ROTATIONALLY SYMMETRIC 1-HARMONIC MAPS FROM D^2 TO S^2

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Abstract. We consider rotationally symmetric 1-harmonic maps from D^2 to S^2 subject to Dirichlet boundary condition. We prove that the corresponding energy – a degenerate non-convex functional with linear growth – admits a unique minimizer, and that the minimizer is smooth in the bulk and continuously differentiable up to the boundary. We also show that, in contrast with 2-harmonic maps, a range of boundary data exists such that the energy admits more than one smooth critical point: more precisely, we prove the existence of a unique (up to scaling and symmetries) global solution to the corresponding ode, which turns out to be oscillating, and characterize the minimizer and the smooth critical points of the energy as monotone, respectively non-monotone, branches of such solution.

Contents

1. Introduction and main results

Given a domain Ω in $\mathbb{R}^N$, a real constant $p \geq 1$, a smoothly embedded compact submanifold (without boundary) M of $\mathbb{R}^{N+1}$ and a mapping $u : \Omega \to M$, let

$$E_p(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx$$

(if $p = 1$, E_1 is the total variation of u). The so-called p-harmonic flow for E_p (i.e. the L^2-gradient flow for E_p with constraint of values in M) is given by

$$u_t = -\pi_u(-\text{div } (|\nabla u|^{p-2}\nabla u)) \,, \tag{1}$$

where π_v denotes the orthogonal projection of $\mathbb{R}^{N+1}$ onto the tangent space T_vM of M at $v \in M$. Because of π_v the values of a solution of (1) are constrained to remain in M if they are in M initially.

Here we shall consider the case in which $N = 2$, Ω is the unit disk ($\Omega = D^2 = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1^2 + x_2^2 \le 1\}$) and M is the unit sphere ($M = S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1^2 + x_2^2 + x_3^2 = 1\}$). In this case (1) may be explicitly written as

$$u_t = \text{div}\,(|\nabla u|^{p-2}\nabla u) + u|\nabla u|^p. \tag{2}$$

Equation (2) arises in various contexts, such as multi-grain problems [14], theory of liquid crystals [12], ferromagnetism [9] and image processing [16]. In particular, for $p = 1$, (2) may be seen as a constrained gradient system of total variation:

$$u_t = \text{div}\left(\frac{Du}{|Du|}\right) + u|Du|. \tag{3}$$

Equation (3) was proposed in [17, 18] as a tool to denoise color images by smoothing the chromaticity data while the contrast is being preserved. The unconstrained version of (3) – the so-called total variation flow – corresponds to the gray image denoising. This problem is by now well understood and there is a vast literature about it (see the monograph [2] and the references therein). However, much less is known about the constrained problem (3).

The Dirichlet problem for (2) has been widely studied over the last decades; referenced discussions of the cases $p = 2$, $p \in (1, \infty)$ and $p = 1$ may be found e.g. in [3, 4], [13, 15] and [10], respectively. In particular, it is shown in [8] that for $p = 2$ and suitable boundary data, classical solutions cease to exist in finite time. This result was recently extended to $p = 1$ in [11]. In both papers, the prototypical case of rotational symmetry and constant (in time) Dirichlet boundary conditions is considered. A rotationally symmetric solution of (3) has the form

$$\begin{aligned} u(t,x) = \Big(\frac{x_1}{r}\sin h(t,r), \frac{x_2}{r}\sin h(t,r), \cos h(t,r)\Big)\,, \\ r = |x|\,,\; x = (x_1, x_2) \in D^2 \end{aligned} \tag{4}$$

while a constant Dirichlet boundary condition turns into the constraint $h(t, 1) = h(0, 1) = \ell$. Upon (4), the energy functional E_1 takes the form

$$\mathcal{E}(h) := \int_0^1 \sqrt{r^2 h_r^2(r) + \sin^2 h(r)}\, dr, \tag{5}$$

and after an routine computation, equation (3) becomes

$$h_t = \left(h_r^2 + \frac{\sin^2 h}{r^2}\right)^{-\frac{3}{2}} \cdot$$
$$\cdot \left(\frac{\sin^2 h}{r^2}\left(h_{rr} + \frac{2h_r}{r} - \frac{\sin 2h}{2r^2}\right) + \frac{h_r^2}{r}\left(h_r - \frac{\sin 2h}{r}\right)\right). \tag{6}$$

By comparison with suitable sub-solutions, in [11] it is shown that a solution of (6) ceases to be smooth in finite time if the initial datum h_0 is such that $h_0(0) = 0$, $h_0(r) \in (0, \pi)$ for $r \in (0, 1)$, and $h_0(1) = \pi$. Noticeably, for the same initial datum the solution of (2) with $p = 2$ will keep smooth for all times [7]. Such striking contrast enforces the intuition that the strong singular character of (3) should give raise to more complex phenomena with respect to the heat flow of harmonic maps, especially concerning the formation and the profile of singularities. It also naturally raises the question whether the constant $\pi = h_0(1)$ plays any specific role in this respect.

As a step towards a better understanding of these issues, in this note we will investigate the minimizers of $\mathcal{E}$ subject to $h(1) = \ell$, as well as its smooth critical points, i.e. the steady states of (6). In fact, by the periodicity of $\sin^2 h$ and the symmetry with respect to $\frac{\pi}{2}$ (i.e. $\sin(\pi - h) = \sin(h)$), we may assume without loss of generality that

$$\ell \in (0, \frac{\pi}{2}] \tag{7}$$

(we also rule out the trivial case $\ell = 0$). We incorporate the boundary condition by letting

$$\mathcal{G}(h) = \begin{cases} \mathcal{E}(h) & \text{if } h(1) = \ell \\ +\infty & \text{otherwise.} \end{cases}$$

In Section 2 we shall prove the following representation formula for the relaxation of $\mathcal{G}$ in $BV((0,1))$:

Proposition 1. *The relaxation of $\mathcal{G}$ in $BV((0,1))$ is given by*

$$\overline{\mathcal{G}}(h) = \overline{\mathcal{E}}(h) + |\ell - h(1)|,$$

where

$$\overline{\mathcal{E}}(h) = \int_0^1 \sqrt{r^2(h_a')^2 + \sin^2(h)}\, dr + \int_0^1 r\, |dh_c| + \sum_{r \in J_h} r|h^+(r) - h^-(r)|.$$

Here and throughout, primes denote differentiation with respect to the independent variable; as usual, the measure h' is decomposed into its absolutely continuous part $h_a'\, dr$, its Cantor part dh_c and its jump part dh_j supported in J_h, and the relaxation $\overline{\mathcal{J}}$ of a functional $\mathcal{J}$ in $BV(I)$ is defined as

$$\overline{\mathcal{J}}(h) := \inf \left\{ \liminf_{n\to\infty} \mathcal{J}(h_n) : \ h_n \in W^{1,1}(I),\ h_n \to h \text{ in } L^1(I) \right\}.$$

The first main result is the following:

Theorem 1. *Upon (7), and up to a reflection with respect to $h = \frac{\pi}{2}$ if $\ell = \frac{\pi}{2}$, there exists a unique solution h to*

$$h \in \text{argmin } \{\overline{\mathcal{G}}(h) : \ h \in BV((0,1))\}. \tag{P}$$

Furthermore $h \in C^\infty((0,1]) \cap C^1([0,1])$, h is positive and nondecreasing in $(0,1]$, $h(0) = 0$, $h(1) = \ell$, and h solves

$$h'' = \frac{\sin 2h}{2r^2} - \frac{2h'}{r} + \frac{rh'^2}{\sin^2 h}\left(\frac{\sin 2h}{r} - h'\right). \tag{ODE}$$

Here and throughout, by a solution to an ode we mean a twice-differentiable function which satisfies the ode point-wise.

Of course, Theorem 1 implies the existence of a unique (up to symmetries) minimizer of $\mathcal{G}$. However, it turns out that $\mathcal{G}$ is genuinely non-convex at least for ℓ sufficiently close to $\frac{\pi}{2}$. Indeed, in our second main result we provide the existence of smooth critical points of $\mathcal{G}$ which are different from the absolute minimizer.

Theorem 2. *There exists a global solution $h \in C^\infty((0,\infty)) \cap C^1([0,\infty))$ to (ODE) which satisfies the following properties:*

a) *$h(0) = 0$, $h(r) \to \frac{\pi}{2}$ as $r \uparrow \infty$;*
b) *the counter-image of the critical points of h consists of increasing sequence $r_n \uparrow \infty$ such that $h(r_{2n}) \in (\frac{\pi}{2}, \pi)$ are local maxima, $h(r_{2n+1}) \in (0, \frac{\pi}{2})$ are local minima, and $|h(r_n) - \frac{\pi}{2}|$ is decreasing.*

Furthermore:

c) *for any $\ell \in (0, \frac{\pi}{2}]$, the minimizer h_ℓ of $\overline{\mathcal{G}}$ is given by $h_\ell(r) = h(\alpha r)$ (or $h_\ell(r) = \pi - h(\alpha r)$ if $\ell = \frac{\pi}{2}$) for a suitable $\alpha > 0$;*
d) *any non-constant global solution $\hat{h}$ to (ODE) has the form*

$$\hat{h}(r) = k\pi \pm h(\alpha r) \text{ for some } \alpha > 0, \ k \in \mathbb{Z}.$$

Note that (d) characterizes all global non-constant solutions to (ODE) as suitable rescalings of h: its qualitative properties are summarized in Figure 1.

Due to the scaling invariance of (ODE) with respect to r, Theorem 2 shows that, in addition to $h \equiv \frac{\pi}{2}$ and to the minimizer given by Theorem 1, there exist infinitely many other critical points for $\mathcal{E}$ subject to $h(1) = \frac{\pi}{2}$. This marks another substantial difference with respect to 2-harmonic maps. Also in contrast with 2-harmonic maps, Theorem 2 excludes the existence of global monotone solutions connecting 0 to $\frac{\pi}{2}$.

Coming back to the question of the role played by π, Theorem 2 suggests that a classical solution of (6) subject to $h = h_0$ on the parabolic boundary, with $h_0(0) = 0$ and $h_0(r) \in (0,\pi)$ for $0 < r \leq 1$, should cease to exist in finite time

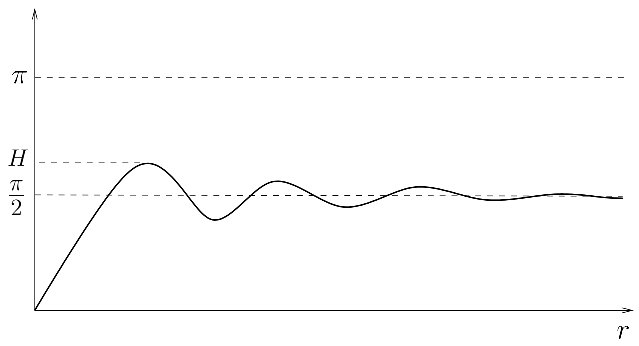

FIGURE 1. The qualitative profile of the global solutions to (ODE).

at least provided $h_0(1) > H = \max h$, where h is the profile given by Theorem 2 (see Figure 1). Whether or not this expected lower bound may be improved to $h_0(1) > \frac{\pi}{2}$ is intuitively less clear and relates to the stability of the smooth critical points.

The paper is organized as follows: in section 2 we prove Proposition 1. In section 3 we derive basic properties of any minimizer: positivity, monotonicity, continuity and attainment of the boundary value. In section 4 we use these properties, combined with regularity theory, to construct a smooth minimizer. In Section 5 we characterize the behavior at $r = 0$ of solutions to (ODE). The proof of Thoerem 1 is completed in section 6 by showing uniqueness of monotone solutions to (ODE). Finally, in section 7 we prove Theorem 2.

2. The relaxed functional

In this section we prove Proposition 1. The relaxation $\overline{\mathcal{G}}$ of $\mathcal{G}$ is characterized by:

(a) $\overline{\mathcal{G}}$ is lower semi-continuous;
(b) for any $h \in BV((0,1))$ there exist $\{h_n\} \subset W^{1,1}((0,1))$ with $h_n(1) = \ell$ such that $h_n \to h$ in $L^1((0,1))$ and
$$\overline{\mathcal{G}}(h) = \lim_{n\to\infty} \mathcal{G}(h_n).$$

To see (a), let us introduce $\mathcal{E}_A : L^1((0,A)) \to \mathbb{R}$ defined by
$$\mathcal{E}_A(h) := \int_0^A \sqrt{r^2 h'^2 + \sin^2 h}\, dr.$$

By [1, Theorem 6.1], its relaxation is represented by
$$\overline{\mathcal{E}}_A(h) = \int_0^A \sqrt{r^2 (h'_a)^2 + \sin^2(h)}\, dx + \int_0^A r\, |dh'_c| + \sum_{r\in J_h} r|h^+(r) - h^-(r)|.$$

Given $h \in BV((0,1))$, we consider $h^* \in BV((0,2))$ given by

$$h^*(r) = \begin{cases} h(r) & \text{if } 0 < r \le 1 \\ \ell & 1 < r < 2 \end{cases}$$

Then

$$\overline{\mathcal{E}}_2(h^*) = \overline{\mathcal{E}}(h) + |\sin \ell| + |h(1) - \ell| = \overline{\mathcal{G}}(h) + |\sin \ell|,$$

and (a) is implied by the lower semi-continuity of $\overline{\mathcal{E}}_2$.

In order to prove (b), let $h \in BV((0,1))$; we first claim that a sequence $\{f_n\} \subset W^{1,1}((0,1))$ exists such that

$$f_n(1) = h(1),\ f_n \to h \text{ in } L^1((0,1)) \text{ and } \overline{\mathcal{E}}(h) = \lim_{n\to\infty} \mathcal{E}(f_n). \tag{8}$$

Indeed, as $\overline{\mathcal{E}}_1 = \overline{\mathcal{E}}$ is the relaxed functional of $\mathcal{E}_1 = \mathcal{E}$, a sequence $\tilde{f}_n \in W^{1,1}((0,1))$ exists such that $\tilde{f}_n \to h$ in $L^1((0,1))$ and

$$\overline{\mathcal{E}}(h) = \lim_{n\to\infty} \mathcal{E}(\tilde{f}_n)$$

A version of the Slicing Lemma of De Giorgi given in [6, Lemma 2.7] then permits find a sequence $f_n \in W^{1,1}((0,1))$ such that $f_n(1) = h(1)$, $f_n \to h$ in $L^1((0,1))$ and

$$\limsup_{n\to\infty} \mathcal{E}(f_n) \le \liminf_{n\to\infty} \mathcal{E}(\tilde{f}_n) = \mathcal{E}(h).$$

Together with the lower semi-continuity of $\overline{\mathcal{E}}$, this proves (8).
Let now

$$g_n(x) = n(\ell - h(1)) \left(x - 1 + \frac{1}{n} \right)_+ \quad \text{and} \quad h_n = f_n + g_n.$$

Then

$$\begin{aligned} \overline{\mathcal{G}}(h_n) &= \overline{\mathcal{E}}(h_n) \\ &= \int_0^{1-\frac{1}{n}} \sqrt{(rf_n')^2 + \sin^2 f_n} + \int_{1-\frac{1}{n}}^1 \sqrt{(r(f_n' + g_n'))^2 + \sin^2(f_n + g_n)} \\ &\le \int_0^{1-\frac{1}{n}} \sqrt{(rf_n')^2 + \sin^2 f_n} + \int_{1-\frac{1}{n}}^1 (r|f_n'| + r|g_n'| + 1) \\ &\le \mathcal{E}(f_n) + \int_{1-\frac{1}{n}}^1 |g_n'| + \frac{1}{n} \\ &= \mathcal{E}(f_n) + |\ell - h(1)| + o_n(1) \end{aligned}$$

which proves (b) completes the proof of Proposition 1.

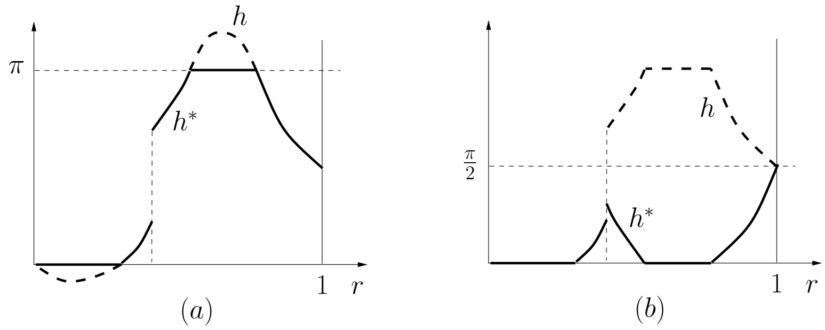

FIGURE 2. Cutting h at levels 0 and π decreases the energy (a), and reflecting it below $\frac{\pi}{2}$ does not increase it (b).

3. Properties of minimizers

In this section we prove the following result.

Proposition 2. *Assume (7). Then, up to a reflection with respect to* $h = \frac{\pi}{2}$ *if* $\ell = \frac{\pi}{2}$*, any solution* h *to problem (P) is nondecreasing, continuous in* $(0, 1]$ *with* $h(1) = \ell$*, and positive in* $(0, 1]$.

We split the proof into Lemmas. We begin with two straightforward observations based on periodicity and symmetry of $\sin^2 h$ (see Figure 2).

Lemma 1. *Assume (7). Then for any* $h \in BV((0, 1))$*, the function*

$$h^*(r) = \min\{\pi, \max\{0, h(r)\}\}$$

is such that $\overline{\mathcal{G}}(h^*) < \overline{\mathcal{G}}(h)$ *unless* $h = h^*$.

Lemma 2. *Assume (7). Then for any* $h \in BV((0, 1); [0, \pi])$*, the function*

$$h^*(r) = \frac{\pi}{2} - \left|\frac{\pi}{2} - h\right|$$

is such that $\overline{\mathcal{G}}(h^*) \le \overline{\mathcal{G}}(h)$.

Next, we show that the energy may be strictly decreased if h is not monotone (see Figure 3 (a)).

Lemma 3. *Assume (7). If* $h \in BV((0, 1); [0, \frac{\pi}{2}])$ *is not nondecreasing, there exists* $h^* \in BV((0, 1); [0, \frac{\pi}{2}])$ *such that* h^* *is nondecreasing and* $\overline{\mathcal{G}}(h^*) < \overline{\mathcal{G}}(h)$.

Proof. Since h is not nondecreasing, $r_1 \in (0, 1)$ exists such that

$$h(r_1) > h_* = \inf_{(r_1,1)} h. \tag{9}$$

We are going to show that the energy may be decreased by "cutting the hill" around r_1 with a straight line (see Figure 3 (a)). Take a minimizing sequence

$$(r_1, 1) \ni r_n \to r_2 \in [r_1, 1] \text{ such that } h(r_n) \downarrow h_* \text{ as } n \uparrow \infty. \tag{10}$$

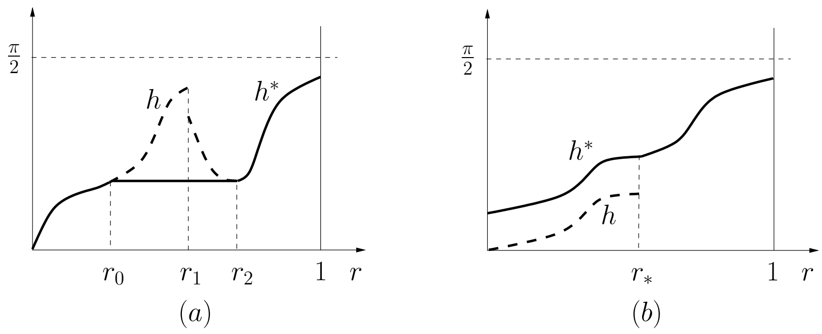

FIGURE 3. Cutting a hill (a), as well as healing a discontinuity (b), decreases the energy.

Let also

$$r_0 = \inf\{r \le r_1 : h(\rho) > h_* \text{ for all } \rho \in [r, r_1]\}. \tag{11}$$

Note that

$$\lim_{r \to r_0^-} h(r) \overset{(11)}{\le} h_*. \tag{12}$$

If $r_2 = r_0 (= r_1)$, we would have by (10) that $\lim_{r\to r_1^+} h(r) = h_*$, which together with (11) would contradict (9). Therefore $r_2 - r_0 > 0$. Let then

$$h^*(r) = \begin{cases} h_* & r \in (r_0, r_2) \\ h(r) & \text{elsewhere.} \end{cases}$$

If $r_2 < 1$, we have

$$\begin{aligned}
&\overline{\mathcal{G}}(h) - \overline{\mathcal{G}}(h^*) \\
&= \int_{r_0}^{r_2} \sqrt{r^2 (h_a')^2 + \sin^2(h)}\, dx + \int_{r_0}^{r_2} r\, d|h_c'| - (r_2 - r_0) \sin h_* \\
&\quad + \sum_{r \in J_h \cap [r_0, r_2]} r|h^+(r) - h^-(r)| - r_0 |h_* - h^-(r_0)| - r_2 |h^+(r_2) - h_*| \\
&\overset{9),(11)}{>} r_0 \left(|h^+(r_0) - h^-(r_0)| - |h_* - h^-(r_0)| \right) \\
&\quad + r_2 \left(|h^+(r_2) - h^-(r_2)| - |h^+(r_2) - h_*| \right) \\
&\overset{(11),(12)}{=} r_0 \left(h^+(r_0) - h_* \right) + r_2 \left(|h^+(r_2) - h^-(r_2)| - (h^+(r_2) - h_*) \right) \\
&\ge 0.
\end{aligned}$$

In the last inequality, we have used the fact that, by (10), either $h^+(r_2) = h_*$ or $h^-(r_2) = h_*$. This proves Lemma 3 if $r_2 < 1$. Drawing the conclusion if $r_2 = 1$ is simpler and we omit it. □ □

We continue by ruling out jumps in the bulk:

Lemma 4. *Any solution h to problem (P) belongs to $C((0,1])$ and satisfies $h(1)=\ell$.*

Proof. By Lemmas 1-3, we may assume without loss of generality that h is nondecreasing and $h \le \frac{\pi}{2}$. We first observe that

$$h(1) \le \ell. \tag{13}$$

Indeed, if by contradiction $h(1) > \ell$, one may argue as in the proof of Lemma 3: letting

$$r_0 = \inf\{r \le 1 : h(\rho) > \ell \text{ for all } \rho \in (r,1]\} < 1,$$

the function

$$h^*(r) = \begin{cases} h(r) & r \in (0, r_0) \\ \ell & \text{elsewhere} \end{cases}$$

would have less energy. We now show that

$$h \in C((0,1]) \text{ and } h(1) = \ell. \tag{14}$$

Suppose by contradiction that (14) is false. Then, by (13), either

- h has a jump at $r_* \in (0,1)$; then, let $A = h^+(r_*) - h^-(r_*)$

or

- $h(1) < \ell$; then, let $r_* = 1$ and $A = \ell - h(1)$.

We are going to show that the energy may be decreased by moving the discontinuity down to $r=0$. Namely, for $r \le r_*$ we raise h (see Figure 3 (b)) by A:

$$\theta(r) := \begin{cases} h(r) + A & \text{if } 0 < r < r_* \\ h(r) & r_* < r < 1 \end{cases}$$

Since $A > 0$, $h \ge 0$ and $A + h \le \frac{\pi}{2}$, we see that for $r \in (0, r_*)$

$$\begin{aligned} & \sqrt{r^2(h')^2 + \sin^2\theta} - \sqrt{r^2(h')^2 + \sin^2 h} \\ = \; & \frac{\sin^2(h+A) - \sin^2(h)}{\sqrt{r^2(h')^2 + \sin^2(h+A)} + \sqrt{r^2(h')^2 + \sin^2 h}} \\ \le \; & \frac{\sin^2(h+A) - \sin^2 h}{\sin(h+A) + \sin h} \\ = \; & \sin(h+A) - \sin h \\ < \; & \sin A \\ < \; & A. \end{aligned}$$

Hence

$$\begin{aligned} \overline{\mathcal{G}}(\theta) &= \overline{\mathcal{G}}(h) + \int_0^{r_*} \left(\sqrt{r^2(h')^2 + \sin^2\theta} - \sqrt{r^2(h')^2(r) + \sin^2 h(r)} \right) - r_* A \\ &< \overline{\mathcal{G}}(h), \end{aligned}$$

and we have found a contradiction. Hence (14) holds true and the proof is complete. □ □

In view of Lemmas 3 and 4, the proof of Proposition 2 is complete once we have shown that:

Lemma 5. *Assume (7). Up to a reflection with respect to* $\frac{\pi}{2}$ *if* $\ell = \frac{\pi}{2}$*, any solution* h *to problem (P) is positive in* $(0,1]$.

Proof. By Lemmas 1-3 and 4, we assume without loss of generality that $h \in [0, \frac{\pi}{2}]$ is nondecreasing and continuous. Assume by contradiction that $a > 0$ exists such that $h(a) = 0$ and $h(r) > 0$ for $a < r \leq 1$. By Lemma 4, $a < 1$. Let then $\varepsilon > 0$ such that $a - \varepsilon \geq 0$ and $a + \varepsilon \leq 1$. We first notice that

$$\begin{aligned}\int_0^{a+\varepsilon} r\left(\sqrt{r^2(h_a')^2+\sin^2 h}dr + |dh_c|\right) &\geq \int_a^{a+\varepsilon} r\left(|h_a'|\,dr + |dh_c|\right)\\ &\geq ah(a+\varepsilon).\end{aligned}$$

On the other hand, if we take

$$\theta(r) = 2\arctan\left(\frac{\tan(\frac{h(a+\varepsilon)}{2})r}{(a+\varepsilon)}\right),$$

we have

$$\int_0^{a+\varepsilon}\sqrt{r^2(\theta')^2+\sin^2\theta}\,dr = -\sqrt{2}(a+\varepsilon)\frac{\ln(\frac{\cos(h(a+\varepsilon))+1}{2})}{\tan(h(a+\varepsilon)/2)}.$$

Therefore, letting

$$h^*(r) = \begin{cases}\theta(r) & 0 < r < a+\epsilon\\ h(r) & \text{elsewehere}\end{cases}$$

and recalling that h is continuous, we obtain that

$$\begin{aligned}\overline{\mathcal{G}}(h^*) - \overline{\mathcal{G}}(h) &\leq -\sqrt{2}(a+\varepsilon)\frac{\ln(\frac{\cos(h(a+\varepsilon))+1}{2})}{\tan(h(a+\varepsilon)/2)} - ah(a+\varepsilon)\\ &= -a\left(\sqrt{2}(1+o_\varepsilon(1))\frac{\cos h(a+\epsilon)-1}{h(a+\epsilon)} + h(a+\epsilon)\right)\\ &= -ah(a+\epsilon)\left(-\frac{\sqrt{2}}{2}+1+o_\epsilon(1)\right) \quad \text{as } \epsilon \downarrow 0.\end{aligned}$$

Hence h cannot be a minimizer. □ □

4. Existence of a smooth minimizer

In this section we prove the following result:

Proposition 3. *Assume (7). There exists a solution* $h \in C^\infty((0,1])$ *to problem (P), and* h *is also a solution of (ODE).*

We split the proof into Lemmas. Existence of a minimizer follows immediately from Lemmas 1-3:

Lemma 6. *There exists a solution* h *to problem (P).*

Proof. Let $h_j \in BV((0,1))$ be a minimizing sequence. By Lemmas 1-3, we may assume that each h_j is nondecreasing and $h_j \in [0, \frac{\pi}{2}]$. Then,

$$\|h_j\|_{BV((0,1))} = \|h_j\|_{L^1((0,1))} + |Dh|((0,1)) \leq 2h_j(1) \leq \pi.$$

Thus, we can extract a subsequence, still denoted by $\{h_j\}$ such that $h_j \to h^* \in BV((0,1))$ in $L^1((0,1))$. Having in mind the lower semi-continuity of $\overline{\mathcal{G}}$, we obtain that h^* is a solution to problem (P). □ □

In order to improve continuity of h (as given by Proposition 2) to Lipschitz continuity, we shall apply a regularity result proved in [5]. This is made possible by the fact that any minimizer is positive in $(0,1]$, i.e. $\overline{\mathcal{G}}$ is non-degenerate in the bulk. The following holds:

Lemma 7. *For any solution $\overline{h}$ of (P) and any $0 < a < b \leq 1$, there exists a solution $h^*_{a,b} \in W^{1,\infty}((a,b))$ of*

$$\begin{aligned} h \in \operatorname{argmin}\{\mathcal{F}_{a,b}(h) = \textstyle\int_a^b \sqrt{r^2(h')^2 + \sin^2 h}\, dr \ : \\ h \in W^{1,1}((a,b)), h(a) = \overline{h}(a), h(b) = \overline{h}(b), h' \geq 0\}. \end{aligned} \tag{15}$$

Proof. Via the smooth transformation

$$g(x) = \frac{1}{L}\log(\tan(\frac{h(be^{-Lx})}{2})),$$

we see that (15) is equivalent to

$$\begin{aligned} g \in \operatorname{argmin}\{\mathcal{G}_{a,b}(g) = \int_0^1 \frac{2Le^{-Lx}e^{Lg}}{1+e^{2Lg}}\sqrt{1+|g_x|^2}\, dx \ : \\ g \in W^{1,1}((0,1)), g(0) = s_b, g(1) = s_a, g' \geq 0\} \end{aligned}$$

where

$$L = \log b - \log a,\ s_b = \frac{1}{L}\log(\tan(\frac{\overline{h}(b)}{2})),\ s_a = \log(\tan(\frac{\overline{h}(a)}{2})).$$

Applying [5, Theorem 5.1], we obtain that there exists a solution $g^* \in W^{1,\infty}((0,1))$ to

$$\begin{aligned} g \in \operatorname{argmin}\{\mathcal{G}_{a,b}(g) + |\arctan(e^{g(0)}) - \arctan(e^{s_b})| \\ + \tfrac{a}{b}|\arctan(e^{g(1)}) - \arctan(e^{s_a})| : \quad g \in W^{1,1}((a,b)),\ g_x \geq 0\} \end{aligned}$$

which implies the existence of a solution $h^*_{a,b} \in W^{1,\infty}(a,b))$ to

$$\begin{aligned} h \in \operatorname{argmin}\{\mathcal{F}_{a,b}(h) + a|h(a) - \overline{h}(a)| + b|h(b) - \overline{h}(b)| : \\ h \in W^{1,1}((0,1)),\ h' \geq 0\}. \end{aligned} \tag{16}$$

To complete the proof, it remains to show that

$$h^*_{a,b}(a) = \overline{h}(a) \text{ and } h^*_{a,b}(b) = \overline{h}(b).$$

To this aim, let

$$\begin{aligned} \overline{h}_n \in W^{1,1}((a,b)),\ \overline{h}_n(a) = \overline{h}(a),\ \overline{h}_n(b) = \overline{h}(b) \\ \text{such that } \overline{h}_n \to \overline{h} \text{ in } BV((a,b)) \end{aligned}$$

and

$$\tilde h(r) := \begin{cases} \overline{h}(r) & r \in (0,1)\setminus(a,b) \\ h^*_{a,b}(r) & r\in(a,b) \end{cases}$$

Note that

$$\int_a^b \sqrt{r^2(\overline{h}_n')^2+\sin^2\overline{h}_n}\,dr \to \int_a^b \sqrt{r^2(\overline{h}_a')^2+\sin^2\overline{h}}\,dr + \int_a^b r|d\overline{h}_c|.$$

Therefore

$$\begin{aligned}\overline{\mathcal{G}}(\tilde h) &= \overline{\mathcal{G}}(\overline{h}) - \int_a^b \sqrt{r^2(\overline{h}_a')^2+\sin^2\overline{h}}\,dr - \int_a^b r|d\overline{h}_c| \\ &\quad +\mathcal{F}_{a,b}(h^*_{a,b}) + a|\overline{h}(a)-h^*_{a,b}(a)| + b|\overline{h}(b)-h^*_{a,b}(b)| \\ &= \overline{\mathcal{G}}(\overline{h}) - \int_a^b \sqrt{r^2(\overline{h}_n')^2+\sin^2\overline{h}_n}\,dr + o_n(1) \\ &\quad +\mathcal{F}_{a,b}(h^*_{a,b}) + a|\overline{h}(a)-h^*_{a,b}(a)| + b|\overline{h}(b)-h^*_{a,b}(b)| \\ &\le \mathcal{G}(\overline{h}) + o_n(1),\end{aligned}$$

where the last inequality follows from the fact that $h^*_{a,b}$ is a solution to (16). Then $\tilde h$ is also a minimizer of $\overline{\mathcal{G}}$, hence it is continuous by Lemma 4. □ □

We may use Lemma 7 to construct a Lipschitz continuous minimizer from a given one. It will turn out useful in the next section to also fix a value in the interior.

Lemma 8. *For any solution $\overline{h}$ to problem (P) and any $r_0 \in (0,1)$, there exists a solution $\tilde h$ to problem (P) such that $h \in W^{1,\infty}_{loc}((0,1])$ and $\tilde h(r_0) = \overline{h}(r_0)$.*

Proof. Let $a_0 = 1$, $a_1 = r_0$, and a_n such that $a_n \downarrow 0$. Let $h_0 = \overline{h}$, $h^*_{a_n,a_{n-1}}$ as given by Lemma 7, and

$$h_n(r) = \begin{cases} h_{n-1}(r) & r\in(0,1)\setminus(a_n,a_{n-1}) \\ h^*_{a_n,a_{n-1}}(r) & r\in(a_n,a_{n-1}). \end{cases}$$

By Lemmas 4 and 7, $h_n(r) \in W^{1,\infty}([a_n,1]) \cap C((0,1])$, and by construction, $h_n \to \tilde h$ in $W^{1,\infty}_{loc}((0,1])$ with $\tilde h(r_0) = \overline{h}(r_0)$. To see that $\tilde h$ is a minimizer of $\overline{\mathcal{G}}$, it suffices to show that

$$\overline{\mathcal{G}}(h_n) \le \overline{\mathcal{G}}(h_{n-1}) \le \overline{\mathcal{G}}(\overline{h}) \quad \text{for all } n \tag{17}$$

and that

$$h_n \to \tilde h \quad \text{in } L^1((0,1)). \tag{18}$$

For (17), let $f_j \in W^{1,1}((a_n,a_{n-1}))$ such that $f_j(a_n) = h_{n-1}(a_n)$, $f_j(a_{n-1}) = h_{n-1}(a_{n-1})$ and $f_j \to h_0$ in $BV((a_n,a_{n-1}))$. Then

$$\begin{aligned}\overline{\mathcal{G}}(h_n) - \overline{\mathcal{G}}(h_{n-1}) &= \mathcal{F}_{a_n,a_{n-1}}(h^*_{a_n,a_{n-1}}) - \overline{\mathcal{F}}_{a_n,a_{n-1}}((h_0)) \\ &= \mathcal{F}_{a_n,a_{n-1}}(h^*_{a_n,a_{n-1}}) - \lim_{j\to\infty}\mathcal{F}_{a_n,a_{n-1}}(f_j),\end{aligned}$$

and since $h^*_{a_n,a_{n-1}}$ is a minimizer, the right-hand side can not be positive. For (18), we just note that h_n is a Cauchy sequence, since for $n \geq m$

$$\int_0^1 |h_n - h_m| \leq \int_0^{a_{m-1}} 2\ell = o_m(1).$$

Therefore $\tilde{h}$ is a solution to problem (P). □ □

We are now ready to complete the proof of Proposition 3.

Proposition 3. Existence of a minimizer is guaranteed by Lemma 6. Let then $\tilde{h}$ as given by Lemma 8. It is straightforward to check that for all $a \in (0,1)$ $\tilde{h}$ is also a solution to

$$h \in \operatorname{argmin}\{\mathcal{F}_{a,1}(h) = \textstyle\int_a^b \sqrt{r^2(h')^2 + \sin^2 h}\, dr \ : \\ h \in W^{1,1}((a,1)), h(a) = \tilde{h}(a), h(1) = \ell\}.$$

Then, since $\tilde{h} \in W^{1,\infty}((a,1))$, $\tilde{h}$ is a point-wise solution to (ODE), hence it is smooth. □ □

5. Local behavior at $r = 0$

In this section we prove the following result:

Proposition 4. *Let h be a non-constant solution to (ODE) in $(0,\varepsilon)$ for some $\varepsilon > 0$, such that $h \in (0,\pi)$. Then $h \in C^1([0,\varepsilon))$ and*

$$\lim_{r\to 0^+} h(r) = 0 \ \textit{or} \ \lim_{r\to 0^+} h(r) = \pi. \tag{19}$$

A few smooth coordinate transforms will turn out to be useful here as well as int the next section. With the first one, we pass to logarithmic coordinates by letting

$$f(t) := h(e^{-t}). \tag{20}$$

In terms of f, (ODE) reads as

$$f'' = f' + \frac{(f')^2}{\sin^2 f}(f' + \sin(2f)) + \frac{\sin(2f)}{2}. \tag{21}$$

For $f \in (0,\pi)$, we may define

$$w(t) := \begin{cases} (\tan f(t))^{-1} & \text{if } f(t) \neq \frac{\pi}{2} \\ 0 & \text{if } f(t) = \frac{\pi}{2}. \end{cases} \tag{22}$$

Note that

$$w'(t) = -\frac{1}{\tan^2 f}\frac{1}{\cos^2 f} f' = -\frac{f'}{\sin^2 f} = -\frac{e^t h'(e^t)}{sin^2(h(e^t))}. \tag{23}$$

In terms of w, (21) reads as

$$w'' = w' + \frac{(w')^3}{1+w^2} - w. \tag{24}$$

We first prove (19), which is implied by the following:

Lemma 9. *Let $w(t)$ be a non-constant solution of (24) in $(t_\epsilon, +\infty)$ for some $t_\epsilon \in \mathbb{R}$. Then*

$$\lim_{t\to\infty} w^2(t) + (w')^2(t) = +\infty.$$

Proof. We have

$$\frac{d}{dt}(w^2 + (w')^2)) = 2(w')^2 \left(1 + \frac{(w')^2}{1+w^2}\right) \geq 2(w')^2. \tag{25}$$

In particular,

$$\exists \lim_{t\to\infty} w^2(t) + (w')^2(t) = C^- \in (0, \infty]$$

and

$$w^2(t) + w^2(t) \geq w^2(t_0) + w'^2(t_0) + \int_{t_0}^t (w')^2 \tag{26}$$

for all $t_0 > t_\varepsilon$. Assume by contradiction that $C < \infty$. Then, we claim that

$$\lim_{t\to\infty} (w')^2(t) = 0. \tag{27}$$

Indeed, if not we would have $(w')^2(t_n) \geq C_1 > 0$ for a subsequence; choosing $t_0 = t_n$ in (26) with n sufficiently large, we would have

$$(w')^2(t) \geq o_n(1) + \int_{t_n}^t (w')^2\, dt, \quad (w')^2(t_n) \geq C_1,$$

yielding $(w')^2 \to \infty$ in contradiction with $(w')^2 \leq C$. Therefore (27) holds, implying that

$$\lim_{t\to\infty} w^2(t) = C \neq 0.$$

Thus

$$\lim_{t\to\infty} w''(t) = -C$$

in contradiction with (27). Hence $C = \infty$. □ □

In view of Lemma 9, w is strictly monotone for t sufficiently large. Hence we may define its inverse

$$t(w): \ w(t(w)) = w \tag{28}$$

and the function

$$g(w) := \frac{w}{w'(t(w))}. \tag{29}$$

In terms of g, (24) reads as

$$g' = \frac{1}{w}\left(g^3 - g^2 + g - \frac{w^2}{1+w^2}\right). \tag{30}$$

The large-w behavior of g is at the core of our argument for the regularity of h:

Lemma 10. *Let g be a positive solution of (40) in (w_ε, ∞) for some $w_\varepsilon \in \mathbb{R}$. Then*

$$\lim_{w\to+\infty} g(w) = 1^-. \tag{31}$$

Proof. We claim that

$$g(w) \le 1 \text{ for all } w \in (w_\varepsilon, \infty). \tag{32}$$

Assume on the contrary that $g(w_0) = 1+\varepsilon$ for some $\varepsilon > 0$ and $w_0 > w_\varepsilon$. Since, by (40),

$$g' > \frac{1}{w}(g^3 - g^2 + g - 1) = \frac{1}{w}(1 + g^2)(g - 1),$$

we would then have that $g'(w_0) > 0$. Hence $g(w) > 1 + \varepsilon$ for all $w > w_0$, and therefore

$$g'(w) > \frac{g^2}{w}\varepsilon.$$

Integrating this inequality, we see that

$$g(w) > \frac{1}{\frac{1}{g(w_0)} - \varepsilon \log w}$$

which shows that $g(w)$ would blow up at a finite w, a contradiction. This proves (32).

To complete the proof, it suffices to show that $\liminf_{w\to+\infty} g(w) \ge 1$. Assume on the contrary that a sequence $w_n \uparrow \infty$ exists such that $g(w_n) \le 1 - \varepsilon$ for some $\varepsilon > 0$, and choose n so large that $\frac{w_n^2}{1+w_n^2} \ge 1 - \frac{\varepsilon}{2}$. Note that

$$\begin{aligned} g' &\le \frac{1}{w}\left[(1 + g^2)(g - 1) + \frac{\varepsilon}{2}\right] \\ &< \frac{1}{w}\left[g - 1 + \frac{\varepsilon}{2}\right] \qquad \text{for all } w : \ \frac{w^2}{1 + w^2} \ge 1 - \frac{\varepsilon}{2} \text{ and } g(w) < 1. \end{aligned}$$

Hence $g'(w_n) < 0$, $g(w) \le 1 - \varepsilon$ for all $w > w_n$, and moreover

$$g'(w) < -\frac{\varepsilon}{2w}.$$

After integration, this yields

$$g(w) < g(w_n) - \frac{\varepsilon}{2}\log(w)$$

in contradiction with the positivity of g. □ □

We are now ready to prove Proposition 4.

Proposition 4. The limit (19) follows immediately from Lemma 9. Up to exchanging h with $\pi - h$, we may assume that $h \downarrow 0$, so that (9) reads as

$$\lim_{t\to\infty} w(t) = \infty, \quad \lim_{t\to\infty} w'(t) = \infty. \tag{33}$$

In particular, for t sufficiently large w is increasing, so that (28) and (29) make sense, and g is positive. To complete the proof, it suffices to show that

$$\exists \lim_{r\to 0^+} h'(r) \in \mathbb{R}. \tag{34}$$

We write

$$h'(r) \sim \frac{h'(r)}{\cos^2 h(r)} = \frac{rh'(r)}{\sin h(r)\cos h(r)} \cdot \frac{\tan h(r)}{r} \quad \text{as } r \downarrow 0. \tag{35}$$

On one hand, we have

$$\begin{aligned} \lim_{r\to 0^+} \frac{rh'(r)}{\sin h(r)\cos h(r)} &\overset{(20)}{=} \lim_{t\to\infty} -\frac{f'(t)}{\sin f(t)\cos f(t)} \\ &\overset{(23)}{=} \lim_{t\to\infty} \frac{w'(t)}{w(t)} \\ &\overset{(33),(28)}{=} \lim_{w\to\infty} \frac{w}{w'(t(w))} \\ &\overset{(31)}{=} 1^-. \end{aligned} \tag{36}$$

On the other hand, we have

$$\left(e^{-t}w(t)\right)' = e^{-t}\left(w'(t) - w(t)\right) \overset{(31)}{\geq} 0 \quad \text{for } t \gg 1,$$

and therefore

$$\frac{1}{w(t)e^{-t}} \leq C \quad \text{for } t \gg 1,$$

which in terms of h means that

$$\frac{\tan h(r)}{r} \leq C \quad \text{for } r \ll 1. \tag{37}$$

Plugging (36) and (37) into (35) and passing to the limit yields (34) and completes the proof of Proposition 4. □

6. Uniqueness of the minimizer

Uniqueness of the smooth minimizer is the central result of this section:

Proposition 5. *Assume (7). Then, there exists a unique positive, non-constant and non-decreasing solution to*

$$\begin{cases} h'' = \dfrac{\sin 2h}{2r^2} - \dfrac{2h'}{r} + \dfrac{r(h')^2}{\sin^2 h}\left(\dfrac{\sin 2h}{r} - h'\right) & r \in (0,1) \\ h(1) = \ell. \end{cases} \tag{38}$$

Furthermore h is increasing.

As a consequence, we obtain Theorem 1:

Theorem 1. By Propositions 2, 3 and 4, there exists a solution h to problem (P), and furthermore $h \in C^\infty((0,1]) \cap C^1([0,1])$ with $h(0) = 0$ and $h(1) = \ell$, h is positive and nondecreasing in $(0,1]$, and h solves (ODE). In order to show that h is unique, let $\overline{h}$ be any solution to problem (P): by Proposition 3, $\overline{h} \in C((0,1])$ with $\overline{h}(1) = \ell$, hence we may assume by contradiction that $r_0 \in (0,1)$ exists such that $\overline{h}(r_0) \neq h(r_0)$.
By Lemma 8, there exists a minimizer $\tilde{h} \in C^\infty_{loc}((0,1])$ of $\overline{\mathcal{G}}$ such that $\tilde{h}(r_0) = \overline{h}(r_0)$ and $\tilde{h}$ is a solution of (ODE). By Propositions 2 and 4, $\tilde{h}(1) = \ell$, $\tilde{h}$ is positive and non-decreasing in $(0,1]$, and $\tilde{h}$ is non-constant since $\tilde{h}(0) = 0$. Hence, by Proposition 5, $\tilde{h} \equiv h$, in contradiction with $\tilde{h}(r_0) = \overline{h}(r_0) \neq h(r_0)$. As a consequence h is increasing, and the proof is complete. □ □

The rest of the section is concerned with the proof of Proposition 5. We pass to logarithmic coordinates by (20). Then (38) reads as

$$\begin{cases} f'' = f' + \dfrac{(f')^2}{\sin^2 f}(f' + \sin(2f)) + \dfrac{\sin(2f)}{2} & t \in (0,\infty) \\ \lim\limits_{t\to\infty} f(t) = 0, \ f(0) = \ell \leq \dfrac{\pi}{2} \end{cases} \tag{39}$$

with f positive, non-constant and non-increasing.

Lemma 11. *Assume (7). Then any positive and non-increasing solution of (39) is decreasing.*

Proof. Suppose by contradiction that there exists $t_0 \in (0, +\infty)$ such that $f'(t_0) = 0$. We have $0 < f(t_0) < \frac{\pi}{2}$ (otherwise $f \equiv \frac{\pi}{2}$). Then $f''(t_0) = \frac{\sin(2f(t_0))}{2} > 0$ and therefore f will be increasing, a contradiction. Hence $f'(t) < 0$ for all $t \in (0,+\infty)$. □ □

Since $f \in (0, \frac{\pi}{2})$ in $(0,\infty)$, we may perform the transformations (22) and (29). In terms of g, (39) turns into

$$g' = \frac{1}{w}\left(g^3 - g^2 + g - \frac{w^2}{1+w^2}\right) \quad w \in (\hat{\ell}, \infty) \tag{40}$$

with g positive, where

$$\hat{\ell} = \begin{cases} \frac{1}{\tan \ell} & \text{if } \ell \in (0, \frac{\pi}{2}) \\ 0 & \text{if } \ell = \frac{\pi}{2}. \end{cases} \tag{41}$$

This reformulation is particularly useful since its solutions satisfy a comparison principle:

Lemma 12. *Assume (7) and (41). If g_1 and g_2 are two positive solutions of (40) and $g_1(\hat{\ell}) > g_2(\hat{\ell})$, then*

$$(g_1 - g_2)' > \frac{1}{2w}(g_1 - g_2) \quad \textit{for all } w > \hat{\ell}. \tag{42}$$

In particular, $g_1 > g_2$ in $(\hat{\ell}, \infty)$.

Proof. We have

$$\begin{aligned}(g_1 - g_2)' &= \frac{1}{w}(g_1^3 - g_2^3 - (g_1^2 - g_2^2) + g_1 - g_2) \\ &= \frac{g_1 - g_2}{w}(g_1^2 + g_1 g_2 + g_2^2 + g_1 + g_2 + 1) \\ &= \frac{g_1 - g_2}{w}\left[\left(g_1 - \frac{1}{2}\right)^2 + \left(g_2 - \frac{1}{2}\right)^2 + g_1 g_2 + \frac{1}{2}\right]\end{aligned}$$

Hence $(g_1 - g_2)'(\hat{\ell}) > 0$, and (42) follows. □ □

We are now ready to prove Proposition 5.

Proposition 5. Let h_1 and h_2 be two different positive, non-constant and non-decreasing solutions of (38). Then $h_1'(1) \neq h_2'(1)$, so that without loss of generality the corresponding solutions g_1 and g_2 to (40) are such that $g_1(\hat{\ell}) > g_2(\hat{\ell})$. By Lemma 12

$$(g_1 - g_2)' \geq \frac{g_1 - g_2}{2w} \qquad \text{for all } w > \hat{\ell}. \quad \text{for all } w > \hat{\ell}.$$

After integrating, we get for a fixed $w_0 > \hat{\ell}$

$$\frac{g_1(w) - g_2(w)}{g_1(w_0) - g_2(w_0)} \geq \sqrt{\frac{w}{w_0}}.$$

Hence

$$\lim_{w \to \infty} (g_1(w) - g_2(w)) = \infty,$$

in contradiction with Lemma 10. □ □

7. Smooth critical points

In this section we prove Theorem 2. The existence of the profile is based on the analysis of the continuation of solutions to problem (P) beyond $r = 1$. This amounts to studying the following problem, which follows from (24) upon exchanging the t-axis direction:

$$\left\{\begin{array}{ll} w'' = -w' - \dfrac{(w')^3}{1 + w^2} - w & t > 0 \\ w(0) = 0, \ w'(0) = -\alpha. & \end{array}\right. \tag{43}$$

The following holds:

Lemma 13. *For any $\alpha > 0$ there exists a unique global solution to (43). Furthermore*

$$\lim_{t \to \infty} w(t) = 0 \tag{44}$$

and the counter-image of the critical points of w consists of an increasing sequence $t_n \uparrow \infty$ such that $w(t_{2n}) < 0$ are local minima, $w(t_{2n+1}) > 0$ are local maxima, and $|w(t_n)|$ is decreasing.

Proof. Let

$$\mathcal{W}(w) = w^2 + (w')^2.$$

Since

$$\frac{d}{dt}\mathcal{W}(w(t)) = 2ww' + 2w'w'' = -2w'^2 - \frac{2w'^4}{1+w^2} < 0, \tag{45}$$

we have that $\|w\|_{C^1} < \alpha$, and therefore the solution to (43) is global. The oscillatory properties of w (43) are based on the following observations:

(A) Given a point s_* such that $w(s_*) = 0$ and $w'(s_*) \neq 0$, there exists $t_* \in (s_*, \infty)$ such that $w'(t_*) = 0$ and $w(t_*) < 0$ if $w'(s_*) < 0$ [resp. $w(t_*) > 0$ if $w'(s_*) > 0$], and t_* is a local minimum [resp. maximum] point for w.

(B) Given a point t_* such that $w'(t_*) = 0$ and $w(t_*) \neq 0$, there exists $s_* \in (t_*, \infty)$ such that $w(s_*) = 0$ and $w'(s_*) > 0$ if $w(t_*) < 0$ [resp. $w'(s_*) < 0$ if $w(t_*) > 0$].

To see (A), let

$$g(t) := \frac{w(t)}{w'(t)}.$$

Then (43) reads as

$$\begin{aligned} g' &= 1 - \frac{ww''}{w'^2} = 1 + \frac{w}{w'^2}(w' + \frac{w'^3}{1+w^2} + w) \\ &= 1 + g + g^2 + \frac{ww'}{(1+w^2)}, \end{aligned}$$

with $g(s_*) = 0$ and $g'(s_*) = 1$. Therefore g is locally increasing, hence positive, which means that $ww' > 0$ and that $g' \geq 1 + g^2$. Consequently, g remains positive as long as it is defined and blows up to ∞ at a finite t. This means that $t_* \in (s_*, \infty)$ exists such that $w'(t_*) = 0$. The sign of $w(t_*)$ follows from the positivity of g. Moreover, from (43), we get that $w''(t_*) > 0$ [resp. $w''(t_*) < 0$] and therefore t_* is a local minimum [resp. maximum] for w.

To see (B), let

$$j(t) := \frac{w'(t)}{w(t)}.$$

Then (43) reads as

$$j' = \frac{w''}{w} - \frac{w'^2}{w^2} = -j - \frac{j^3w^2}{1+w^2} - 1 - j^2.$$

We have $j(t_*) = 0$ and so $j'(t_*) = -1$. Therefore j is locally decreasing, hence negative, with $j' \leq -1 - j^2$. Therefore j is negative as long as it is defined, and blows down to $-\infty$ at a finite t, that is, $s_* \in (t_*, \infty)$ exists such that $w(s_*) = 0$. The sign of $w'(s_*)$ is determined by that of j.

It follows from an iterated application of (A) and (B), starting from $s_* = s_0 = 0$, that an increasing sequence t_n exists such that $w'(t_n) = 0$, with $w(t_{2n}) < 0$ local minima, $w(t_{2n+1})$ local maxima, and w monotone in between, with $w(s_n) = 0$, $s_n \in (t_n, t_{n+1})$. We have that

$$\lim_{n\to\infty} t_n = \infty.$$

Indeed, else we would have $t_n \uparrow T$, $s_n \uparrow T$, and therefore $w(T) = w'(T) = 0$, which is impossible since w is not constant. Furthermore, it holds that

$$|w(t_{n+1})| < |w(t_n)|.$$

Else, we would have $\mathcal{W}(t_n) \geq \mathcal{W}(t_{n+1})$, which recalling (45) would imply $\mathcal{W}' = 0$, i.e. $w' = 0$, in (t_n, t_{n+1}), a contradiction. Finally, we observe that by (45) there exists a constant $C \geq 0$ such that $\lim_{t\to+\infty} \mathcal{W}(t) = C$. If by contradiction $C > 0$, we would have that $\mathcal{W}(t_n) = w^2(t_n) \geq C^2$. Thus,

$$2C^2 \leq |w(x_n)| + |w(x_n + 1)| = |w(x_{n+1}) - w(x_n)| = \int_{x_n}^{x_{n+1}} |w'|,$$

which implies that

$$\int_0^\infty |w'| = +\infty.$$

On the other hand, we see from (45) that $\mathcal{W}'(t) \leq -2(w')^2$, and therefore

$$\mathcal{W}(t) \leq \mathcal{W}(0) - \int_0^t w'^2 \leq \mathcal{W}(0) - \left(t\int_0^t w'\right)^{\frac{1}{2}} \leq \mathcal{W}(0)(1-\sqrt{t}),$$

in contradiction with $\mathcal{W} \geq 0$. Hence $C = 0$ and (44) follows. □ □

In order to charactrize global solutions to (ODE), we shall also need to rule out that solutions may smoothly cross $h = k\pi$, the values where the equation becomes singular (which is conceivable for a generic ode, think of $h'' = h'^2/2h$, which is solved by $h = r^2$).

Lemma 14. *There exists no nontrivial solution h of (ODE) such that $h(r_0) = 0$ for some $r_0 > 0$.*

Proof. Since h is non-trivial, $r_1 \in (0,\infty)$ exists such that $h(r_1) \neq 0$. We assume without loss of gnerality that $h(r_1) \in (0, \pi/2]$, pass to logarithmic coordinates by letting $f(t) = h(e^t)$, and define

$$F(t) = \log(\tan(f/2)).$$

A simple computation shows that F solves

$$F'' = (1 + (F')^2)\left(\frac{1-e^{2F}}{1+e^{2F}} - F'\right)$$

in a neighbourhood of $t_1 = \ln r_1$, with $F(t_1) \in (-\infty, 0]$. Note that

$$\frac{d}{dt}\left(\frac{1}{2}\ln(1 + (F')^2) + G(F)\right) = -(F')^2,$$

where $G(F) = \log(1 + e^{2F}) - F$ is non-negative and such that $G(F) \to \infty$ as $|F| \to \infty$. Therefore F is defined for all $t > t_1$, which in terms of h means that $h(r) > 0$ for all $r > r_1$. Hence, it remains to show that

$$h(r) > 0 \text{ for all } r < r_1.$$

If not, we may assume without loss of generality that $F(t) \to -\infty$ as $t \downarrow 0$. Note that

$$F'' < (1 + (F')^2)(1 - F').$$

Note also that $F' \to +\infty$ at least for a sequence. Hence there exists a point t_0 such that $F'(t_0) \geq 2$. At t_0, $F'' < 0$, hence $F' > 2$ and $F'' < 0$ for all $t < t_0$, and therefore $F' \uparrow +\infty$ as $t \downarrow 0$. Integrating

$$\frac{F''}{(1 + F'^2)(F' - 1)} < -1,$$

we see that

$$-\frac{1}{F'^2} + C \lesssim -t \text{ as } t \downarrow 0,$$

that is

$$F'^2 \lesssim \frac{1}{1+t} \text{ as } t \downarrow 0.$$

Then F would be finite at $t = 0$, a contradiction. □ □

We are now ready to prove Theorem 2.

Theorem 2. Let $h^* \in C^\infty((0,1])$ be the solution to (P) with $h^*(1) = \frac{\pi}{2}$, as given by Theorem 1. Since h^* is increasing, ${h^*}'(1) = \alpha > 0$ (${h^*}'(1) = 0$ would imply $h^* \equiv \frac{\pi}{2}$, whereas $h^*(0) = 0$). Consider now the Cauchy problem

$$\left\{ \begin{array}{ll} h'' = \dfrac{\sin 2h}{2r^2} - \dfrac{2h'}{r} + \dfrac{r(h')^2}{\sin^2 h}\left(\dfrac{\sin 2h}{r} - h'\right) & r > 1 \\ h(1) = \frac{\pi}{2},\ h'(1) = \alpha. & \end{array} \right. \tag{46}$$

By standard ode theory, (46) admits a maximal solution. We pass to logarithmic coordinates by letting $f(t) := h(e^t)$, $t \in (0, \infty)$. As long as $f \in (0, \pi)$, we may define as in section 5

$$w(t) := \left\{ \begin{array}{ll} (\tan f(t))^{-1} & \text{if } f(t) \neq \frac{\pi}{2} \\ 0 & \text{if } f(t) = \frac{\pi}{2}. \end{array} \right.$$

In terms of w, (46) turns into (43), which by Lemma 13 admits a unique global solution. Hence the solution of (46) is global, too, and moreover $h \in (0, \pi)$. Patching this solution with h^* and translating the properties of w back to h immediately gives the existence of a global solution h which satisfies (a) and (b) of Theorem 2.

To prove (c), for $\ell < \frac{\pi}{2}$ we scale r so that $\hat{h}(\hat{r}) = h(\alpha\hat{r})$ satisfies $\hat{h}(1) = \ell$ with $\alpha < 1$. By the scale invariance of (ODE), $\hat{h}$ is also a solution, and $\hat{h}$ is increasing and positive in $(0, 1)$. Hence, by Proposition 5, it coincides with the minimizer h_ℓ of $\overline{\mathcal{G}}$.

To complete the proof, it remains to characterize global non-constant solutions $\hat{h}$ to (ODE). Take one of them: by Lemma 14, we may assume without loss of generality that $\hat{h} \in (0, \pi)$. We claim that

$$\exists\ r_1 : \hat{h}'(r_1) = 0. \tag{47}$$

If not, $\hat{h}$ is monotone. Up to exchanging $\hat{h}$ with $\pi - \hat{h}$, we may assume that $\hat{h}$ is non-decreasing. By Proposition 4 $\hat{h} \to 0$ as $r \to 0$, hence $\hat{h}(\varepsilon) \leq \frac{\pi}{2}$ for some $\varepsilon > 0$. Then $\tilde{h}(r) = \hat{h}(\varepsilon r)$ is a positive, non-decreasing and non-constant solution to (38). On the other hand, we may also scale r so that $\overline{h}(r) = h(\alpha r)$ satisfies $\overline{h}(1) = \hat{h}(\varepsilon)$ with $\alpha < 1$. Therefore $\tilde{h}$ and $\overline{h}$ are two positive, non-constant and non-decreasing solutions of (38) in $(0, 1)$ with the same boundary value. By Proposition 5 they coincide in $(0, 1)$, hence they coincide everywhere. But this is absurd since $\overline{h}$ oscillates, and (47) holds.

Next, we show that

$$\exists\ r_0 \leq r_1 : \hat{h}'(r_0) = 0 \text{ and } \hat{h} \text{ si monotone in } (0, r_0). \tag{48}$$

Up to exchanging $\hat{h}$ with $\pi - \hat{h}$, we may assume by Proposition 4 that $h(r) \to 0$, which means that $\varepsilon > 0$ exists such that $0 < h(r) < \frac{\pi}{2}$ for all $0 < r < \varepsilon$. It follows from the equation that all critical points in $(0, \varepsilon)$ have positive second derivative, i.e. are local minima, which is impossible. Therefore there is no critical point in $(0, \varepsilon)$, and (48) follows from (47).

To conclude, up to exchanging $\hat{h}$ with $\pi - \hat{h}$ we may assume by (48) that $\hat{h}$ is non-decreasing in $(0, r_0)$, with $\hat{h}'(r_0) = 0$. If $\hat{h}(r_0) = \frac{\pi}{2}$ then $\hat{h}$ would be a constant, and if $\hat{h}(r_0) < \frac{\pi}{2}$ then $\hat{h}''(r_0) > 0$. Both are impossible, and therefore $\hat{h}(r_0) > \frac{\pi}{2}$. Then, again by Proposition 4, $r_* < r_0$ exists such that $\hat{h}(r_*) = \frac{\pi}{2}$. Scaling r so that $r_* = 1$ and applying Proposition 5 with $\ell = \frac{\pi}{2}$ we conclude that $\hat{h} \equiv h$. □ □

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