PITFALLS CAUSED BY "EVIDENT" PROPERTIES IN A DYNAMIC GEOMETRY ENVIRONMENT

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Sometimes diagrams produced by dynamic geometry systems hide "evident" properties. It follows that people involved in open problem activities do not conjecture this kind of properties and, moreover, submit proofs which contain bugs due to the oversight of facts which appear to be "evident" in computer generated diagrams. In this paper we introduce the notion of "evident property in a dynamic geometry diagram". We show that this properties can create pitfalls not only to high school students but also to mathematically mature people, like pre- service and in-service high school mathematics teachers.

KEY WORDS: Plane Geometry, Dynamic Geometry Software, Computers in mathematics teaching.

1. INTRODUCTION

In order to give a proof in Euclidean geometry which can really be mastered by a student it is important to use good diagrams. There are many good examples of faulty proofs in geometry based on wrong diagrams. See, for example, Dubnov (1963), Ball and Coxeter (1987). Moreover in the proof of a theorem one has often to examine several diagrams, for considering different cases or for getting a better insight.

Dynamic geometry software can be very useful because one can draw good diagrams and, by using the dragging facility, produce several diagrams at once.

When used in open problem activities (see Arsac *et al.* (1988)), dynamic geometry software displays all its strength. The student, accustomed to reading in a text a proof and just memorizing it, begins to work and think like a mathematics researcher. See for example Schattschneider and King (1997), Furinghetti *et al.* (2001), Furinghetti and Paola (2003).

Open problem activities within a dynamic geometry environment "coupled with suitable tasks, may provide an opportunity [for students] to develop a basis for a fuller appreciation of the nature and purpose of mathematical proof", Hoyles and Jones (1998), p. 1123. See also Olivero (1999), Hadas *et al.* (2000), Hanna (2000), Laborde (2000), Mariotti (2000), Jones (2000), Mariotti (2001), Mariotti (2005).

A teacher can in fact ask students to draw a diagram using dynamic geometry software, to examine it carefully, to make conjectures and test them.

Unfortunately sometimes students do not make the conjecture they are supposed to make by their teacher.

In Sinclair (2003a) the author considers this problem while studying students exploring pre-constructed dynamic geometry diagrams. She shows some examples of questions to which students give trivial responses because they were too vague. She points out that in such cases, rather than making the question more pointed, which could affect the cognitive demand of the task, the teachers should "work on visual reasoning skills to help them [the students] focus on details", loc. cit., p. 23. In Sinclair (2003b) the author claims also that sometimes students do not use the hints that an accurate dynamic geometry diagram gives because they are usually warned by the teachers not to trust figures that are not necessarily accurate. She claims that teachers should "focus student's attention on the difference between textbook diagrams and dynamic geometry sketches", loc. cit. p. 197.

On the other hand, we have evidences that with some kind of geometric configurations secondary school students trust dynamic geometry diagrams but do not see what they are supposed to see. We give an example taken from Accascina *et al.* (2004).

We gave to a group of 9th grade, 15 years old students, accustomed to use Cabri géomètre, the following worksheet (translated from Italian):

Draw with Cabri:

a point A, a line r passing though A, a line s passing through A and perpendicular to line r; a point B on r different from A, a point C on s different from A, the segment BC.

Now draw the perpendicular bisectors of segments AB and AC.

Drag points A, B and C.

Do you note some property which holds all the time?

Almost all the students made the correct drawing. See Figure 1.

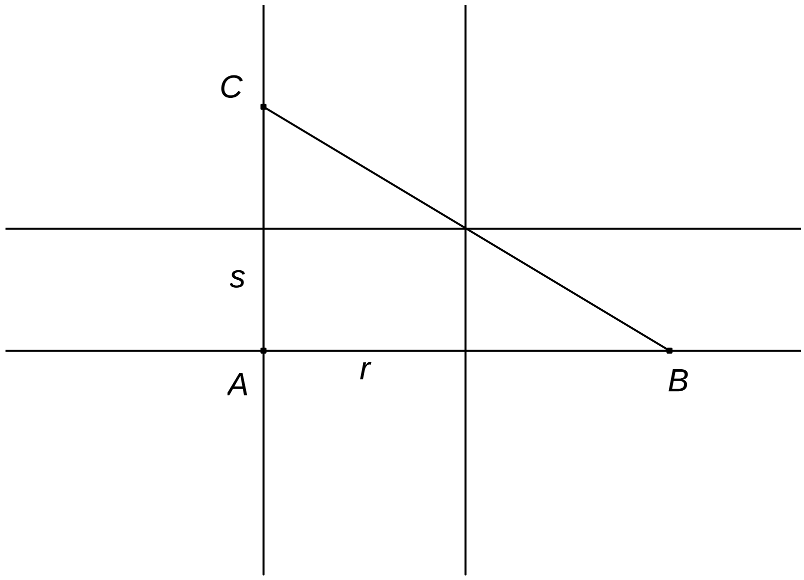

Figure 1. The perpendicular bisectors of the legs of a right triangle

We wanted them to conjecture that the perpendicular bisectors intersect in the hypotenuse.

Most of them where attracted by the dynamicity of the diagram. At the beginning they dragged points A, B and C without a particular aim (the “wandering dragging” of Arzarello *et al.* (2002)). Then gradually their attention was attracted by the internal triangle and they began dragging the points B and C on the lines r and s respectively (“bound dragging” of Arzarello *et al.* (2002)) in order to see if, as a student wrote, “the internal rectangle was all the time a rectangle”. Some of them tried to give a proof of this.

None of the students pointed out that the perpendicular bisectors intersect in the hypotenuse. Why?

One should think that the students did non point it just because they considered it “obvious”, in the sense that they knew this property and therefore they did not consider worth mentioning it.

However, by reading their comments we have evidences that this is not the case. Moreover we have examples that most students at their university’s first year and even graduated in mathematics do not know this property, see *Accascina et al.* (2004).

We claim that students did not point out that property of the perpendicular bisector because the property was ***evident*** in the diagram in the sense of the following definition:

A property is evident in a dynamic geometry diagram if:

1) the property continues to hold while dragging (i.e. it is an invariant):
2) for spotting the property in the diagram one does not need to add new elements or to measure segments or angles.

For example the property that “the internal rectangle [is] all the time a rectangle” is a property NOT evident in the Cabri diagram of Figure 1 since in order to see it one has realize that four angles are right.

Hefendehl-Hebeker (1994), Seyffart (1995), Hölz (2001) describe cases where students do not spot properties which their teacher supposed to be seen. All of these properties are “evident” in the sense of our definition.

In this paper we show some experiments which give evidence that also pre-service teachers and in service teachers face the problem of “evident” properties in diagrams produced by dynamic geometry software.

We also discuss how it is possible to avoid this problem modifying the task given to students.

In the appendix we see that the problem of “evident” properties appears also in 3D dynamic geometry diagrams.

2. THE EXPERIMENTS WITH THE FERMAT CIRCLES

a) Experiment with pre-service teachers

Participants and organization. We undertook the experiments on this topic in the Academic Years 2003-04 and 2004-05 with pre-service teachers who already had a four year degree in mathematics or physics and were attending the first of a two year course of postgraduate school in secondary (high school) teaching (SSIS). Each year the pre-service teachers where subdivided in two groups: one constituted by pre-service teachers in Math and Physics, the other by pre-service teachers in Applied Mathematics. Therefore we repeated the same experiment with four different classes. The pre-service teachers of each class were enrolled in a 12 hour lab course subdivided into 4 meetings. Each meeting was divided in two sessions of 1 hour and half each with a 30 minutes break. They worked in pairs. Each pair had access to a PC. We made use of a projector linked to a PC. The pre-service teachers were asked to write after each meeting at least half a page with a description of the most significant points of the meeting. The first 3 meetings focused on the pros and cons of using a Computer Algebra System in secondary teaching. Derive was used. The last meeting was on Dynamic Geometry Systems. Almost none of the pre-service teachers had worked with them. Cabri Géomètre was used. The activities of the last meeting were described in worksheets which were chosen from Accascina and Margiotta (2002 and 2003), a proposal of an activity of problem solving with Cabri. Each worksheet was handed to students after the completion of the proceeding one.

Getting acquainted with Cabri. The first five worksheets were prepared to introduce Cabri. They begun with the description of main commands of Cabri. Then some problems were posed:

- Draw two points *A* and *B* and one of the equilateral triangles which have an edge on the segment *AB* (Euclid's Elements, book 1, prop. 1). Write a macro for it.
- Draw two points *A* and *B*, the segment *AB*, its midpoint and its perpendicular bisector (Euclid's Elements, book 1, prop. 10). Do not use the Cabri commands for the midpoint and perpendicular bisector.

- Draw a point *A*, a straight line *r*, the straight line passing through *A* and perpendicular to *r* (Euclid's Elements, book 1, prop. 11 and 12). Do not to use the Cabri commands for the perpendicular line.
- Draw three points *A*, *B* and *C*, the triangle *ABC* and its circumcircle (Euclid's Elements, book 4, prop. 5). Write a macro for it.

On the fifth worksheet we gave the usual solution to the last problem. The circumcircle is constructed by drawing the circle with its centre in the intersection *O* of the perpendicular bisectors of two edges of the triangle and passing through one of the vertices of the triangle, in this case A.

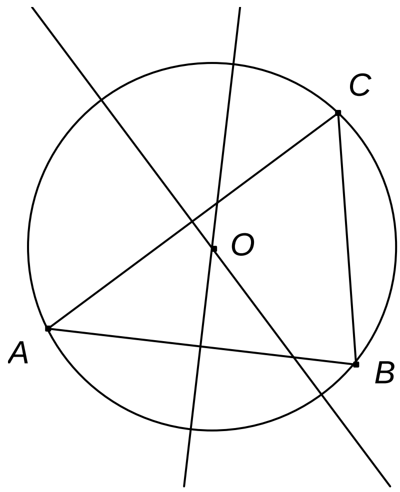

Figure 2. The construction of the circumcircle.

We made the pre-service teachers aware that when we gave the same problem to 9th grade high school students, they were able to draw the circumcircle using Cabri because they studied it in the middle school. But high school students were not able to prove that the constructed circle, which passes through the point *A* by construction, passes also through the other two vertices *B* and *C* of the triangle. High school students usually do not even understand the teacher's request of a proof since the Cabri drawing passes the "dragging test" described in Arzarello *et al.* (2002). We pointed out to pre-service teachers that the proof rests on the fact that the three perpendicular bisectors are not in general position (they intersect in a point).

Well before the break all pre-service teachers were able to use Cabri properly.

The core of the experiment. We handed the following Worksheet 6.

```
Draw a triangle ABC. Construct on the edge AB the equilateral
triangle ABC′ externally to triangle ABC, draw the Circle c1
which circumscribes ABC′. Repeat the same construction on the
edges BC and AC. You obtain the equilateral triangles BCA′ e
AB′C and the circles c2 and c3 which circumscribe them. Which
are the properties of this figure? Write the conjectures in the
order you make them.
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All pre-service teachers used the macros they constructed before and quickly drew the Figure 3.

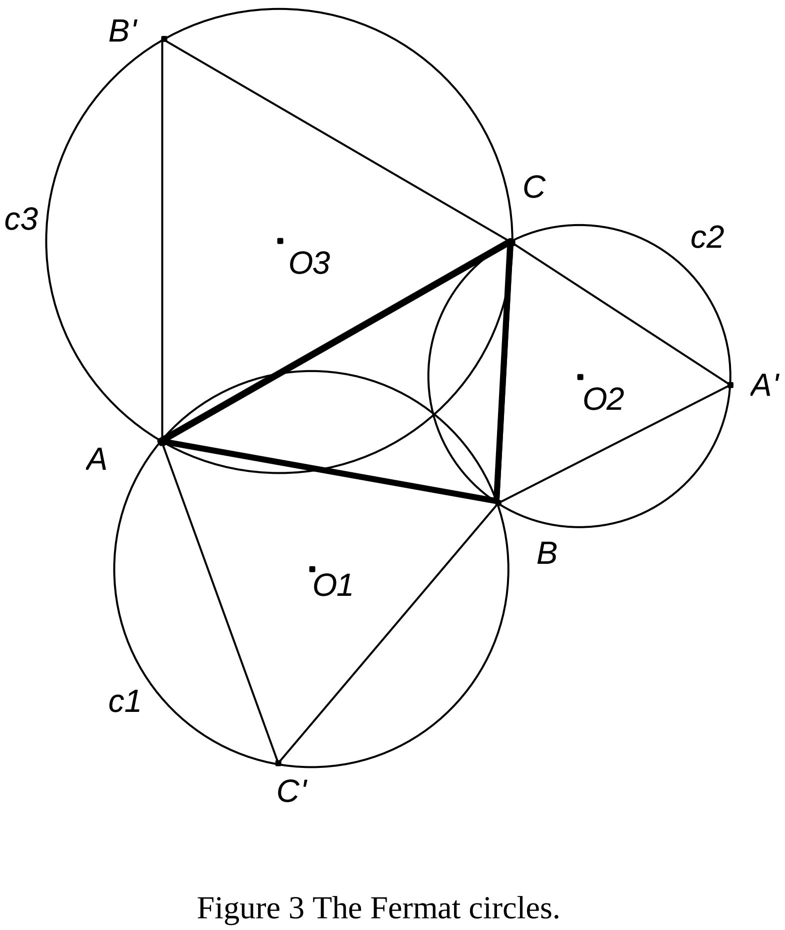

Figure 3 The Fermat circles.

This configuration is a well known. See for example Coxeter (1961), Posamentier and Salkind (1970), Sinclair (2002). The book of Posamentier and Salkind has the picture of it in the cover. None of the pre-service teachers knew the problem before.

Pre-service teachers spent 15 minutes before the break looking for conjectures. All of them got very involved in the problem. Many continued investigating during the break.

The configuration has many properties: the circles *c1*, *c2* and *c3* (which are called Fermat circles) intersect at a point *F* (which is called the Fermat point); the triangle *O1O2O3* is equilateral (Napoleon Theorem); the straight lines *CC'*, *AA'* and *BB'* intersect in *F*; the segments *CC'*, *AA'* and *BB'* are congruent; the segments *CC'*, *AA'* and *BB'* are perpendicular to the edges *O2O3*, *O1O3*, *O1O2* respectively of the triangle *O1O2O3*. See Figure 4.

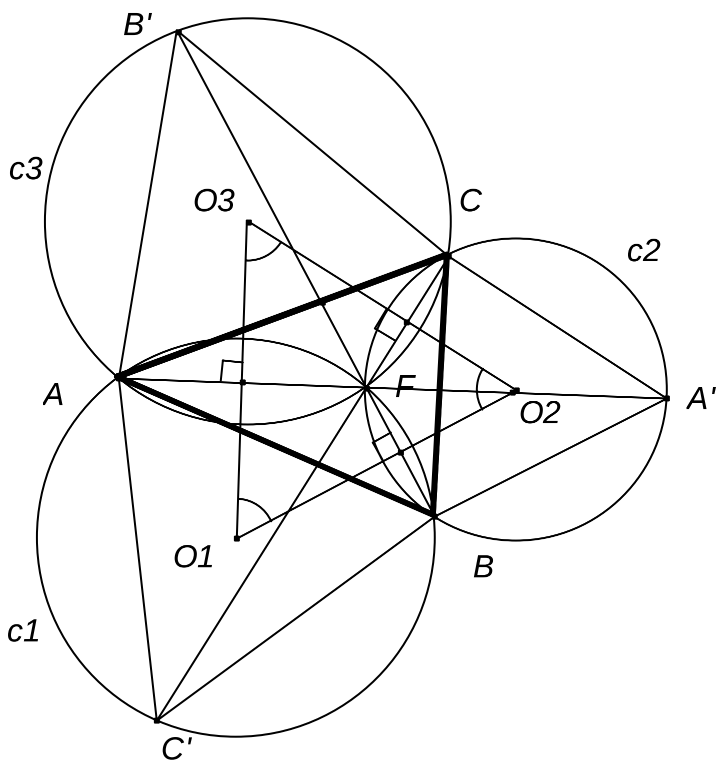

Figure 4. The properties of Fermat circles.

It can also be seen that the point *F* is interior to the triangle *ABC* if and only if all the angles of the triangle are less of 120°.

Almost all of the pre-service teachers conjectured Napoleon Theorem. This is not an "evident" property of the diagram shown in Figure 3 because, in order to detect it, they had to draw, at least mentally, a triangle and measure either its sides or its angles.

We proposed this problem because we were interested in finding out if pre-service teachers would have noticed the "evident" property that the three Fermat circles are not in general position (they do intersect at a point). This property is “evident” in the Cabri diagram shown on Figure 3.

Many did not notice it. Many conjectured some not “evident” properties of the Fermat point without noticing explicitly that this point exists. They wrote, for example:

The point of intersection of the three circles is the incentre of the triangle ABC (Wrong). *The circles intersect in a point interior to the triangle* (Wrong).

	2003-04		**2004-05**	
	Math. & Fis.	**Appl. Math**	**Math. & Fis.**	**Appl. Math**
No conjectures on Fermat point	5	8	10	2
Conjectures on properties of Fermat point but without any conjecture on its existence	6	4	3	5
The conjecture on the existence of Fermat point is not the first in their list	3	4	10	7
The first conjecture in their list is on the existence of Fermat point	12	4	9	8
Total	**26**	**20**	**32**	**22**

Table 1. Pre-service teachers’ answers to the Fermat circles problem

Table 1 shows that in the problem of Fermat circles the pre-service teachers acted similarly to the high school students in the problem of the intersection of the intersection of the legs of a right triangle described in the first section. Recall that pre-service teachers were advised just few minutes earlier about this kind of behaviour of school students.

We discussed with the pre-service teachers on their answer in the first 15 minutes after the break. They were very impressed by their behaviour. In the description of the meeting Silvia wrote: *Starting from the famous sentence “Geometry is the art of good reasoning on a bad picture”, I would continue saying that Cabri offered me the possibility of not reasoning over a well done picture. Why do I say this? I do realise that the sentence is strong but this is what happened to me today in the laboratory. Looking at Fermat circles, I hardly understood that the triangle with* *vertices on the centres of those circles is equilateral and I was not able to see anything else. Well, maybe with a bad figure I would not even understand this.*

b) Experiments with in-service teachers

A similar experiment was done during two seminars with two different groups of in-service teachers.

In the first seminar the audience consisted of middle school mathematics and sciences teachers. In the second seminar the audience was of secondary school mathematics teachers.

In both seminars, after one hour of seminar on Cabri, the teachers were given a worksheet were the problem was stated as in worksheet 6 and the diagram shown in Figure 3 was given. The same diagram was shown by a projector connected to a PC. The speaker dragged the points A, B and C following the teachers' requests. Therefore, in order to make their conjectures, the in-service teachers could use the static diagram on their worksheets and use the dynamic diagram asking the speaker to drag some point. The teachers worked for 10 minutes to give the following answers.

	Middle S. Teachers	**High S. Teachers**
No conjectures on Fermat point	12	13
Conjectures on properties of Fermat point but without any conjecture on its existence	3	2
The conjecture on the existence of Fermat point is not the first one in their list	0	0
The first conjecture in their list is on the existence of Fermat point	0	4
Total	**15**	**19**

Table 2. In-service teachers' answers to the Fermat circles problem

Table 2 shows that none of the middle school teachers pointed out some property of the intersection of the Fermat circles. Only few among the high school teachers did it.

The results of in-service teachers are much worse than the ones of pre-service teachers.

This could be due by the fact that the possibility to manipulate a dynamic geometry diagram gives much more insight than the possibility to see a manipulated diagram.

The difference between middle and high school teachers should be referred to the fact that most of the middle school mathematics and science teachers, like unfortunately in all Italian middle schools, graduated in subjects different from mathematics or physics (like biology, geology and so on). On the contrary all the high school mathematics teachers graduated in mathematics or physics.

3. THE EXPERIMENTS ON CIRCUMSCRIBYNG A TRIANGLE WITH AN EQUILATERAL TRIANGLE

We did the experiments with the same pre-service teachers of section 2 a) after the conclusion and discussion of the topic on Fermat circles. In this experiment we asked the pre-service teachers to discuss all together, instead of writing their answers. We gave the following problem:

`Given a triangle` *`ABC`*`, draw an equilateral triangle` *`A*B*C*`* `which circumscribes it, like in Figure 5.`

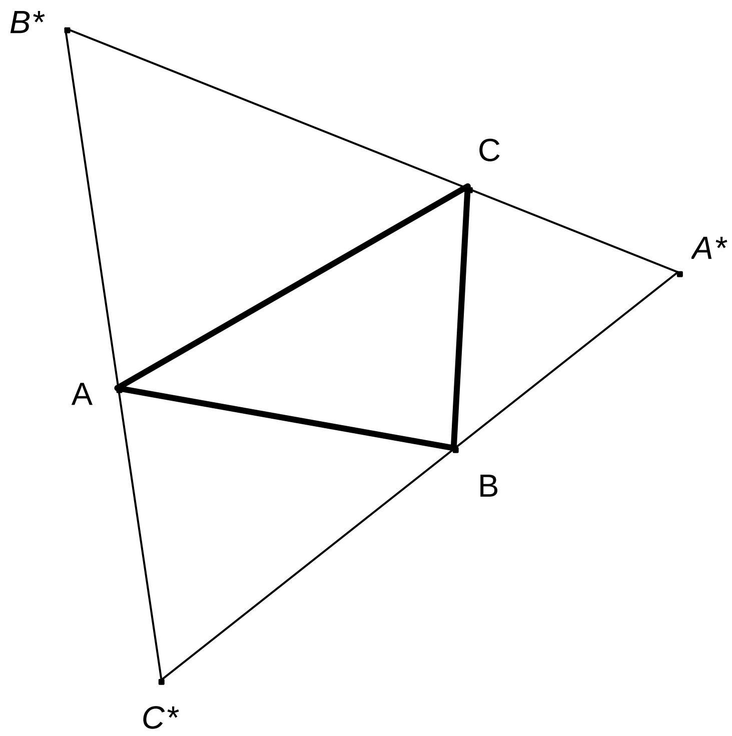

Figure 5. An equilateral triangle *A*B*C** which circumscribes the triangle ABC.

In all of the experiments no one, as we expected, was able to give an answer. Therefore we gave them a hint with Figure 6.

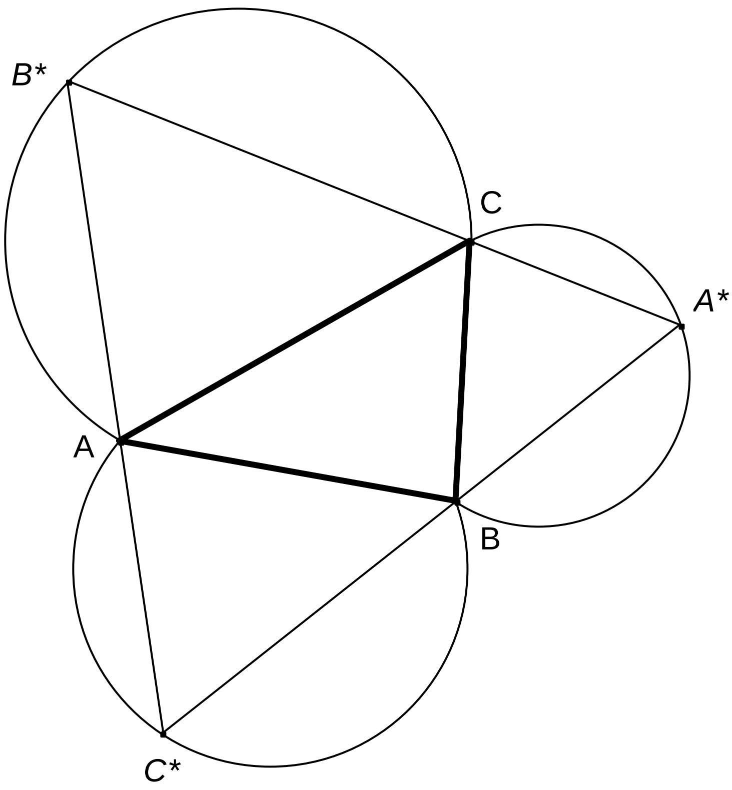

Figure 6 . A first hint.

Later, since no one was still able to provide an answer, we gave another hint with Figure 7.

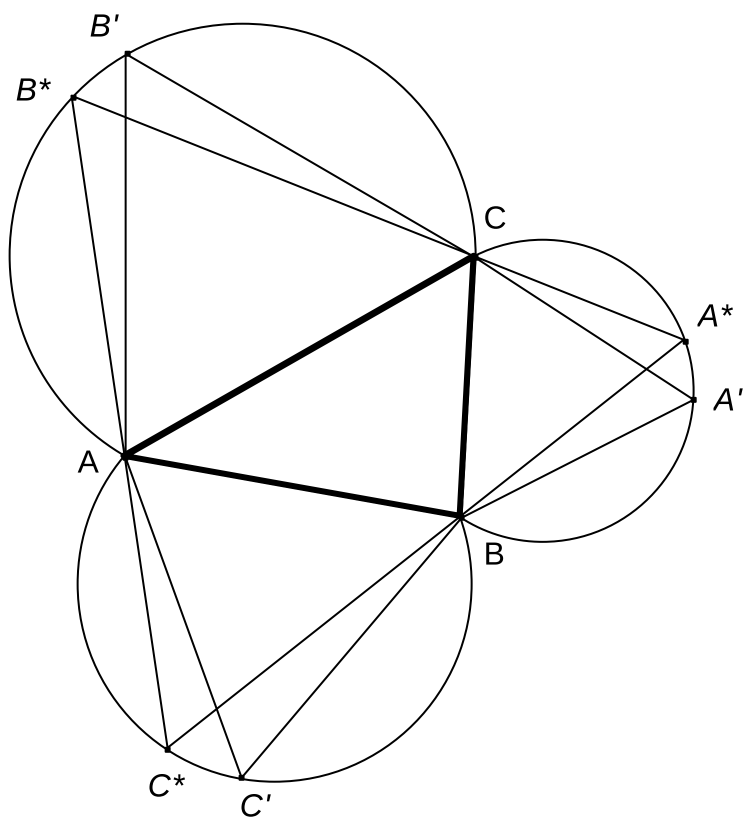

Figure 7. A second hint.

After seeing this diagram, everything was clear to pre-service teachers. The arcs are part of Fermat circles. The angle in *A** (*B** and *C** resp.) is equal to the angle in *A’* (*B’* and *C’* resp.). Therefore it measures 60°. The construction of the equilateral triangle *A*B*C** is easily done. We choose a point *C** on a suitable part of the Fermat circle and from it we draw the rays passing through *A* and *B* respectively. There are therefore infinitely many triangles *A*B*C**. Then we asked:

`Did we solve completely our problem? Did we really prove that there exist equilateral triangles circumscribed to the triangle` *ABC*?

At the beginning, in all four experiments, the students did not respond. In one of the experiments a student said after a while:

"I would say "yes" but the preceding experiences and the way you posed the question make me think that the correct response is "no". But what else is missing?"

In another experiment a student found the solution. Luca, another student, in his comments to the meeting, described what happened: *"To all of us the construction looked so beautiful that it should be exact … [later] a student observed that, although all the angles of the triangle we constructed were of 60°, it was still needed to be proved that the last constructed edge contains indeed the vertex of the original triangle; this is evident in the drawing but from the drawings one does not have a mathematical proof. Yet another time one must not be cheated by the drawing; if the teacher's apprentices fail, what will happen to students?"*

In the other two experiments nobody found the solution.

Therefore in the four experiments just one of the 98 pre-service teachers pointed out the fact that one has to prove that the points B*, C and A* are collinear.

Note that this property is again an "evident" property in the Cabri diagram, in the sense of our definition of section 1.

This "evident" property causes big obstacle once one tries to prove it.

Since the Cabri diagram shows that the point C is on the line B*A*, one usually mixes up the line B*C with the line CA*. It therefore convenient to leave the Cabri diagram and work with a "wrong" diagram, where the points A*, C and B* do not appear collinear. We could draw a diagram by hand. Otherwise we can use Cabri and construct a configuration where the three triangles outside the triangle ABC are not equilateral. See Figure 8. In the proof anyway we suppose that the three triangles are equilateral and therefore the angles with vertices A*, B* and C* measure 60°.

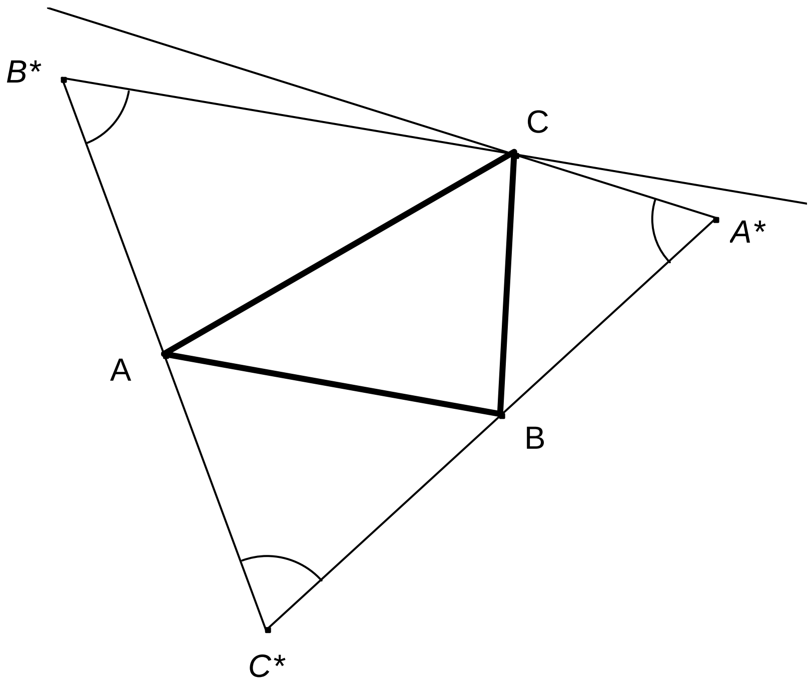

Figure 8. A “wrong” diagram helps in the proof.

Leung and Lopez-Real (2002) show a similar example where two 16 year-old students founded the need to draw “wrong” diagrams. One of them said that one of the disadvantages of a dynamical geometry environment is that “it cannot draw a wrong picture”.

4. EDUCATIONAL IMPLICATIONS

Several consequences follow from the fact that properties which are “evident” in dynamic geometry diagrams are usually not seen by students.

Students, at school and university level, involved in open problem activities do not usually see what the teacher wants them to see and, moreover, submit proofs which contain bugs due to the oversight of “evident” properties in dynamic geometry diagrams.

In order to overcome these problems, the teacher should propose more general situations. For example, going back to our experiments:

1) In section 1 we have discussed experiments where most of 9th grade students did not point out that the bisectors of the legs of a right triangle intersect in the hypotenuse. We can avoid this

fact by asking to draw two bisectors of a general triangle and asking when their intersection is inside or outside the triangle;

2) In section 2 we have discussed experiments where most of pre-service and in-service teachers did not point out that the three Fermat circles intersect in a point. We can avoid this behaviour by asking to study the configuration shown in Figure 9.

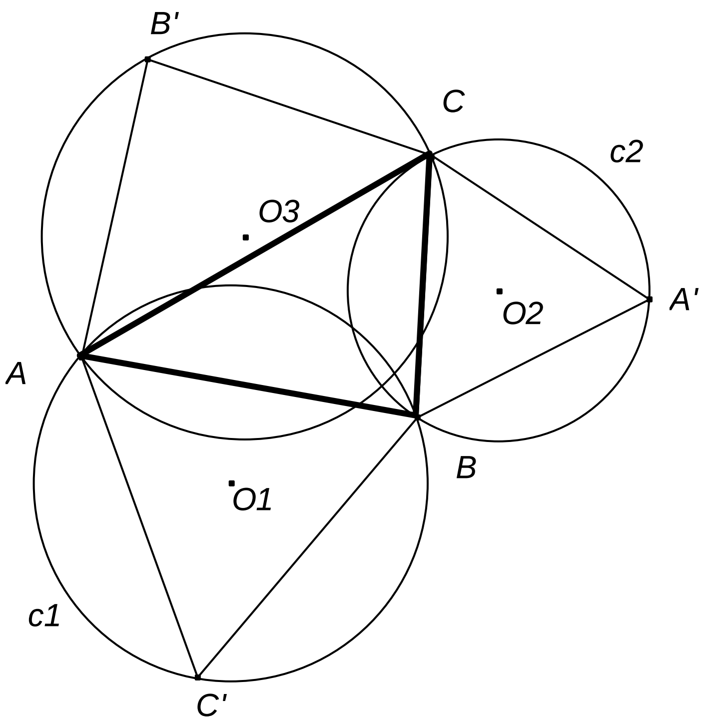

Figure 9. A slight generalization of the Fermat circles. One of the three triangles is general.

This is a slightly more general configuration of the one of Fig 3 representing the Fermat circles: one of the three equilateral triangles outside the triangle *ABC* is substituted by a general triangle. The point *B'* can be now freely dragged. By dragging it the students will see how the circle *c3* moves and can now conjecture that the three circles intersect in a point when the angle with vertex in B' measures 60°. The next step is easy: the points A', B' and C' do not need to be vertices of equilateral triangles: it is sufficient that the three angles with vertices A', B' and C' measure 60°.

Once this problem has been solved, teacher could go push students further on. One could ask to examine the configuration shown on Figure 10 where the three triangles outside the triangle *ABC* are general. The point *A'*, *B'* and *C'* can be freely dragged and the circles *c1*, *c2* and *c3* move this then. The students could conjecture that the three circles intersect in a point when the sum of the measure of the three angles with vertices in *A'*, *B'* and *C'* is equal to 180°. A See Accascina *et al*. (2004) for details.

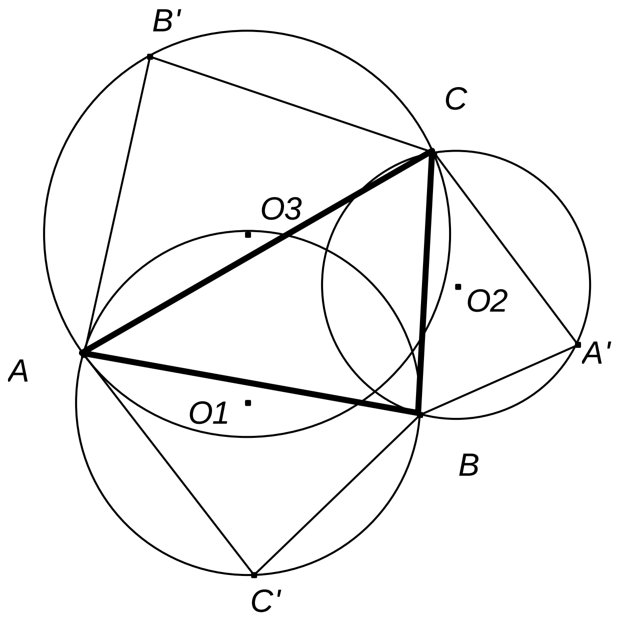

Figure 10. A generalization of the Fermat circles. All the three triangles are general.

3) In section 3 we have seen experiments where almost all pre-service teachers did not point out in the Cabri diagram shown in Figure 7 that they needed to prove the "evident" property that the point C belongs to the line A*B *. In this case it is not possible to generalize the configuration, at least in a natural way.

In order to avoid these behaviours, professors and teachers should let their students become aware of the pitfalls due to "evident" properties in dynamic geometry diagrams.

The only way to obtain it is to expose students (also mathematically mature students) to many diagrams displaying "evident" properties.

5. CONCLUSIONS

We have shown examples of Cabri diagrams in which some properties were not pointed out by some of the users.

We believe that all these properties have not been pointed out because they were “evident” properties of the diagrams. We recall that a property is “evident” in a dynamical geometry diagram when it continues to hold while dragging (i.e. it is an invariant) and for spotting it in the diagram one does not need to add new elements or to measure segments or angles.

We have shown that “evident” properties in a dynamic geometry diagram cause pitfalls to high school students and also to mathematically mature people, like pre- service and in-service high school mathematics teachers, while they are involved in open problem activities.

In order to overcome these pitfalls, the teacher should propose more general situations in such a way that the properties not pointed out because they are “evident” in a diagram, hold only in particular cases. This means that, while applying to the new task, one will produce dynamic geometry diagrams where the property we are interested in is not “evident” because it is not an invariant.

We have also shown that sometimes an “evident” property may cause problems even when it has been detected. In fact the evidence of the property in the diagram causes big obstacles when one tries to use the diagram to prove the property. We suggest that in this case one has to give up the dynamic geometry diagram and use hand-made “wrong” diagrams.

6. APPENDIX. THE 3D DYNAMIC GEOMETRY DIAGRAMS.

In 3D the situation is more complicated than in 2D since the 3D dynamic geometry diagrams do not represent faithfully the original 3D configuration. See Accascina, G. and Rogora, E. (preprint).

Anyway we have some preliminary evidence that also 3D dynamic geometry diagrams can hide “evident” properties.

We made an experiment with 8 pre-service teachers and to 19 in-service teachers to show this.

The 8 pre-service teachers were attending in the Academic Year 2004-05 the second of the two year course of the postgraduate school in secondary teaching described in section 2a. All of them during their first year did the experiments described in section 2a and 3.

The same experiment was done in the seminar with high school math teachers just after the first experiment on Fermat circles described in section 2b.

We showed to all of them the 3D Cabri reproduced in Figure 10 representing a regular tetrahedron and its four altitudes and asked them to conjecture some properties of the four altitudes.

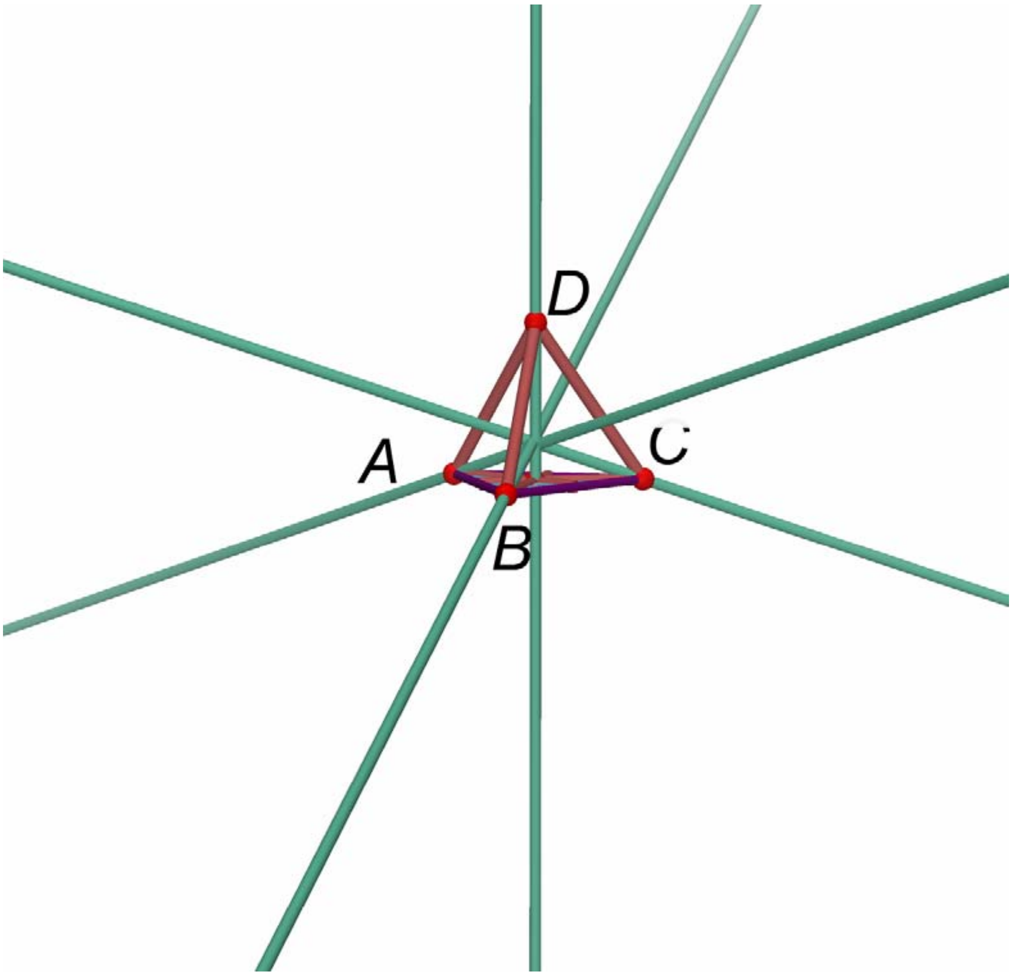

Figure 11. The altitudes of a regular polyhedron.

Teachers could change the point of view of the diagram by themselves (the pre-service teachers) or by asking to the seminar speaker to do it (the in-service teachers).

We wanted to see if all the teachers did point out the evident fact that the 4 altitudes intersect in a point.

Two out of 8 pre-service teachers and 7 out of 19 in-service high school teachers did not made any remark on it.

It is remarkable that when we asked to conjecture some properties of the four altitudes and although there are not many properties to be conjectured, in both experiments not all the people noticed the "evident" property that the altitude are not in general position. They graduated in mathematics or physics and should have known that four lines in space do not usually intersect in a point.

In order to make this property likely to be pointed out one can ask to examine a more general tetrahedron where the vertex C is substituted by a point C' which, instead of being fixed, can be dragged on the plane passing through C and parallel to the base ABD of the tetrahedron. See Figure 12.

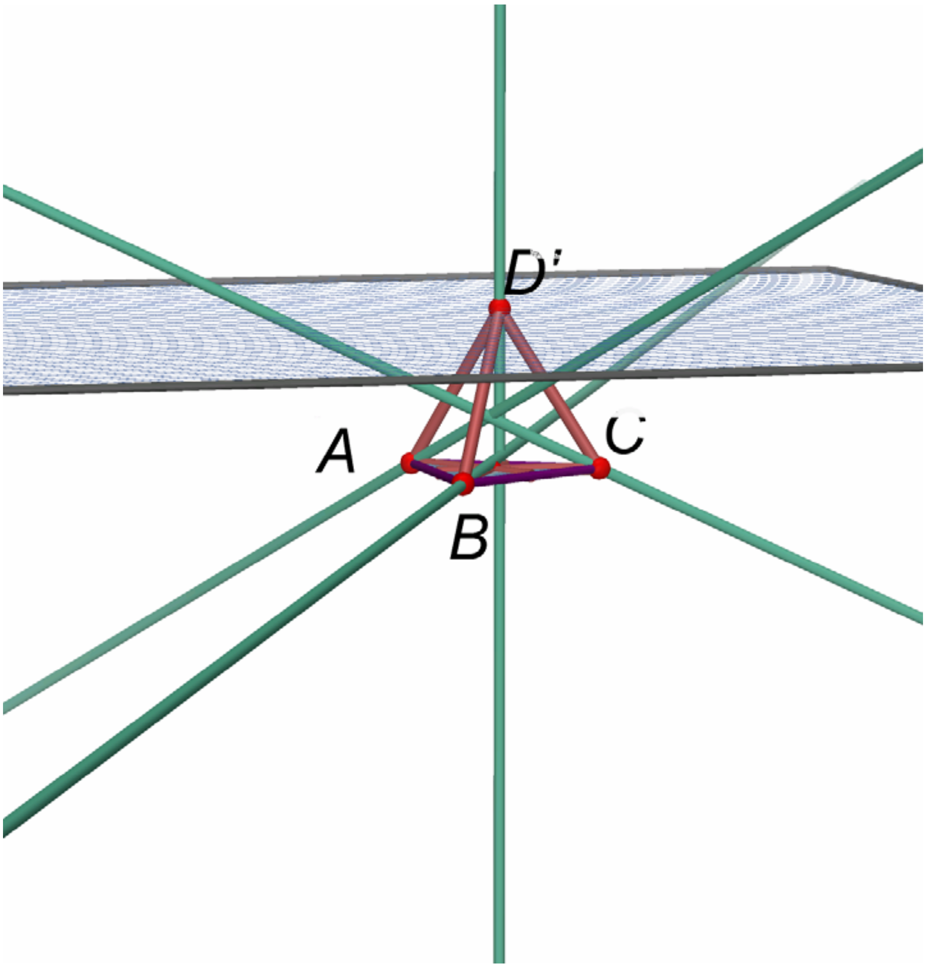

Figure 12. The altitudes of a pyramid with an equilateral triangle as base.

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