

Towers of isospectral manifolds

Manlio Bordoni, Verita Marino*

July 2, 2007

Abstract

Given two isospectral not isometric manifolds, we construct a new couple of such manifolds as the total spaces of two Riemannian submersions with totally geodesic fibers isometric to the given ones and of basis any other given manifold. By iteration, we obtain families of isospectral not isometric manifolds.

1 Introduction

We shall assume, in this paper, that all the manifolds are connected, compact and boundaryless. The *spectrum* of a Riemannian manifold (M, g) is the sequence of the eigenvalues of the Laplace-Beltrami operator $\Delta_M = \Delta(M, g)$ acting on C^∞ functions defined on M :

$$(1) \quad \text{Spec}(\Delta_M) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow +\infty\}$$

each eigenvalue is repeated according to its multiplicity, which is finite.

From the definition itself, it follows that two *isometric* manifolds are also *isospectral*, but *the converse is not true*, that is to say there exist *isospectral but not isometric manifolds*. The first example of such manifolds was given in 1964 by J. Milnor [9] with two 16-dimensional flat tori. After him, several examples were produced by many mathematicians. A fundamental contribution in the isospectrality problem is due to T. Sunada in his famous paper [12] in 1985. Subsequently, the problem was: does isospectrality imply at least local isometry? Also the answer to this question is negative, the first counterexamples in this sense were given by C. S. Gordon, D. Webb and S. Wolpert in 1992, [6]. For a survey on isospectrality, see for instance C. S. Gordon, [5].

*Work supported by M.U.R.S.T. of Italy

Inspired by the paper [3] of G. Besson and the first author, we illustrate a general method to build up, starting from two isospectral not isometric manifolds, new couples of such manifolds, obtained as total spaces of Riemannian submersions over any basis manifold, with totally geodesic fibers isometric to the two manifolds considered at the beginning. The iteration of this process allows to build up, on any basis manifold, a couple of towers of isospectral not isometric manifolds of higher and higher dimensions: this justifies the title of the paper.

In section 2, the preliminaries are presented and it is stated the main theorem (2.1) to construct the first couple of isospectral not isometric manifolds; the proof of this theorem is performed in section 3. We end by illustrating how to build up the towers and by showing that Sunada's examples are suitable for this construction.

2 The main theorem

Let (B, j) , (F, g_F) be two Riemannian manifolds of dimensions n and r respectively, and let G be the (compact) Lie group of the isometries of (F, g_F) . Consider the fibration π associated to the principal bundle $p : P \rightarrow B$ of structural group G on the basis B with fibers diffeomorphic to F :

$$\begin{array}{ccc} P \times F & \xrightarrow{\text{diag}} & M = P \times_G F \\ \text{pr}_1 \downarrow & & \pi \downarrow \\ P & \xrightarrow{p} & B \end{array}$$

where pr_1 is the projection on the first factor, and diag is the diagonal action of the group G on the product $P \times F$. Recall that G acts freely without fixed points on the right on P and $B = P/G$. Moreover G acts freely without fixed points on $P \times F$ by the diagonal action. The quotient of $P \times F$ by diag is a $(n+r)$ -dimensional manifold, denoted $P \times_G F$; in the sequel we shall write briefly M for $P \times_G F$. It is well known that the fibration $\pi : M \rightarrow B$ is a submersion with totally geodesic fibers, see S. Kobayashi and K. Nomizu [8], vol. I p. 54 and B. O'Neill [10].

Choose a suitable biinvariant metric on G , this gives a Riemannian metric on P which makes the projection map p a Riemannian submersion with totally geodesic fibers isometric to F , see J. Cheeger [4]. The product metric on $P \times F$ passes to the quotient and it gives a Riemannian metric g on $M = P \times_G F$. In this case all the fibers $F_x = \pi^{-1}(x)$, $x \in B$, are regular r -dimensional submanifolds of M , and they, endowed with the metric g_x induced by g , are isometric to

the model fiber (F, g_F) , see B. O'Neill [10], R. Hermann [7].

With respect to the Laplacians, L. Bérard Bergery and J. P. Bourguignon introduce in [1] a decomposition of Δ_P as a sum of two operators:

$$(2) \quad \Delta_P = \Delta_P^h + \Delta_P^v$$

the so called horizontal and vertical Laplacians, corresponding to the natural decomposition of the metric. Namely, the vertical Laplacian is defined to be the operator

$$(\Delta_P^v f)(x) = (\Delta_{P_x}(f \upharpoonright_{P_x}))(x)$$

where P_x is the fiber of p through $x \in B$; in other words, Δ_P^v is the Laplacian of the fibers of p . The operator

$$\Delta_P^h = \Delta_P - \Delta_P^v$$

is the horizontal Laplacian. In general, these two operators are not elliptic, however they have discrete spectra and their eigenvalues have finite multiplicity. When the fibers are totally geodesic, as in the case under consideration, vertical and horizontal Laplacians commute then the eigenvalues of Δ_P are (not all) the sums of the eigenvalues of each of them. The eigenvalues of Δ_P^h are in substance the ones of Δ_B , and all of them appear in the sums; we shall see later that the eigenvalues of the vertical Laplacian do not take part in our problem.

Consider now two Riemannian manifolds $(F_1, g_{F_1}), (F_2, g_{F_2})$ with isometry groups G_1 and G_2 respectively. The previous construction gives two Riemannian submersions

$$\pi_1 : (M_1, g_1) \longrightarrow (B, j) \quad \text{and} \quad \pi_2 : (M_2, g_2) \longrightarrow (B, j),$$

with totally geodesic fibers isometric to (F_1, g_{F_1}) and (F_2, g_{F_2}) respectively:

$$\begin{array}{ccc} P_1 \times F_1 & \xrightarrow{\text{diag}} & M_1 \\ pr_1 \downarrow & & \pi_1 \downarrow \\ P_1 & \xrightarrow{p_1} & B \end{array} \quad \begin{array}{ccc} P_2 \times F_2 & \xrightarrow{\text{diag}} & M_2 \\ pr_1 \downarrow & & \pi_2 \downarrow \\ P_2 & \xrightarrow{p_2} & B \end{array}$$

The main result of this paper is:

Theorem 2.1 *Let (B, j) be any Riemannian manifold, and let $(F_1, g_{F_1}), (F_2, g_{F_2})$ be two isospectral manifolds, and assume that their isometry groups G_1, G_2 are isomorphic as abstract groups. Then the manifolds $M_1 = P_1 \times_{G_1} F_1$ and $M_2 = P_2 \times_{G_2} F_2$, endowed with the induced metrics g_1 and g_2 , are also isospectral.*

3 Proofs

In the proof of 2.1, an important tool is the theory of complex irreducible unitary representations of groups.

Lemma 3.1 *Two isomorphic compact Lie Groups G_1, G_2 have the same complex irreducible unitary representations.*

Proof. Let $\alpha : G_1 \rightarrow G_2$ be an isomorphism and let $\rho : G_1 \rightarrow GL(S)$ be a complex irreducible unitary representation. For $g_1 \in G_1$ denote $g_2 = \alpha(g_1)$ and define $\sigma : G_2 \rightarrow GL(S)$ by setting

$$\sigma_{g_2} = \sigma(g_2) = \sigma(\alpha(g_1)) := \rho(g_1) \quad \text{i.e. } \sigma = \rho \circ \alpha^{-1}.$$

The representation σ is irreducible. Indeed, if there exists a non trivial subspace V of S which is G_2 -invariant, one has $\sigma_{g_2}(v) \in V \forall v \in V$ and $\forall g_2 \in G_2$. This implies $V \ni \sigma_{g_2}(v) = \sigma_{\alpha(g_1)}(v) = \rho_{g_1}(v) \forall g_1 \in G_1$ (because α is an isomorphism) and $\forall v \in V$, so V is G_1 -invariant, in contradiction with the assumption that ρ is irreducible. Moreover, σ is unitary as

$$\langle \sigma_{g_2}(v), \sigma_{g_2}(w) \rangle = \langle \rho_{g_1}(v), \rho_{g_1}(w) \rangle = \langle v, w \rangle$$

for any $v, w \in S$ ($\langle \cdot, \cdot \rangle$ denotes the inner product in S).

□

Denote \mathcal{R} the set of complex irreducible unitary representations of the groups G_1 and G_2 : they are finite dimensional, since the groups are compact. As G_1 and G_2 act by isometries on (F_1, g_{F_1}) and (F_2, g_{F_2}) resp., we can consider the canonical decompositions of the spaces $L^2(F_1)$ and $L^2(F_2)$ in isotypical components (they are L^2 -orthogonal)

$$(3) \quad L^2(F_1) = \oplus_{\rho \in \mathcal{R}} L^2_{\rho}(F_1) \quad , \quad L^2(F_2) = \oplus_{\rho \in \mathcal{R}} L^2_{\rho}(F_2).$$

We have also another decomposition of these two spaces. Let us call $\mu_1 < \mu_2 < \dots$ the eigenvalues of the Laplace-Beltrami operators Δ_{F_1} and Δ_{F_2} : as the manifolds F_1 and F_2 are isospectral, Δ_{F_1} and Δ_{F_2} have the same eigenvalues and each of them has the same multiplicity. Thus, the eigenspaces $E_{F_1}(\mu_i)$ and $E_{F_2}(\mu_i)$ have the same (finite) dimension and they are consequently isomorphic. The spaces $L^2(F_j)$ have also the following L^2 -orthogonal decompositions:

$$(4) \quad L^2(F_1) = \oplus_i E_{F_1}(\mu_i) \quad , \quad L^2(F_2) = \oplus_i E_{F_2}(\mu_i).$$

As G_1 and G_2 act by isometries on (F_1, g_{F_1}) and (F_2, g_{F_2}) resp., we have the similar decompositions in isotypical components

$$(5) \quad E_{F_1}(\mu_i) = \oplus_{\rho \in \mathcal{R}} E_{F_1}^{\rho}(\mu_i) \quad , \quad E_{F_2}(\mu_i) = \oplus_{\rho \in \mathcal{R}} E_{F_2}^{\rho}(\mu_i).$$

As $E_{F_1}(\mu_i)$ and $E_{F_2}(\mu_i)$ are isomorphic, these two decompositions coincide. Notice that for any fixed $\rho \in \mathcal{R}$, one has

$$(6) \quad E_{F_j}^\rho(\mu_i) = E_{F_j}(\mu_i) \cap L_\rho^2(F_j) \quad j = 1, 2,$$

this means that not all the representations ρ appear in a fixed eigenspace and that the same ρ may appear in different eigenspaces. The number of times in which a fixed ρ appears in the eigenspaces $E_{F_1}(\mu_i)$ and $E_{F_2}(\mu_i)$ is the same.

We consider also the decomposition in isotypical components of $L^2(P_j)$, $j = 1, 2$:

$$(7) \quad L^2(P_1) = \oplus_{\rho \in \mathcal{R}} L_\rho^2(P_1) \quad , \quad L^2(P_2) = \oplus_{\rho \in \mathcal{R}} L_\rho^2(P_2).$$

For a fixed d -dimensional representation $\rho \in \mathcal{R}$, consider two isomorphic irreducible non trivial d -subspaces V_1 and V_2 of $L_\rho^2(P_1)$ and $L_\rho^2(P_2)$ which are G_1 - , resp. G_2 -, invariant. For the same ρ , consider also two isomorphic irreducible non trivial d -subspaces W_1 and W_2 of $L_\rho^2(F_1)$ and $L_\rho^2(F_2)$ which are G_1 - , resp. G_2 -, invariant. By Schur's lemma, there exists a unique (up to a complex number of modulus 1) equivariant isometry $\alpha_{V_j W_j}$ between V_j and W_j , $j = 1, 2$.

Definition 3.2 Let $\Psi^j = (\psi_1^j, \dots, \psi_d^j)$ be an orthonormal basis of V_j and let $\Phi^j = (\varphi_1^j, \dots, \varphi_d^j)$, with $\varphi_i^j = \alpha_{V_j W_j}(\psi_i^j)$, be the corresponding orthonormal basis of W_j . Define the scalar product

$$(8) \quad \langle \Psi^j, \Phi^j \rangle = \frac{1}{d} \sum_{i=1}^d \psi_i^j \overline{\varphi_i^j}$$

(the functions ψ_i^j and φ_i^j are lifted on the product $P_j \times F_j$ by the canonical projections).

It is easy to verify that the function $\langle \Psi^j, \Phi^j \rangle$, $j = 1, 2$, is invariant under the action of G_j on $P_j \times F_j$, and that it does not depend on the choice of the basis Ψ^j , thus it pass to the quotient $M_j = P_j \times_{G_j} F_j$; in the sequel we shall consider $\langle \Psi^j, \Phi^j \rangle$ as a function on M_j .

Theorem 3.3 The family of the functions $\langle \Psi^j, \Phi^j \rangle$ defined by (8) associated to all the possible pairs (V_j, W_j) is a Hilbertian basis of the space of the L^2 -functions on $P_j \times F_j$ invariant under the action of G_j , i.e. the space $L^2(M_j)$.

Proof. Let $L_{G_j}^2(P_j \times F_j)$ denote the space of L^2 -functions on $P_j \times F_j$ invariant under the action of G_j , and consider the decomposition (cf. (3), (7))

$$L^2(P_j \times F_j) = L^2(P_j) \otimes L^2(F_j) = \sum_{\rho, \rho' \in \mathcal{R}} L_\rho^2(P_j) \otimes L_{\rho'}^2(F_j).$$

Then for $\eta \in L^2_{G_j}(P_j \times F_j)$ the classical relations on the characters (see for instance [11], p. 28,48) give that the orthogonal projection of η on $L^2_\rho(P_j) \otimes L^2_{\bar{\rho}}(F_j)$ is equal to zero if $\rho' \neq \bar{\rho}$, that is to say

$$L^2_{G_j}(P_j \times F_j) = \sum_{\rho \in \mathcal{R}} L^2_\rho(P_j) \otimes L^2_{\bar{\rho}}(F_j).$$

Moreover, it is clear by computing L^2 -scalar products that for different choices of the couples (V_j, W_j) the corresponding functions $\langle \Psi^j, \Phi^j \rangle$ are L^2 -orthogonal. □

V_j and W_j are in fact included in some eigenspaces of Δ_{P_j} and Δ_{F_j} resp., related to certain eigenvalues $\alpha^j + \beta^j \in \text{Spec}(\Delta_{P_j})$ with $\alpha^j \in \text{Spec}(\Delta_{P_j}^h)$ and $\beta^j \in \text{Spec}(\Delta_{P_j}^v)$, and $\mu^j \in \text{Spec}(\Delta_{F_j})$ respectively:

$$(9) \quad \Delta_{P_j} \psi^j = \Delta_{P_j}^h \psi^j + \Delta_{P_j}^v \psi^j = \alpha^j \psi^j + \beta^j \psi^j \quad , \quad \Delta_{F_j} \varphi^j = \mu^j \varphi^j.$$

Theorem 3.4 *The family of the functions $\langle \Psi^j, \Phi^j \rangle$ is a Hilbertian basis of $L^2(M_j)$ consisting in eigenfunctions of Δ_{M_j} . The eigenvalue corresponding to the eigenfunction $\langle \Psi^j, \Phi^j \rangle$ is $\alpha^j + \mu^j$.*

For the proof, see [3]; in this paper it is also given the following explicit formula for the multiplicity of a given eigenvalue ν^j of $\text{Spec}(\Delta_{M_j})$:

$$(10) \quad \text{mult}(\nu^j) = \sum_{\rho \in \mathcal{R}} \sum_{\alpha^j + \mu^j = \nu^j} m_{\alpha^j}(\rho) m_{\mu^j}(\bar{\rho})$$

with $\alpha^j \in \text{Spec}(\Delta_{P_j}^h)$, $\mu^j \in \text{Spec}(\Delta_{F_j})$ and where $m_{\alpha^j}(\rho)$, resp. $m_{\mu^j}(\bar{\rho})$, is the multiplicity of the representation ρ , resp. $\bar{\rho}$, in the eigenspace of $\Delta_{P_j}^h$, resp. Δ_{F_j} , related to the eigenvalue α^j , resp. μ^j . It is clearly finite.

Notice that not all sums $\alpha^j + \mu^j$ are eigenvalues of Δ_{M_j} : if α^j correspond to a representation ρ , μ^j must correspond to $\bar{\rho}$, and all unitary irreducible representations of G_j appear in $L^2(P_j)$, but not all do in $L^2(F_j)$. Notice also that the vertical Laplacian $\Delta_{P_j}^v$ does not give any contribution in $\text{Spec}(\Delta_{M_j})$.

4 Conclusion

Theorem 3.4 and formula (10) show that, when the manifolds F_1, F_2 are isospectral and not isometric, the manifolds M_1, M_2 defined in section 2 have the same eigenvalues, each of them with the same multiplicity, that is they are isospectral and not isometric too. The metrics

g_1, g_2 defined on M_1, M_2 and such that the projections π_1, π_2 on B are Riemannian submersions with totally geodesic fibers isometric to F_1, F_2 respectively, decompose each in horizontal and vertical part. The horizontal parts are the same, because both are isometric to the metric of the basis B , while the vertical parts are the metrics of the fibers. Therefore the isometry groups of M_1 and M_2 are isomorphic because they must respect the decomposition in horizontal and vertical parts (these are orthogonal), and they come from the group of the isometries of B and from the groups G_1 and G_2 of the isometries of F_1 and F_2 , which are isomorphic. Consequently, the process of construction of isospectral manifolds can be repeated starting from B and M_1, M_2 . By iteration, we get so two families of isospectral manifolds fibered on B , in each of them the fibers are the manifolds obtained in the previous step: these families are the "towers".

Example 4.1 Consider now the isospectral manifolds of Sunada [12]. Let us recall that, given a 4-dimensional manifold M_0 with fundamental group G and whose universal covering is denoted $p : M \rightarrow M_0$, Sunada builds up the intermediate coverings $p_1 : F_1 = M/H_1 \rightarrow M_0$ and $p_2 : F_2 = M/H_2 \rightarrow M_0$ having as fundamental groups two *non conjugate* subgroups H_1, H_2 of G , and he shows that, when H_1 and H_2 are not isomorphic, then the manifolds $(F_1, p_1^*g_0)$ and $(F_2, p_2^*g_0)$ are isospectral but not isometric for *any* metric g_0 on M_0 . However, also in the case in which H_1 and H_2 are isomorphic as abstract groups, such manifolds are isospectral and not isometric for *some* metric g_0 on M_0 . For instance, this happens when the isometry group of the universal covering space M of M_0 , endowed with the metric p^*g_0 coincides with the covering transformation group G . In this case, the isometry groups $G_1 = G/H_1$ of $(F_1, p_1^*g_0)$ and $G_2 = G/H_2$ of $(F_2, p_2^*g_0)$ are isomorphic as abstract groups since H_1 and H_2 are. Then, we can use the manifolds F_1, F_2 to apply our construction.

References

- [1] Bérard Bergery, L.; Bourguignon, J.P. : *Laplacians and Riemannian submersions with totally geodesic fibers*, Illinois Journal of Math., **26**, n.2 (1982), 181–200
- [2] Besson, G.: *A Kato type inequality for Riemannian submersions with totally geodesic fibers*, Annals of Global Analysis and Geometry **Vol. 4**, n. 3 (1986), 273–289
- [3] Besson, G.; Bordonì, M.: *On the spectrum of Riemannian submersions with totally geodesic fibers*, Rend. Mat. Acc. Lincei **s. 9**, v. 1 (1990), 335–340

- [4] Cheeger, J. : *Some examples of manifolds of non negative curvature*, J. Diff. Geom. **8** (1972), 623-628
- [5] Gordon, C. S. : *Survey of isospectral manifolds*, Dillen, Franki J. E. (ed) et al., Handbook of differential geometry. Volume I. Amsterdam: North-Holland (2000), 747-778
- [6] Gordon, C. S.; Webb, D.; Wolpert, S. *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. 110 (1992), no. 1, 1–22.
- [7] Hermann, R.: *A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle*, Proc. Amer. Math. Soc., **vol. II** (1960), 236–242
- [8] Kobayashi, S.; Nomizu, K.: *Foundations of differential Geometry, Vol. I*, Wiley Interscience, New York-London 1963
- [9] Milnor, J.: *Eigenvalues of the Laplace operators on certain manifolds*, Proc. Nat. Acad. Sci. USA **51** (1964), 542
- [10] O’Neill, B. : *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469
- [11] Serre, J. P. : *Représentations linéaires des groupes finis*, Hermann, Paris 1971
- [12] Sunada, T. : *Riemannian coverings and isospectral manifolds*, Ann. of Math. **121** (1985), 169-186

Manlio Bordoni
 Università degli Studi di Roma "La Sapienza"
 Dipartimento Me.Mo.Mat.
 Via A. Scarpa 16
 00128 Roma - Italia
 bordoni@dmmm.uniroma1.it

Verita Marino
 Università della Calabria
 Dipartimento di Matematica
 Ponte P. Bucci, Cubo 31B
 87036 Arcavacata di Rende (Cs) - Italia
 marino@unical.it