates to the *brick-coloring*, while the notions of degree and chromatic index are respectively rephrased as the *degree* (δ) and the *chromatic number* (ρ) of the wall. Thus, the above inequality is equivalent to $\rho \leq 2\delta - 1$. Although only non-overlapping hypergraphs can be interpreted as walls, we provide an algorithm which transforms - in polynomial time - an overlapping hypergraph into a non-overlapping one having the same chromatic index (Proposition 2.7).

We term *coherent* any wall arising from a non-overlapping interval hypergraph. Proposition 2.4 suggests to partition coherent walls into finitely many classes, according to the chromatic number, and to possibly enlighten some interesting property of each class. In Section 3 we move a step in the above direction, by focusing on coherent walls of maximal chromatic number, and in particular on walls of degree 2. We prove that a further assumption - rather natural - on any coherent wall of degree 2 and chromatic number 3, forces the wall to contain a previously defined sub-wall. In other words, under that assumption there exists a unique 2-*critical* wall attaining the upper bound.

2 Some basic properties of the arc-coloring

Firstly, we show that in the case of digraphs the chromatic index collapses to the degree.

Property 2.1. If H is a digraph, then $q(H) = \Delta(H)$.

Proof. Let us consider the bipartite graph G, of degree $\Delta(H)$, whose edges are the ordered pairs of nodes forming the arcs of H. König's Theorem guarantees that $\Delta(H)$ colors are enough for an edge-coloring of G. Every such coloring can be easily interpreted as an arc-coloring of H. □

On the other hand, by allowing sufficiently large tails it is rather easy to exhibit directed hypergraphs, of any degree greater than 1, whose chromatic number is any number not smaller than the degree (we leave to the reader the proof of this fact). However, if we restrict the analysis to the subclass of *non-overlapping, interval* hypergraphs, then the chromatic index turns out to be upper bounded for every fixed degree.

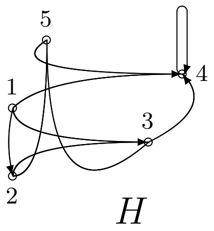
Definition 2.2. A directed hypergraph H is *non-overlapping* if there exists no pair of arcs of H sharing the same head and some node of the tails. Otherwise, H is *overlapping* and any pair of such arcs is termed *overlapping* as well.

Definition 2.3. A directed hypergraph H is an *interval (directed)* hypergraph if there exists a linear ordering $\leq$ of its nodes, such that every tail of H is a closed interval with respect to $\leq$.

Notice that none of the two above defined classes contains the other.

Proposition 2.4. If H is a non-overlapping interval hypergraph, then $q(H) \leq 2\Delta(H) - 1$. Moreover, there exist non-overlapping, interval hypergraphs H_d, of arbitrary degree d, such that $q(H_d) = 2d - 1$.

As mentioned in the Introduction, the proof of the above proposition will be rendered more fluent by interpreting hypergraphs as particular adjacency matrices. It will be soon clear that such rephrasing can be performed if and only if the hypergraph is non-overlapping. The effort required for introducing the relevant notions is small, in comparison with the usefulness of adjacency matrices and in view of the crucial role of non-overlapping hypergraphs (see the end of this section). In Figure 2 we show how to associate a non-overlapping hypergraph H to the corresponding adjacency matrix P_H. Notice that the nodes of H need to be numbered, and that the representation depends on the given numbering. Also notice that if H is a digraph, P_H reduces to the transposed adjacency matrix of H.



$$E(H) = \{(\{1\}, 2), (\{1, 2\}, 3), (\{1, 5\}, 4), (\{3\}, 4), (\{4\}, 4), (\{2, 3\}, 5)\}$$

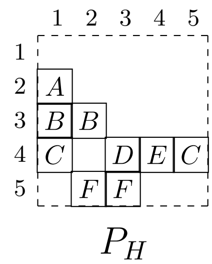


Figure 2: A labelled directed hypergraph and the related adjacency matrix

We formalize these particular adjacency matrices as follows.

Definition 2.5. A *wall* P is a partial chessboard whose squares have been labelled under the condition that every label occurs in a unique row of P. Its *degree*, $\delta(P)$, is the largest number of distinct labels in the same row or column. Every maximal set of squares having the same label is a *brick*.

It is clear that $\Delta(H) = \delta(P_H)$ for every non-overlapping hypergraph H. Using the language of walls, the arc-coloring of a non-overlapping hypergraph translates as follows.

Definition 2.6. A *(brick)-coloring* of a wall P is the assignment of a positive integer (*color*) to each square of P, in such a way that: 1) All the squares of any fixed brick are given the same color; 2) No color occurs in more than one brick in every fixed row; 3) No color occurs in more than one square in every fixed column. The *chromatic number* of P, denoted by $\rho(P)$, is the least number of colors required for a coloring of P.

It is straightforward to see that $q(H) = \rho(P_H)$. We are now ready for the

proof *of Proposition 2.4.* Let the nodes of H be numbered in such a way that every tail is a closed interval. Notice that every brick of P_H is composed of adjacent squares and that the inequality in the claim translates to $\rho(P_H) \leq 2\delta(P_H) - 1$. We prove this inequality by induction on the length of the smallest chessboard containing P_H. If the length is equal to 1, then $\rho(P_H) = \delta(P_H)$ and we are done. If the length is greater than 1, let us consider the rightmost column, say c, of the chessboard. Due to the induction hypothesis, the wall obtained by removing c from P_H admits a coloring in at most $2\delta(P_H) - 1$ colors. It remains to extend to c this partial coloring. The square bricks of c are the only squares to be managed, because every other square legally inherits the color of the corresponding left square. Let us choose a square brick among the h square bricks lying in c (we assume that $h \geq 1$). The number of available colors is not smaller than $2\delta(P_H) - 1 - (\delta(P_H) - 1) - (\delta(P_H) - h) = h$. Therefore, we can actually color this brick. If $h \geq 2$, by replacing h with $h - 1$ in the above formula we see that a further brick of c is colorable. We proceed analogously, until $h = 1$.

The examples in Figure 3 - and their obvious generalization - show that the bound is attained for all the values of $\delta(P_H) \geq 2$. Finally, if $\delta(P_H) = 1$, the inequality trivially reduces to $\rho(P_H) = 1$.

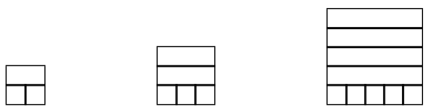


Figure 3: Walls attaining the upper bound

□

In the sequel, walls having all bricks consisting of adjacent squares will be termed *coherent*. As shown in the above proof, such walls arise from non-overlapping interval hypergraphs endowed with a suitable labelling of the nodes.

Every given arc-coloring of an overlapping hypergraph can be interpreted as a particular arc-coloring of a non-overlapping hypergraph. In order to state the relevant result, we provide some terminology. If $z \in V(H)$, the *basic outdegree* of z is the integer $bas\delta^+(z) = |\{v \in V(H)\colon z \in tail(\varepsilon), v = head(\varepsilon)\,\exists \varepsilon \in E(H)\}|$. The *basic degree* of z is $bas\delta(z) = \max(\delta^-(z), bas\delta^+(z))$. The *basic degree* of H is $bas\Delta(H) = \max_{z \in V}\ (bas\delta(z))$. Clearly, if H is non-overlapping then $bas\delta^+(z), bas\delta(z), bas\Delta(H)$ coincide with $\delta^+(z)$, $\delta(z)$, $\Delta(H)$ respectively. Notice that $bas\delta^+(z) \leq \delta^+(z)$ for all z, and that equality holds for all the nodes if and only if H is non-overlapping.

Proposition 2.7. Every fixed overlapping hypergraph H can be changed into a non-overlapping hypergraph H' such that
1) There exists a bijection $f : E(H) \to E(H')$.
2) Every arc-coloring γ of H induces an arc-coloring γ' of H', defined as $\gamma \circ f^{-1}$ and, conversely, every arc-coloring γ' of H' induces an arc-coloring γ of H, defined as $\gamma' \circ f$. In particular, $q(H) = q(H')$.
3) $\Delta(H') = bas\Delta(H)$.
4) The algorithm which transforms H into H' requires polynomial time with respect to $|H|$.

Proof. We explain how to change H into H' in the specific case depicted in Figure 4. As the general constrution is easily obtainable from this example, we leave it to the reader.

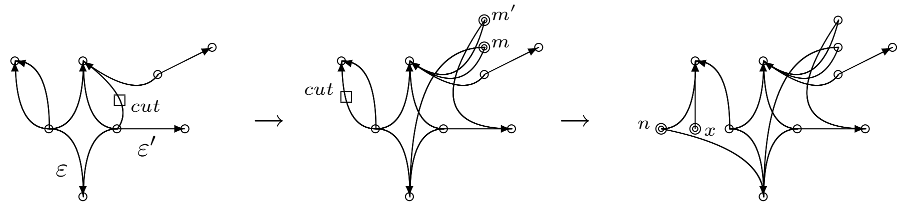


Figure 4: From an overlapping hypergraph to a non-overlapping one

We arbitrarily choose an overlapping pair among the occurring ones, and remove one node from the tail of some arc involved in the overlap. Having introduced the nodes m, m', we add them to $tail(\varepsilon)$, $tail(\varepsilon')$ respectively, and - both - to the reduced tail, so as to restore the chromatic constraints which were lost after the removal of the node. Now we pass to another overlapping pair. As in this case the tails of the involved arcs are singletons, by removing one node of a fixed tail we actually remove the whole arc; this forces us to introduce the node x, so as to restore the arc. Subsequently we add a new node, n, to $tail(\varepsilon)$ and to the tail $\{x\}$, so as to restore the chromatic constraints (in our example n need not be introduced, but in many cases a node like x is not present).

The bijection f is defined as the map sending each arc to the same one, possibly modified or removed-restored by the above procedure. The second claim follows easily by the definition of the procedure. To prove the third claim, firstly notice that $bas\Delta(H) \geq 2$ (indeed, H has at least one overlapping pair, whence some node of H is the head of at least two arcs). Furthermore, all the nodes introduced at a fixed step have degree 1 or 2, and this degree is preserved until the end of the procedure. Finally, the outdegree of each removed node is decreased by 1 and its basic outdegree is not modified. Since the algorithm stops precisely when $bas\delta^+(z) = \delta^+(z)$ for all the nodes z, the third claim is proved.

We do not provide the elementary proof that the whole procedure runs in polynomial time. □

3 The classification of walls: an example

The brick-coloring seems a valid tool for investigating the previously defined arc-coloring. It is, therefore, desirable to get some more understanding of the chromatic properties of walls. As already done with the degree and the chromatic index, also the notion of *criticality* can be exported from graph theory (see e.g. [7]) as follows: a wall P is termed r*-critical* if $\rho(P) = r + 1$ and the removal of any brick of P lowers the chromatic number. Many questions arising in graph theory can be rephrased in the language of walls, possibly adapted. In particular, notice that Proposition 2.4 highlights a class of walls (coherent walls) which satisfy a generalized version of Vizing's theorem. In fact, the classification of such walls up to their chromatic number resembles the *classification problem* for graphs (that is, the problem of deciding whether or not a given graph of degree d can be edge-colored in d colors). For example, Property 2.1 implies that walls consisting of only square bricks are colorable in as many colors as their degree. Such walls play the same role of bipartite graphs.

In accordance with the above remarks, the rest of this section is devoted to coherent walls having the largest chromatic number among the ones of fixed degree. Given two coherent walls P, Q, we write $P \sqsupset Q$ if Q can be obtained by removing some rows and/or columns of P. Let us denote by C_d the generic wall of degree $d \geq 2$ as defined in Figure 3. Clearly, if $P \sqsupset C_{\delta(P)}$ then $\rho(P) = 2\delta(P)-1$, but the converse is not true in general (see Figure 5).

By imposing a simple constraint on the brick incidence, and by restricting the analysis to walls of degree 2, it is possible to "locally" characterize the walls attaining the upper bound ($\rho = 3$).

Definition 3.1. A wall is termed *connected* if it is coherent and if the union of

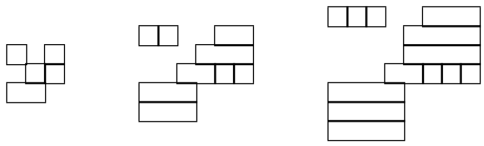


Figure 5: $\rho(P) = 2\delta(P) - 1, \;\; P \not\sqsupset C_{\delta(P)}$.

the closed squares of P (in the euclidean topology) is connected and cannot be disconnected by any finite set of points.

Notice that the left wall in Figure 5 is not connected.

Proposition 3.2. Let P be a connected wall of degree 2. If $\rho(P) = 3$, then $P \sqsupset C_2$.

Proof. We show that the negation of the thesis implies that $\rho(P) = 2$, provided P is connected and has degree 2. The proof is by induction on the height, say h, of the wall. If $h = 1$ then, trivially, $\rho = 2$. If $h \geq 2$, let us denote by P' the wall obtained by removing the lowest row, say r, of P. If P' is connected, we use induction to color P' in two colors. Subsequently, we extend the coloring to r without needing a further color; this is indeed allowed because P' has a stairs-like shape, as represented in the left side of Figure 6 (we leave to the reader the easy proof of this fact).

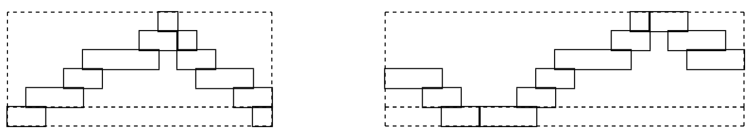


Figure 6: The two cases

Otherwise, if P' is not connected, then it consists of two stairs-like walls (see the right side of Figure 6). The induction hypothesis implies that the two sub-walls can be colored in two colors. By possibly reversing the colors of one sub-wall, we obtain a coloring of P' which can be extended to a coloring of P in two colors. $\square$

Because the connectedness assumption does not have the same effect on coherent walls of larger degree, we leave it as an open question to find some other constraint which forces a coherent wall P of fixed - or, hopefully, of any - degree greater than 2 to contain some $C_{\delta(P)}$ (up to permutations of rows and coloumns) whenever $\rho(P) = 2\delta(P) - 1$.

A different question is, for example, the classification of all 2-critical, coherent walls of degree 2, not necessarily connected. Notice that, due to Proposition 3.2, the only 2-critical, connected wall of degree 2 - up to permuting rows and to possibly shortening bricks - is C_2. We plan to settle the above question in a future paper.

Some further topics on directed hypergraphs and walls have been dealt with in [12, 13].

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