STEADY AND QUASI-STEADY THIN VISCOUS FLOWS NEAR THE EDGE OF A SOLID SURFACE

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ABSTRACT. A new approach is proposed to the description of thin viscous flows near the edges of a solid surface. For a steady flow, the lubrication approximation and the no-slip condition are assumed to be valid on most of the surface, except for relatively small neighborhoods of the edges, where an universality principle is postulated: the behavior of the liquid in these regions is universally determined by flux, external conditions and material properties. The resulting mathematical model is formulated as an ordinary differential equation involving the height of the liquid film and the flux as unknowns, and analytical results are outlined. The form of the universal functions which describe the behavior in the edge regions is also discussed, obtaining conditions of compatibility with lubrication theory for small fluxes. Finally, an ordinary differential equation is introduced for the description of moderate time asymptotic profiles of a liquid film which flows off a bounded solid surface.

1. INTRODUCTION

The height $h(x,t)$ of a thin liquid film spreading over a horizontal solid surface is described by the equation

$$\partial_t h + \kappa\, \partial_x (h^3 \partial_x^3 h) = 0, \tag{1.1}$$

where x is the spatial coordinate, t is time, and $\kappa = \gamma/(3\mu)$, where γ is the surface tension and μ is the viscosity. Equation (1.1) is derived as an approximation of the Navier-Stokes equation, on the basis of four main assumptions:

(LR1) The typical vertical length-scale is much smaller than the typical horizontal length-scale.
(LR2) The evolution is slow, in the sense that the horizontal velocity is a slowly modulated Poiseuille field.
(LR3) The Laplace formula $p = -\gamma k$ is valid, where p is the hydrodynamic pressure and k is the mean curvature of h.
(NS) The tangential velocity of the liquid in points of contact with the solid is zero.

The first three assumptions constitute the lubrication regime (LR), while the fourth is the no-slip condition. By careful asymptotic expansions – see the comprehensive review by Oron, Davis and Bankoff [13] and the references therein –, they yield to the *lubrication approximation* of Navier-Stokes equations: an evolution equation for the thickness h of the liquid film, which in one space dimension is given by equation (1.1). To our knowledge, the only rigorous justification of lubrication approximation has been provided by Giacomelli and Otto [6]: in the regime of "complete wetting" [3], i.e. assuming a zero contact angle condition at triple junctions, they prove that suitably rescaled solutions to the two-dimensional Hele-Shaw flow (with curvature) in half-space converge to zero-slope solutions to an equation of the form (1.1), with mobility $m(h) = h$ instead of $m(h) = h^3$.

Let us briefly review the problem of the spreading of a finitely extended liquid film over an infinitely extended horizontal solid surface. It is nowadays well known that for a Newtonian liquid (NS) leads to the following paradox: an infinite amount of energy is needed for the extension of the film. This has been proved by Dussan V. and Davis [4] in the context of theoretical fluid dynamics, and can also be observed directly from (1.1) via an asymptotic expansion near the boundary of the support. The usual approach to solve this paradox (first introduced by Greenspan [7]) is to replace (NS) with different, ad-hoc laws which allow for a small tangential velocity of the liquid at the liquid-solid interface [13]. Such laws lead to lubrication approximations of the type

$$\partial_t h + \kappa\, \partial_x[(h^3 + \beta^{3-n} h^n)\partial_x^3 h] = 0, \tag{1.2}$$

where usually $n \in \{1, 2\}$, and β is a positive parameter. To our knowledge, there is no quantitative characterization of β except that it has to be small with respect to the average thickness of the film. Equations of the type (1.2) have been extensively analyzed by many authors, and we refer e.g. to [1, 8] for a list of references.

An entirely different approach was proposed by Barenblatt, Beretta and Bertsch [2]. It is based on the observation that the purely fluid–mechanical description might not be valid in the vicinity of triple junctions (interfaces between liquid, solid and air), due to the dominant role which cohesive forces play at small thicknesses, thus violating (LR3). And even if (LR3) were valid, the averaged vertical component of the velocity could be not negligible, and the slope could become large, thus violating (LR1) and (LR2). Also, molecular dynamics computations by Thompson and Troian [14] suggest that (NS) can be violated in this region, too. Therefore, it was postulated the existence of a small *contour region* near the interface liquid-solid-gas which is *autonomous*: that is, its structure is identical for all films given their velocity, their material components and external conditions. The contour region and the *basic region* – where (LR) and (NS) hold and the evolution is governed by (1.1) – are separated by an interface, where continuity of height, tangential force and velocity are imposed. According to de Gennes [3], a *precursor region* could also exist ahead of the contour region. Without introducing any ad-hoc law, this new concept acts as a general, consistent setting for the explicit introduction of cohesive forces in the modelling of the evolution of contact lines.

The aim of the present paper is to extend this idea to the case of thin film flow on a bounded horizontal solid surface, with particular attention to the description of the flow near its edge. An instructive setting for such investigation is the following schematization of a "dam problem". As shown in Fig. 1, a liquid is confined in a reservoir by a dam of width ℓ, and flows with constant flux $-q$ over the dam itself, due to a positive asymptotic level a at the right of the dam. Assuming $a \ll \ell$, the height h of the liquid over the dam is described by the lubrication approximation, except for neighborhoods of the two edges (see Fig. 2). There, as in the previous case, the vertical component of velocity will not be negligible, the slope could become large and Laplace formula could not describe properly the distribution of cohesive forces. Hence, following [2] we introduce two autonomous small *edge regions* in the vicinity of the edges, according to the following basic assumptions:

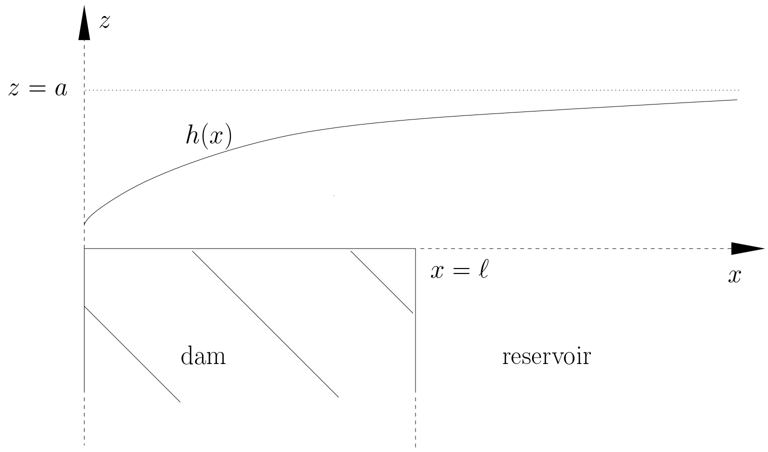

FIGURE 1. Schematization of the dam problem

I. Universality Principle. For given materials (solid, liquid, gas), given external conditions (temperature, pressure), given flux, the structure of each edge region is *universal* – that is, the height of the film and the distribution of forces inside the edge region are identical for all films.

II. Smallness Condition. The longitudinal size δ of each edge region is small in comparison with the size of the basic region (where (LR) and (NS) are valid).

In Section 2 we will show how these assumptions lead to the following nonlinear boundary value problem for the pair (q, h):

$$(1.3)\qquad (\mathbf{I})\begin{cases} \kappa h^3 h''' = -q, \quad h > 0 \qquad x \in (0,\ell) \\ h(0) = h_0(q) \\ -\gamma h(0)h''(0) = G_0(q) \\ h(\ell) = h_1(q) \\ -\gamma h(\ell)h''(\ell) = G_1(q). \end{cases}$$

The functions $h_i(q)$ and $G_i(q)$ are, in view of the Universality Principle, universal functions of the flux which depend on the external conditions and the material triplet. Limitations and appropriate choices for such functions are discussed in section 3, followed by an outline of analytical results concerning (**I**) in section 4. Finally, in section 5 we consider a different, closely related physical situation: namely the symmetric, quasi-steady flow off a horizontal surface. For this problem, we derive an equation for the spatial profile at intermediate time scales, and outline analytical results.

2. The Model

We consider the following schematization of a dam problem. A dam of width ℓ and semi-infinite height confines a liquid in a reservoir of infinite extent, as shown in Fig. 1. The coordinate system (x, z) has its origin at one edge of the dam, the z-axis points upwards and the x-axis in the direction of the reservoir. We assume

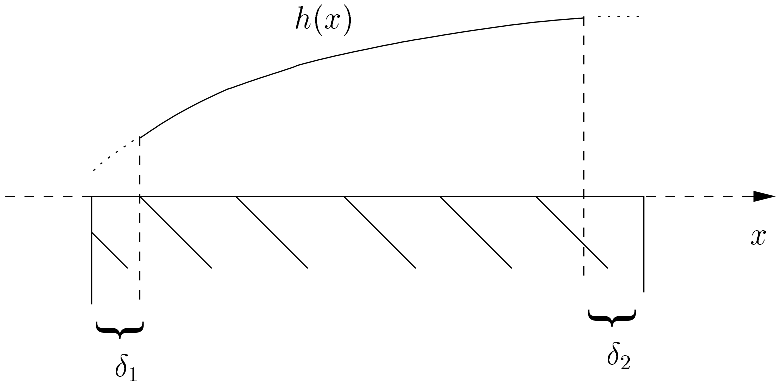

FIGURE 2. Basic region and edge regions.

that, due to a positive asymptotic level a of the liquid in the reservoir, a steady flow is established over the horizontal surface of the dam. Let

$$h : (0, \ell) \longrightarrow (0, \infty)$$

denote the height of the liquid over the surface. If $a \ll \ell$, it is natural to assume that the hypotheses (LR) hold in most of the domain $(0, \ell)$; since the flow is steady, (1.1) becomes

$$(h^3 h''')' = 0. \tag{2.1}$$

Remark 2.1. Assuming the validity of (2.1) up to $h = 0$, existence and stability properties of either positive periodic steady states and equal contact angle steady states have been extensively investigated by Laugesen and Pugh in a series of papers [9, 10, 11, 12], dealing also with more general equations of the form $(f(h)h''' + g(h)h')' = 0$.

Nevertheless, in the vicinity of the edges of the dam the slope could become large, the vertical component of the velocity will not be negligible, and the action of cohesive forces could become dominant, thus breaking the validity of Laplace formula. Therefore, following [2], we divide $(0, \ell)$ into three regions: a *basic region*, where (LR) and (NS) are satisfied and the flow is described by (2.1); and two *edge regions* near the edges of the dam, where (LR) and (NS) can not be applied. The *edge regions* are characterized by the following basic assumptions:

I. Universality Principle. For given materials (solid, liquid, gas), given external conditions (temperature, pressure), given flux $-q$, the structure of each edge region is *universal* – that is, the height of the film and the distribution of forces inside the edge region are identical for all films.

II. Smallness Condition. The longitudinal size δ of each edge region is small in comparison with the size of the basic region.

As shown in Fig. 2, we split $(0, \ell)$ according to the smallness condition:

$$(0, \ell) = (0, \delta_1) \cup (\delta_1, \ell - \delta_2) \cup (\ell - \delta_2, \ell), \quad 0 < \delta_i \ll \ell.$$

In the basic region the lubrication approximation is valid, whence from (2.1)

$$\kappa h^3 h''' = -q, \quad x \in (\delta_1, \ell - \delta_2). \tag{2.2}$$

At the boundary between the basic region and the "exit" edge region $(0, \delta_1)$ we prescribe continuity of the height and equilibrium of tangential forces. In the basic region these quantities are given by h and $-\gamma h h''$, respectively. Indeed, because of (LR2) the curvature k is approximated by $\partial_x^2 h$; hence from (LR3) we have $p = -\gamma \partial_x^2 h$ in lubrication approximation. Therefore, it follows from the Universality Principle that

$$h(\delta_1) = h_0(q), \tag{2.3}$$

$$-\gamma h(\delta_1) h''(\delta_1) = G_0(q), \tag{2.4}$$

where $h_0(q)$ and $G_0(q)$ are universal functions of the flux which depend on the external condition and the given triplet solid-liquid-gas. Applying the same argument at the interface $x = \ell - \delta_2$ between the basic region and the "entrance" edge region, we obtain:

$$h(\ell - \delta_2) = h_1(q), \tag{2.5}$$

$$-\gamma h(\ell - \delta_2) h''(\ell - \delta_2) = G_1(q), \tag{2.6}$$

with $h_1(q)$ and $G_1(q)$ universal functions.

Equation (2.2), together with boundary conditions (2.3)-(2.6), constitute a nonlinear boundary value problem for the pair (q, h). For simplicity, we perform a change of coordinates so that $(\delta_1, \ell - \delta_2) \mapsto (0, \ell)$. In view of the smallness condition $\delta_i \ll \ell$, this transformation is a almost the identity map and thus does not affect the relevant dimensional quantities (in particular, we may neglect that δ_i could depend on q). We obtain:

$$\text{(2.7)} \qquad (\mathbf{I}) \begin{cases} \kappa h^3 h''' = -q, \quad h > 0 \qquad x \in (0, \ell) \\ h(0) = h_0(q) \\ -\gamma h(0) h''(0) = G_0(q) \\ h(\ell) = h_1(q) \\ -\gamma h(\ell) h''(\ell) = G_1(q). \end{cases}$$

Remark 2.2. Boundary conditions (2.3)-(2.6) have been derived assuming a steady flow with positive flux. A natural question is whether it is possible to extend their applicability to the non-stationary case – that is, to consider the time-dependent equation (1.1) with boundary conditions of the type $h(0, t) = h_0(q(t))$. This can certainly be done for sufficiently slowly varying flows (see section 5).

Up to now, we have deliberately avoided any prescription on the functions h_i and G_i. Indeed, we consider this model mainly as part of a new concept in the description of physical phenomena characterized by a transition between continuum mechanics and molecular scales, or between different regimes. In addition, some of the physical mechanisms which govern the edge regions are still very much unknown, and deserve further investigations and comparison with experiments. In this perspective, we allowed for the greatest possible generality in this first stage. In the next section we will analyze in more details limitations and possible choices for universal functions.

3. Investigation of the universal functions

3.1. Limitations on universal functions at the exit edge. The next result is of crucial importance: It implies that it is not possible to simplify the boundary condition $h(0) = h_0(q)$ by setting $h_0 \equiv 0$. Moreover, a limiting procedure which might eventually yield a solution to the differential equation with $h(0) = 0$ and positive flux is also excluded. This result is not related to the properties of the three other universal functions.

Theorem 1. : The value of $h_0(q)$ is positive. *For any $q > 0$, $\kappa > 0$ and $\ell > 0$ a non-negative function $h \in C^3((0, \ell))$ such that*

$$\liminf_{x\to 0^+} h(x) = 0, \qquad \kappa h^3 h''' = -q \quad \text{in } (0, \ell)$$

does not exist.

We argue by contradiction and assume that such an h exists. Since q is positive, h' is concave in $(0, \ell)$. Therefore

$$\lim_{x\to 0^+} h'(x) = c < \infty,$$

and $c \geq 0$ since h is non-negative. Hence

$$h(x) \leq (c + 1)x \quad \text{for all } \ x \in (0, x_0)$$

for a sufficiently small $x_0 \in (0, \ell)$. But this implies that

$$h'''(x) \leq -\frac{q}{\kappa}(c + 1)^{-3}x^{-3} \quad \text{for all } x \in (0, x_0).$$

Therefore

$$\lim_{x\to 0^+} h(x) = -\infty,$$

and we have come to a contradiction.

Let us consider external conditions and material properties as fixed. In order to clarify the relations between h_0 and G_0 – and between them and the universal functions at the entrance edge – it is instructive to consider the example of a power-type behavior as $q \to 0$:

$$\begin{array}{rcl} h_0(q) & = & Aq^m(1 + o(1)) \\ G_0(q) & = & -\gamma D q^s(1 + o(1)) \end{array} \quad \text{as } q \searrow 0 \tag{3.1}$$

for positive constants A, m, s, and for an arbitrary constant D. It turns out that there are limitations on the possible choices of (s, m) to guarantee that solutions to the differential equation are compatible with the scaling assumptions of lubrication theory for any, arbitrarily small value of the flux. Before discussing this issue in detail, let us state the results. Consider any sequence

$$h_q \in C^3((0, \ell) \cap C([0, \ell]), \quad q \searrow 0 \tag{3.2}$$

such that

$$\left\{ \begin{array}{rcl} \kappa h_q^3 h_q''' & = & -q, \quad h_q > 0 \quad x \in (0, \ell) \\ h_q(0) & = & h_0(q) \\ -\gamma h_q(0) h_q''(0) & = & G_0(q). \end{array} \right. \tag{3.3}$$

Then the following holds:

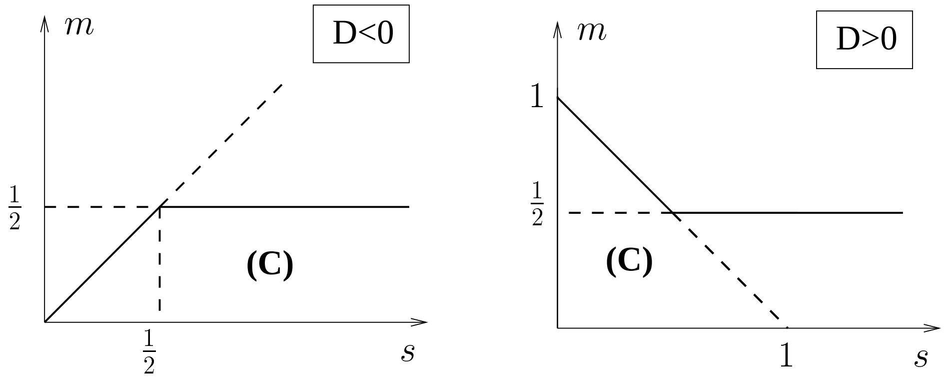

FIGURE 3. In (C), the pair (s, m) is compatible with lubrication regime for arbitrarily small values of the flux.

Theorem 2. : Incompatibility conditions for vanishing flux. *Let $\{h_q\}_{q\searrow 0}$ be any sequence satisfying (3.2), (3.3), and assume that (3.1) holds. If*

$$\begin{aligned} &D<0 \quad \text{and} \quad m>\min\left\{s,\tfrac{1}{2}\right\}, \quad \textit{or} \\ &D=0 \quad \text{and} \quad m>\tfrac{1}{2}, \quad \textit{or} \\ &D>0 \quad \text{and} \quad m>\max\left\{1-s,\tfrac{1}{2}\right\} \end{aligned} \tag{3.4}$$

then

$$\lim_{q\to 0^+} h_q'(0)=\infty.$$

Conditions (3.4) are "optimal" in the following sense:

Theorem 3. : Optimality of conditions (3.4). *Assume that (3.1) hold, and that*

$$\begin{aligned} &D<0 \quad \text{and} \quad m<\min\left\{s,\tfrac{1}{2}\right\}, \quad \textit{or} \\ &D=0 \quad \text{and} \quad m<\tfrac{1}{2}, \quad \textit{or} \\ &D>0 \quad \text{and} \quad m<\max\left\{1-s,\tfrac{1}{2}\right\}. \end{aligned}$$

Then for any $\ell>0$ and $\alpha>0$ there exists a positive constant q_0 (depending upon ℓ, α, A, D, m and s) such that the solutions to the problem

$$(I_\alpha)\quad \left\{ \begin{aligned} \kappa h^3 h''' &= -q \\ h(0) &= h_0(q) \\ h'(0) &= \alpha \\ -\gamma h(0)h''(0) &= G_0(q) \end{aligned} \right. \tag{3.5}$$

are defined and positive in $(0,\ell)$ for any $q\in(0,q_0)$.

The proof of Theorems 2 and 3 is provided in the Appendix. We emphasize that the conclusions of Theorems 2 and 3 are entirely local; in particular, they are independent of the values of h_q and its derivatives at $x=\ell$.

Remark 3.1. In the limiting cases(that is, if an equality holds in (3.4)) the conclusions of Theorem 3 continue to hold for α sufficiently large.

To explain the meaning of the above characterization, let us underline the following: if one considers physical situations in lubrication theory where the flux through the exit edge is an adjustable parameter, then under the conditions of Theorem 2, solutions will be inconsistent with lubrication regime itself for q sufficiently small, in the sense that the slope could be made arbitrarily large *inside the basic region*, thus violating (LR1). This can be interpreted by saying that some external conditions and material properties do not allow for small fluxes in lubrication regime. Therefore, one would be tempted to rule out such seemingly unnatural behavior. But it is important to observe that the situation in problem (**I**) is different: we do not prescribe the flux, which is instead an unknown of the system. As a consequence, it is not legitimate to state a-priori that the solution (q, h) will be incompatible with lubrication theory, unless $q \ll 1$ is explicitly required as a constraint on the solution. Nevertheless, as we shall see in Section 4 in the special case $D = 0$, such compatibility conditions do play a role also in the case of problem (**I**).

3.2. Form of the universal functions at the entrance edge. Let us now turn to the boundary conditions at the entrance edge: we are interested in characterizing the possible forms of the universal functions h_1 and G_1. To this aim, let us consider for a moment the following situation. A wall equipped with uniformly distributed tiny valves is placed at the boundary between the basic region and the entrance edge region – that is, in new coordinates, at $x = \ell$. If the valves are closed, $q = 0$ and a certain force $\overline{G}$ is exerted by the wall on the liquid on the right to keep the system in equilibrium. As the valves are opened, a steady flow through $x = \ell$ with flux $-q$ will be established after a certain transient time. It is natural to expect that the force $\overline{G}(q)$ exerted by the wall continuously decreases with growing q: indeed, an open-valved wall with a lesser mean surface area exerts a smaller force on the liquid on the right to prevent its outflow, and simultaneously allows for a greater incoming flux. Of course, there will be a limiting flux Q – which may also be infinite – which corresponds to zero area of the wall, that is to zero force. Concerning the height $\overline{h}$, though intuition suggests that it should also be a decreasing function of the flux, we think that it is more appropriate to avoid any further specification: we just assume it to be continuous and positive in $[0, Q)$.

Returning to our model, the liquid over the horizontal surface of the dam plays the role of the wall: more precisely, the force $G_1(q)$ that it exerts to keep the system in its equilibrium equals $\overline{G}(q)$, and continuity requires $h_1(q) = \overline{h}(q)$. Hence, we conclude that the universal functions $h_1(q)$ and $G_1(q)$ have the following form:

$$h_1 \in C([0, Q]; [0, \infty)), \quad h_1 > 0 \text{ in } [0, Q) \tag{3.6}$$

and

$$\begin{gathered} G_1 \in C([0, Q]; [0, \infty)) \text{ non-increasing,} \\ G_1(0) = \overline{G}, \quad \lim_{q \to Q^-} G_1(q) = 0. \end{gathered} \tag{3.7}$$

4. Solutions to Problem (I)

In a parallel paper [5], the question of existence of solutions for problem (**I**) is considered. The universal functions h_1 and G_1 at the entrance edge are assumed to have the form specified respectively by (3.6) and (3.7). Concerning the universal functions at the exit edge, as we have seen in Section 3 there are several different, more special proposals which can be discussed. Here we shall assume that h_0 is defined and positive in $(0, Q)$,

$$h_0 \in C([0, Q]; [0, \infty)), \quad h_0 > 0 \text{ in } (0, Q), \tag{4.1}$$

and we shall neglect the tangential force G_0, i.e.

$$G_0(q) \equiv 0, \tag{4.2}$$

assuming that the drag of both edge regions is small in comparison with the drag exerted by the wall. With respect to the behavior of h_0 for small fluxes, it is assumed that it is comparable with $q^{\frac{1}{2}}$ in the sense that the following limit exists:

$$\lim_{q \to 0^+} \frac{q^{\frac{1}{2}}}{h_0(q)} = B, \tag{4.3}$$

with $0 \leq B \leq \infty$. According to Theorems 2 and 3 with $D = 0$, the values $B = 0$ and $B = \infty$ correspond to behaviors of h_0 which are respectively "compatible" and "incompatible" with lubrication approximation asymptotically as $q \to 0$, while $B \in (0, \infty)$ concerns the critical value $m = \frac{1}{2}$. Let us first consider compatible behaviors of h_0:

Theorem 4. (A): The "compatible" case. *Assume that the universal functions satisfy (3.6), (3.7), and (4.1)–(4.3). There exist positive constants* $0 < B_1 \leq B_0 < \infty$ *(depending on* $G_1(0)$, $h_1(0)$, ℓ, γ *and* κ*) such that:*

(i) *for any* $B \in [0, B_0)$ *there exists a solution* (q, h) *of* (**I**)*;*
(ii) *if* $0 \leq B < B_1$*, the solution* h *obtained in (i) is increasing in* $(0, \ell)$ *provided* G_1 *is sufficiently small at a given point* $q^\star$ *(which does not depend on* G_1*).*

Part (*ii*) of the statement gives a sufficient condition for monotonicity of the solution obtained in point (*i*). It requires — roughly speaking — G_1 to be sufficiently small. Let us observe in this respect that considering monotonicity of the solution as a constraint would be misleading, though physically plausible, since continuity of the slope on the boundary between the basic and the edge region is not imposed in the model, and even in this case nothing prevents oscillations of the height inside the entrance edge region. It is more appropriate to view monotonicity as a first approximation of condition (LR2): if $\partial_x h(0)$ is small, (LR2) is not violated as long as $\partial_x h$ is positive (h being concave). In this sense, the aforementioned smallness condition on $G_1(q^\star)$ corresponds to the following: The force exerted by the liquid over the surface on the liquid reservoir, needed to keep the system in equilibrium, should be *achievable* by the liquid over the surface itself; that is, sufficiently small for given materials and external conditions.

Remark 4.1. We conjecture that the solution of problem (**I**) obtained in Theorem 4 (A) (*i*) is unique if G_1 is strictly decreasing, h_0 is increasing and h_1 is decreasing.

The case of "incompatible" behavior of h_0 (in the sense of Theorem 2) is, as one might expect, definitely different:

Theorem 4. (B): The "incompatible" case. *Let the assumption of Theorem 4 (A) hold. Then, with the same constant* B_0*:*

(iii) *if* $B > B_0$*, a solution of* (**I**) *may not exist, depending on the form of* G_1*; in particular, there is no solution if* G_1 *decreases sufficiently fast below a certain value* g_m *(which does not depend on* G_1*);*

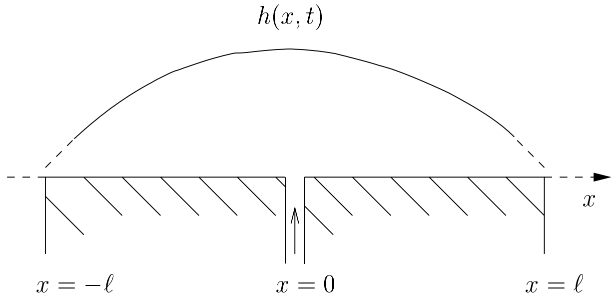

FIGURE 4. Schematization of the flow off a solid surface.

(iv) *if $B=\infty$, there exist functions G_1 satisfying (3.7) and such that* (**I**) *admits at least two solutions (q_1,h_1) and (q_2,h_2); in addition, G_1 can be chosen so that $q_1 \ll 1$ and $(Q-q_2)\ll 1$ (if $Q<\infty$).*

The behavior described by Theorem 4 (B) is intriguing. Part (*iii*) seems to imply that relatively fast convergence to zero of $h_0(q)$ can be balanced only by strong tangential forces at the entrance edge. On the other hand, in this case one solution may very well be incompatible with lubrication regime. Nevertheless, as stated in (*iv*), the possibility to obtain lubrication-compatible solutions in principle exists even if $m>\frac{1}{2}$. However, it is clear that at this point a more refined analysis and qualitative description requires quantitative information and comparison with experiments, both concerning universal functions and the extent of applicability of lubrication theory. For the proof of Theorem 4 we refer to [5].

Remark 4.2. The value of the constants in Theorem 4 might be relevant to applications: B_0 and B_1 can be computed explicitly, and are given by

$$\begin{aligned} B_0 &= \frac{1}{\gamma}\sqrt{\frac{2\kappa}{\ell}}\left(\frac{\ell^2 G_1^2(0)}{2h_1^2(0)}+\gamma G_1(0)\right)^{\frac{1}{2}}, \\ B_1 &= \min\left\{B_0, \frac{2h_1(0)}{\ell}\sqrt{\frac{2\kappa}{\ell}}\right\}. \end{aligned}$$

On the other hand, the constants $q^\star$ and g_m do not depend on G_1, but do depend on all the other parameters and functions.

5. Symmetric flow off a surface

A related problem is that of a thin film which flows off a horizontal solid surface of finite extent $(-\ell,\ell)$. Assume, as shown in Figure 4, that a steady flow is maintained via a constant inflow at $x=0$. As the inflow is stopped we assume that after a relaxation time an intermediate regime is established, characterized as follows:

- the evolution is symmetric with respect to $x=0$;
- the outflow $q(t)$ from the horizontal surface is positive;
- lubrication theory holds everywhere except for two small exit edge regions $(-\ell,-\ell+\delta)$, $(\ell-\delta,\ell)$ near the two edges $x=\pm\ell$;
- the evolution is slow in the sense that $|\partial_t h|\ll h$.

Let us point out that we *assume* the strict positivity of the flux. As a consequence, letting $\ell_1 = \ell - \delta$:

$$\partial_t h + \kappa\, \partial_x (h^3 \partial_x^3 h) = 0 \quad \text{in } (-\ell_1, \ell_1) \tag{5.1}$$

$$\partial_x h(0) = \partial_x^3 h(0) = 0 \tag{5.2}$$

$$h^3 \partial_x^3 h\Big|_{x=\ell_1} = q(t) > 0. \tag{5.3}$$

Following the idea of ordered regime introduced by Boussinesq in the context of nonlinear filtration, we assume that the height h can be described at moderate time scales by asymptotic profiles of the form

$$h(x,t) = X(x)T(t), \quad X \in C^3([0, \ell_1]). \tag{5.4}$$

From (5.1) we obtain

$$\frac{T'}{T^4} = \frac{-\kappa(X^3 X''')'}{X} = -\alpha, \quad \alpha > 0.$$

Without loss of generality, the parameter α can be set to $\frac{1}{3}$, since the rescaling $\hat{X} = \alpha^{-\frac{1}{3}} X$, $\hat{T} = \alpha^{\frac{1}{3}} T$ leaves $h = XT$ invariant. Hence, substituting in (5.1)-(5.3) we obtain:

$$T(t) = (T_0 + t)^{-\frac{1}{3}}, \tag{5.5}$$

$$q(t) = \kappa (T_0 + t)^{-\frac{4}{3}} X^3(\ell_1) X'''(\ell_1), \tag{5.6}$$

$$X = 3\kappa (X^3 X''')', \tag{5.7}$$

$$X'(0) = X'''(0) = 0. \tag{5.8}$$

The constant T_0 depends on the initial conditions preceding the entering of the system into the ordered regime. The assumption that the evolution is slow enforces

$$t \geq t_0 \gg 1. \tag{5.9}$$

On the other hand, it allows to consider the flux q as constant on time steps of lower order, and in particular of order one. Indeed

$$\frac{q(t+\Delta t)}{q(t)} = \left(T_0 + \frac{t + \Delta t}{T_0 + t} \right)^{-\frac{4}{3}} = 1 + o(1).$$

In other words, (5.4) and (5.9) correspond to consider the evolution as quasi-steady. As a consequence, the Universality Principle (which has been postulated only for steady flows) can be assumed to hold also in this case, and therefore the height and the tangential force at $x = \ell_1$ are determined by universal functions of the flux:

$$X(\ell_1) = (T_0 + t)^{\frac{1}{3}} h_0(q(t)) \tag{5.10}$$

$$-\gamma X(\ell_1) X''(\ell_1) = (T_0 + t)^{\frac{2}{3}} G_0(q(t)). \tag{5.11}$$

Problem (5.6)-(5.11) does not correspond to an ordinary differential equation for X except for special cases. Nevertheless, we can proceed in the analysis via some further simplification. As in the previous section, we assume that the tangential force G_0 acting on the exit edge region can be neglected:

$$X''(\ell_1) = 0. \tag{5.12}$$

Let us also assume that h_0 is a positive power of q:

$$h_0(q) = Aq^m,$$

with $A > 0$ and $m > 0$. According to Theorems 2 and 3 with $D = 0$, if $m > 1/2$ the behavior of h_0 is asymptotically incompatible with lubrication regime as $q \searrow 0$. Besides, since we are looking at large time scales, (5.6) implies $q \ll 1$. Combining these two facts we obtain the constraint

$$m \in \left(0, \frac{1}{2}\right].$$

From (5.6) and (5.10) we have

$$(X(\ell_1))^{1-3m} = A\kappa^m (T_0 + t)^{\frac{1-4m}{3}} (X'''(\ell_1))^m.$$

Unless $m = 1/4$, this last condition breaks the homogeneity of the problem. But since

$$(T_0 + t + \Delta t)^{\frac{1-4m}{3}} (X'''(\ell_1))^m = (T_0 + t)^{\frac{1-4m}{3}} (X'''(\ell_1))^m (1 + o(1)),$$

we can consider this quantity as approximately constant at moderate time scales, and we end up with the following ordinary differential equation for X:

$$(\mathbf{II}) \quad \begin{cases} X &= 3\kappa(X^3 X''')', \quad X > 0 \quad \text{in } (0, \ell_1) \\ X'(0) &= X'''(0) = 0 \\ X''(\ell_1) &= 0 \\ (X(\ell_1))^{1-3m} &= C(X'''(\ell_1))^m. \end{cases}$$

To summarize, solutions of problem (**II**) describe the profile of the height h at intermediate time scales:

$$h(x,t) \approx (T_0 + t)^{-\frac{1}{3}} X(x), \quad t \in (t_0, t_0 + \Delta t),$$

where

$$t_0 \gg 1 \quad \text{and} \quad (\Delta t / t_0) \ll 1,$$

with C given by

$$C = A\kappa^m (T_0 + t_0)^{\frac{1-4m}{3}}.$$

In a parallel paper [5], the third author proves that for any $0 < m \leq 1/2$, $\ell_1 > 0$ and $C > 0$ there exists a solution to problem (**II**). It is interesting to observe that the existence result does not carry over to $m > \frac{1}{2}$; unfortunately, however, no counterexample of non-existence is known at present.

6. Conclusions

The steady flow of a thin liquid film over a bounded horizontal surface is described, far from an edge of the surface, by the classical third order equation

$$h^3 h''' = -q.$$

Near an edge, we have proposed a new approach which leads to a class of boundary value problems for the classical third order equation, of the form

$$(\mathbf{I}) \quad \begin{cases} \kappa h^3 h''' &= -q, \quad h > 0 \qquad x \in (0, \ell) \\ h(0) &= h_0(q) \\ -\gamma h(0) h''(0) &= G_0(q) \\ h(\ell) &= h_1(q) \\ -\gamma h(\ell) h''(\ell) &= G_1(q) \end{cases}$$

where h_i and G_i are universal functions of the flux which depend only on external conditions and the material properties. We have shown that it is not possible to simplify the boundary conditions by setting $h_0 = 0$. The analysis of prototypical power-law choices of the universal functions has highlighted the presence of some threshold conditions which, in case of a physical situation where the flux is an adjustable parameter, determine the compatibility with lubrication approximation for arbitrarily small fluxes. These compatibility conditions turn out to be relevant also for problem (I), where the flux is not a parameter but an unkown. In section 5 we have also introduced and discussed a quasi-steady approximation for a special problem.

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A. Appendix

Here we prove Theorems 2 and 3. In the proofs we will make use of the following straightforward observations, which hold for any solution of the differential equation in its existence domain:

$$h''' < 0, \tag{A.1}$$

$$h'' \searrow, \tag{A.2}$$

$$h' \ \text{concave}, \tag{A.3}$$

$$h(x) \le h(0) + h'(0)x + \frac{1}{2}h''(0)x^2. \tag{A.4}$$

Proof of Theorem 2. We argue by contradiction, and assume that for some $\alpha > 0$

$$h_q'(0) \le \alpha$$

for a subsequence. The contradiction will be established once we show that

$$h_q'' \to -\infty \ \text{ locally uniformly in } (0, \ell). \tag{A.5}$$

Note that:

$$\text{if } 0 \le a < b, \text{ and } h_q(x) \le f(x) \text{ in } (a,b), \text{ then}$$
$$h_q''(x) \le h_q''(0) - \frac{q}{\kappa}\int_a^x f^{-3}(\xi)\, d\xi \qquad \text{for all } x \in (0,b). \tag{A.6}$$

Let us consider various different cases.

• $D < 0$ *and* $s < m$. By assumption

$$h_q''(0) \sim \frac{D}{A} q^{s-m} \to -\infty \quad \text{as } q \to 0,$$

thus (A.5) follows immediately from (A.2).

• *$D<0$ and $s\geq m$, or $D\leq 0$.* Using (A.4) we write

$$h_q(x)\leq 2Aq^m+\alpha x \quad \text{in } (0,\ell).$$

This yields by (A.6)

$$h_q''(x)\leq \frac{q}{2\alpha\kappa}\left[(2Aq^m+\alpha x)^{-2}-(2Aq^m)^{-2}\right],$$

hence (A.5) if $m>1/2$.

• *$D>0$.* By assumption

$$h_q(0)\leq 2Aq^m, \quad h_q''(0)\leq \frac{2D}{A}q^{s-m}=:2D_1q^{s-m}$$

for q sufficiently small. Using (A.4), we write

$$h_q(x)\leq q^{-m}x^2\left(2Aq^{2m}x^{-2}+\alpha q^m x^{-1}+D_1q^s\right), \tag{A.7}$$

so that, since $s\geq 0$,

$$h_q(x)\leq A_1q^{-m}x^2 \quad \text{for all } x\geq q^m$$

with $A_1=2A+\alpha+D_1$. We use this bound to control the second derivative via (A.6):

$$h_q''(x)\leq 2D_1q^{s-m}+\frac{1}{5\kappa A_1^3}\left(q^{1+3m}x^{-5}-q^{1-2m}\right),$$

and (A.5) follows if $1-2m<\min\{0,s-m\}$. □

Proof of Theorem 3. Let us first observe, since h_q' is concave, that any solution of (3.5) exists as long as h_q is positive. Thus, it is sufficient to prove a uniform lower bound for h_q in $(0,\ell)$.

• *$D\leq 0$.* By assumption

$$h_q(0)\geq \frac{A}{2}q^m, \quad h_q''(0)\geq \frac{2D}{A}q^{s-m}=:D_1q^{s-m}$$

for sufficiently small q. Since h_q is positive as long as h_q' is positive, the number

$$x_q:=\sup\left\{x\geq 0 \ : \quad h_q'>\frac{1}{4}\alpha \quad \text{in } (0,x)\right\}\in(0,\infty], \tag{A.8}$$

is well defined, and we have

$$h_q(x)\geq \frac{A}{2}q^m+\frac{1}{4}\alpha x, \quad x\in(0,x_q).$$

Using this bound in the differential equation, we obtain:

$$\begin{aligned} h_q'(x) &\geq \alpha+D_1q^{s-m}x-\frac{q}{\kappa}\int_0^x(x-\xi)\left(\frac{A}{2}q^m+\frac{1}{4}\alpha\xi\right)^{-3}d\xi \\ &\geq \alpha+\left(D_1q^{s-m}-\frac{8}{\kappa\alpha A^2}q^{1-2m}\right)x=:\alpha+B(\alpha,q)x \end{aligned}$$

for any $x\in(0,x_q)$. If

$$\begin{aligned} &D_1<0, \quad 1-2m>0, \quad s-m>0, \qquad \text{or} \\ &D_1=0, \quad 1-2m>0, \end{aligned}$$

then $\lim_{q\to 0^+}B(\alpha,q)=0$, and therefore $x_q\geq\ell$ for q sufficiently small. This completes the proof in the case $D\leq 0$. Note (in connection with Remark 3.1) that the limiting cases $m = \frac{1}{2}$ and $s = m$ can be dealt with by choosing α sufficiently large.

• $D > 0$. By assumption

$$h_q(0) \geq \frac{A}{2} q^m, \quad h_q''(0) \geq \frac{D}{2A} q^{s-m} =: D_2 q^{s-m}$$

for q sufficiently small. We argue as in the previous case, obtaining that

$$h_q'(x) \geq \alpha + B(\alpha, q)x \quad \text{in } (0, x_q),$$

where

$$B(\alpha, q) = D_2 q^{s-m} - \frac{8}{\kappa \alpha A^2} q^{1-2m}.$$

If $1 - 2m > 0$ or $1 - 2m > s - m$, then $\lim_{q \to 0^+} B(\alpha, q) \geq 0$ and therefore $x_q \geq \ell$ for q sufficiently small. This completes the proof in the case $D > 0$. Note once again that the limiting cases $m = \frac{1}{2}$ and $s = 1 - m$ can be dealt with by choosing α sufficiently large. □

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