Nonlinear higher-order boundary value problems describing thin viscous flows near edges

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Abstract

Two boundary value problems for nonlinear higher-order ordinary differential equations are analyzed, which have been recently proposed to model steady, respectively quasi-steady, thin viscous flows over a bounded solid substrate. The first problem describes steady states and consists of a third-order ODE with an unknown parameter, the flux; boundary conditions prescribe, at the edges of the substrate, the height of the liquid and the tangential forces within the liquid as functions of the flux itself. For this problem, (non-)existence and non-uniqueness results are proved depending on the behavior, as the flux approaches zero, of the "height-function" (the height prescribed at the edge out of which the liquid flows). These results enforce the notion of "compatible" behaviors of the height-function as those for which a solution always exists. The second problem describes quasi-steady states and consists of a fourth-order ODE with nonlinear boundary conditions coupling the height and the flux: for this problem, we prove the existence of a solution for compatible behaviors of the height-function.

Key words: nonlinear boundary value problems (34B15); lubrication theory (76D08)

1 Introduction and results

The aim of this paper is to analyze two boundary value problems which have been recently proposed in [4] for the description of thin viscous flows over a *bounded* solid surface. Of particular interest is the flow near the edges of the surface. We begin by shortly reviewing the model.

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1.1 The thin-film equation

The simplest model for the capillarity-driven evolution of the height $h(x,t)$ of a liquid film over a horizontal solid surface is given by the *thin-film equation*

$$\partial_t h + \kappa\, \partial_x (h^3 \partial_x^3 h) = 0, \tag{1}$$

where x is the spatial coordinate, t is time, and $\kappa = \gamma/(3\mu)$ is the ratio between surface tension γ and viscosity μ. Equation (1) may be formally obtained as a limit of Navier-Stokes equations on the basis of two main assumptions [31]: the *no-slip condition* and the applicability of *lubrication approximation*, which in turn requires that the typical vertical lengthscale is much smaller than the typical horizontal lengthscale, that the evolution is slow and that the pressure obeys the Laplace condition $p = -\gamma k$, where k is the mean curvature of h. However, as is well known, the applicability of these assumptions gets problematic when the film height approaches zero, due in particular to the dominant role played by the molecular forces between the different materials (solid, liquid, gas). Several options have been proposed to incorporate these small-scale effects into the model: among them slip conditions [17,23], van der Waals-type potentials [14], non-newtonian rheology [34,39], autonomy principle [3]. Without attempting to outline the very vast literature, here we just indicate the book [15], the reviews [14,31], and a few recent papers where referenced discussions may be found on various aspects of asymptotics [13,22,26, see also §1.5], PDEs [1,24], molecular dynamic simulations and diffuse interface models [33].

1.2 A "dam-problem"

The following schematization of a "dam problem" is considered in [4]: liquid confined in a reservoir by a dam of width $\hat{\ell}$ flows with constant flux q over the dam itself, due to a positive asymptotic level a at the right of the dam. Assuming $a \ll \hat{\ell}$, the height h of the liquid over the dam is described by the thin-film equation over most of the dam – the *basic region* where lubrication approximation and the no-slip condition are supposed to be valid – except for neighborhoods of the two edges. There, the vertical component of velocity will not be negligible, the slope may become large and Laplace formula may not describe properly the distribution of cohesive forces. Hence, two *edge regions* are introduced in the vicinity of the edges, according to the following basic assumptions: ,

(A1). Universality principle. For given materials, given external conditions (temperature, pressure, asymptotic height), given flux, the structure of each edge region is *universal* – that is, the height of the film and the distribution of forces inside the edge region are identical for all films.

(A2). Smallness condition. The longitudinal size of each edge region is small in comparison with the size of the basic region.

Describing the flow by (1) in the bulk, and prescribing continuity of the height and equilibrium of tangential forces at the boundaries between basic and edges region, (A1) and (A2) are shown in [4] to lead to the following nonlinear boundary value problem for the pair (q, h):

Problem (I) To find $q \in (0, Q)$ and $h \in C^3([0, \ell]; (0, \infty))$ such that

$$\begin{cases} \kappa h^3 h''' = -q \quad \text{in} \quad (0, \ell) \\ h(0) = h_0(q), \;\; -\gamma h(0) h''(0) = G_0(q), \\ h(\ell) = h_1(q), \;\; -\gamma h(\ell) h''(\ell) = G_1(q). \end{cases}$$

Here h_i and G_i are universal functions which depend only on the external conditions and the material triplet, $\ell = \hat{\ell} - \delta_1 - \delta_2$ (δ_1 and δ_2 being the sizes of the edge regions) and $Q \in (0, \infty]$ is a limiting value of the flux (corresponding to zero tangential force exerted on the liquid to the right of $x = \ell$, see [4]).

1.3 Form of universal functions

The form of the universal function h_0, G_0 at the edges $x = 0$ and $x = \ell$ (the "exit", respectively "entrance" of the liquid since $q > 0$) are discussed in [4]. In particular, at the exit edge the following power-like behaviors are considered:

$$h_0(q) = A q^m (1 + o(1)) \quad \text{and} \quad G_0(q) = -\gamma D q^s (1 + o(1)) \qquad \text{as } q \downarrow 0 \tag{2}$$

for suitable constants $A > 0$, $D \in \mathbb{R}$, $m > 0$ and $s > 0$. In this case, it turns out that if

$$\begin{aligned} &D < 0 \quad \text{and} \quad m > \min\left\{s, \tfrac{1}{2}\right\}, && \text{or} \\ &D = 0 \quad \text{and} \quad m > \tfrac{1}{2}, && \text{or} \\ &D > 0 \quad \text{and} \quad m > \max\left\{1 - s, \tfrac{1}{2}\right\}, && \end{aligned} \tag{3}$$

then solutions to Problem (I) are "incompatible" with lubrication regime for vanishing flux q, in the sense that their slope at $x = 0$ diverges as $q \downarrow 0$. Conditions (3) are shown to be sharp and may be ruled out in physical situations where the flux is known to be "small". Note however that this is not the case in Problem (I), where the flux is unknown: therefore it is not legitimate to exclude a-priori any incompatible form of h_0 and G_0. Nevertheless, as we shall see in §1.6, at least if $D = 0$ such compatibility conditions do play a rôle also in the case of Problem (I).

This discussion shows that the form of universal functions at the exit edge is crucial, and that there are several possible behaviors which can be taken into account. Hereafter we will suppose that h_0 is well defined and positive in $(0, Q)$, i.e.[1]

$$h_0 \in C([0, Q]), \quad h_0(0) = 0, \quad h_0 > 0 \text{ in } (0, Q), \tag{4}$$

and that the tangential force G_0 is negligible:

$$G_0(q) \equiv 0. \tag{5}$$

With respect to the universal function h_1 and G_1 at the entrance edge, the following behaviors are proposed in [4]:

$$h_1 \in C([0, Q]), \quad h_1 > 0 \text{ in } [0, Q), \quad h_1(0) = H_* \in (0, \infty) \tag{6}$$

and

$$G_1 \in C([0, Q]) \text{ non-increasing}, \quad G_1(0) = G_* \in (0, \infty), \quad G_1(Q) = 0. \tag{7}$$

1.4 Quasi–steady states

A related physical situation is that of a thin liquid film which flows out of a plane surface of finite extent $(-\ell, \ell)$ from the edges $x = \pm\ell$ of the surface itself. In [4], its height is assumed to be symmetric with respect to $x = 0$ and described, at suitable intermediate time scales, by asymptotic profiles of the form

$$h(x, t) \approx t^{-\frac{1}{3}} X(x), \quad X : (-\ell, \ell) \to (0, \infty).$$

The flux $q(t)$ is then shown to be small and approximately constant on such scales. Since q is approximately constant, (A1) and (A2) may be applied, and since q is small, compatibility conditions are to be taken into account: assuming again (4) and (5), according to (2) and (3) it is also assumed that

$$h_0(q) = Aq^m, \quad 0 < m \le 1/2. \tag{8}$$

This leads to the following boundary value problem for the profile X:

$$\begin{cases} X = 3\kappa(X^3 X''')', \quad X > 0 \quad \text{in } (0, \ell) \\ X'(0) = X'''(0) = 0 \\ X''(\ell) = 0, \ (X(\ell))^{1-3m} = c_0 (X'''(\ell))^m, \end{cases}$$

where $c_0 = A\kappa^m t_0^{(1-4m)/3}$. Rescaling X and x appropriately, we obtain:

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[1] The notation $f \in C([0, \infty])$ means that $f \in C([0, \infty))$ and $f(x) \to L \in \mathbb{R}$ as $x \to \infty$, in which case we also write $f(\infty) = L$.

Problem (II)**.** To find $u \in C^4([0,L])$, $u > 0$ in $[0,L]$ such that

$$\begin{cases} (u^3 u''')' = u \quad \text{in} \quad (0,L) \\ u'(0) = u'''(0) = 0, \\ u''(L) = 0, \; u^{\frac{1-3m}{m}}(L) = u'''(L). \end{cases}$$

1.5 The mathematical framework

An extensive literature exists which is related to higher-order nonlinear ODEs relevant to the dynamics of thin liquid films. A first class of such equations is of the form

$$f(u)u''' + g(u)u' + k(u) = F, \tag{9}$$

where F can either be a constant if u represents a steady state or a traveling wave profile, or a function of the independent variable if u represent a self-similar mass-preserving profile. In a series of papers [27–30], Laugesen and Pugh have studied in detail existence and stability properties of either positive periodic and equal contact angle solutions to (9) with $k = 0$. For these solutions one necessarily has $F = 0$, that is $q = 0$ in terms of Problem (I). For appropriate choices of the functions involved and of the boundary conditions, well-posedness and/or properties of solutions to (9) have been considered in the contexts of wetting, coating and Tanner's law [2,3,5,6,11,16,18,19,25,36,37], dewetting [8,21,22], blow-up [7,35,41] and shock formation [9,10,13] (see also the references therein and [20,32] for related PDE approaches). In all these cases F (that is q in terms of Problem (I)) is not an unknown of the problem (whereas the solution's domain often is) and the boundary condition are different, too.

Genuine fourth-order problems, such as Problem (II), arise when looking at self-similar profiles of solutions to thin film-type equations which do not preserve mass. They are relevant in the description of rupture and draining phenomena, and asymptotic studies may be found in [12,38,40,42] (see also the references therein). However, both the structure of the equation and the boundary conditions are different from those of Problem (II).

In conclusion, an unknown flux and boundary conditions which depend on it are peculiarities of Problem (I) which do not seem to be encompassed by previous studies. The intermediate asymptotics given by Problem (II) also features non-standard boundary conditions, and in addition no rigorous result seems to be available for that specific equation. The present paper thus aims to provide a first step in the analysis of (non-)existence, (non-)uniqueness and properties of solutions to the two problems.

In analyzing Problem (I), in addition to (4)-(7) we assume that h_0 is comparable to $q^{1/2}$ as $q \downarrow 0$:

$$\lim_{q \to 0^+} \frac{q^{\frac{1}{2}}}{h_0(q)} = B \in [0, \infty]. \tag{10}$$

Note, according to (3) with $D = 0$, that the values $B = 0$ and $B = \infty$ correspond to behaviors of h_0 which are respectively compatible and incompatible with lubrication approximation asymptotically as $q \downarrow 0$, while $B \in (0, \infty)$ relates to the critical case $m = 1/2$. Let us first consider compatible behaviors of h_0:

Theorem 1.1.A (the compatible case) *Assume* (4)-(7) *and* (10)*. Then two constants* $0 < B_1 \leq B_0 < \infty$ *exist such that: ,*

(i) for any $B \in [0, B_0)$ *there exists a solution* (q, h) *to Problem* (I)*;*

(ii) if $0 \leq B < B_1$*, then two constants* $\overline{q} \in (0, Q]$ *and* $\overline{G} \in (0, \infty)$ *(independent of the function* G_1*) exist such that the solution h obtained in (i) is monotone increasing in* $(0, \ell)$ *provided* $G_1(\overline{q}) < \overline{G}$*.*

The case of incompatible behavior of h_0 is, as one may expect, definitely different:

Theorem 1.1.B (the incompatible case) *Assume* (4)-(7) *and* (10)*. Then, with the same constant* B_0 *of Theorem 1.1.A: ,*

(iii) if $B > B_0$*, then two constants* $g_m \in (0, \infty)$ *(independent of the function* G_1*) and* $q_m \in (0, Q]$ *(depending on the function* G_1 *only through* $G_* = G_1(0)$*), exist such that Problem* (I) *has no solution whenever* $G_1(q) < g_m$ *for some* $q \leq q_m$*;*

(iv) if $B = \infty$*, for any* $\varepsilon > 0$ *there exists a function* G_1 *satisfying* (7) *such that Problem* (I) *admits at least two solutions* (q_1, h_1)*,* (q_2, h_2)*, and in addition*

$$q_1 < \varepsilon \quad \textit{and} \quad \begin{cases} (Q - q_2) < \varepsilon & \textit{if} \;\; Q < \infty \\ q_2 > 1/\varepsilon & \textit{if} \;\; Q = \infty. \end{cases}$$

Both parts of Theorem 1.1 are proved in §2. There, we consider the following boundary value problem, where the last boundary condition is removed and $q \in (0, Q)$ is fixed:

$$\begin{cases} \kappa h^3 h''' = -q, \quad x \in (0, \ell) \\ h(0) = h_0(q), \;\; h''(0) = 0, \\ h(\ell) = h_1(q). \end{cases} \tag{11}$$

Letting h_q denote the unique solution of (11), we then analyze the behavior of the functions

$$q \mapsto \mathcal{G}(q) := -\gamma h_q(\ell) h_q''(\ell) \quad \text{and} \quad q \mapsto \mathcal{H}(q) := h_q'(\ell). \tag{12}$$

Existence, non-existence and non-uniqueness of a solution to Problem (I) correspond to the number, if any, of intersections of the graphs of $\mathcal{G}$ and G_1.

Remark 1 The constants B_0 and B_1 can be computed explicitly:

$$B_0 = \frac{1}{\gamma}\sqrt{\frac{2\kappa}{\ell}}\left(\frac{\ell^2 G_*^2}{2H_*^2} + \gamma G_*\right)^{\frac{1}{2}}, \quad B_1 = \min\left\{B_0, \frac{2H_*}{\ell}\sqrt{\frac{2\kappa}{\ell}}\right\}.$$

The constants $\overline{q}$, $\overline{G}$, g_m and q_m depend on the above parameters (γ, κ, ℓ, H_* and, limited to q_m, G_*) and on the functions h_0 and h_1 (through $\mathcal{G}$ and $\mathcal{H}$, see (12)). For their definition we refer to the proof of Theorem 1.1, see in particular (30)-(33).

Though we are not able to prove it, we expect the solution to problem (I) obtained in Theorem 1.1.A to be unique if G_1 is strictly decreasing, h_0 is increasing and h_1 is decreasing. For the discussion and the physical interpretation of the results in Theorem 1.1 we refer to [4].

1.7 Existence of quasi-steady states

In §3 we prove the second main result of this paper, which concerns Problem (II):

Theorem 1.2 *For any $L > 0$ and $0 < m \leq 1/2$ there exists a solution to Problem* (II).

The proof of Theorem 1.2 is based on a two-parameter shooting technique. This strategy has already been used (see e.g. [6,11]) in the analysis of related higher-order ODEs; however, the present case requires different tools since the flux at $y = L$ is not prescribed (in particular, not zero) and depends (nonlinearly) on the solution itself. More precisely, we first consider the initial value problems

$$(\mathrm{II}_{s,\beta})\begin{cases} (u^3 u''')' = u, \quad y > 0 \\ u(0) = s,\ u'(0) = u'''(0) = 0,\ u''(0) = -\beta, \end{cases}$$

and show (see Lemma 3.4 and Lemma 3.5) that their solutions satisfy interesting monotonicity properties with respect to β; in particular

$$0 \leq \beta_1 < \beta_2 \implies \left[(u_1 u_1''') \circ (u_1'')^{-1}\right]^2 < \left[(u_2 u_2''') \circ (u_2'')^{-1}\right]^2 \quad \text{in } [-\beta_1, 0],$$

where u_i solve $(\mathrm{II}_{s,\beta_i})$. This allows to prove that for any $s > 0$ and $0 < m \leq 1/2$ there exists a unique solution of the following boundary value problem:

Problem (II_s)**.** Given $s > 0$, to find $L > 0$ and $u \in C^4([0, L])$, $u > 0$ in $[0, L]$ such that

$$\begin{cases} (u^3 u''')' = u, \quad y \in (0, L) \\ u(0) = s, \ u'(0) = 0, \ u'''(0) = 0 \\ u''(L) = 0, \ u^{\frac{1-3m}{m}}(L) = u'''(L). \end{cases}$$

Finally, we show that any value $L > 0$ can be achieved by varying s. It is noteworthy that such technique does not seem to cover the case $m > 1/2$; however, we were unable to construct an example of non-existence in this case.

2 Steady states

Our starting point is the following well-posedness result for (11):

Proposition 2.1 *Assume* (4)-(7) *and* (10)*. Then for any* $q \in (0, Q)$ *there exists a unique solution* $h_q \in C^3([0, 1])$ *of* (11)*, and*

$$\lim_{q \to q_0} \|h_q - h_{q_0}\|_{C^3([0,\ell])} = 0 \quad \textit{for all} \ \ q_0 \in (0, Q).$$

The proof is provided in the Appendix and follows the lines of the one used in [19] for a related third-order boundary value problem: via the Green's function, we explicitly construct a solution of the associated linear problem and then apply a fixed point argument.

We concentrate on the behavior of the solutions h_q as q varies in $(0, Q)$. In particular, we are interested in the behavior of $h_q(\ell) h_q''(\ell)$. The next observations will be useful in what follows.

Lemma 2.2 *Assume* (4)-(7) *and* (10)*, let* h_q *be the solution of* (11) *and let* $q_0 \in [0, Q]$*. If*

$$\limsup_{(0,Q) \ni q \to q_0} h_q''(\ell) > -\infty,$$

then: ,

(i) a subsequence (not relabeled) exists such that $h_q \to h$ *in* $C^1([0, \ell])$ *as* $q \to q_0$*;*

(ii) if $q_0 = 0$ *then* $B < \infty$ *(see* (10)*);*

(iii) if $q_0 = Q$ *then* $h_0(Q) > 0$*.*

Proof. We have $h_q''(\ell) \geq -C_1$ for a subsequence (not relabeled) and a suitable constant $C_1 > 0$. Since $h_q''(x)$ is decreasing we obtain $\|h_q''\|_{L^\infty((0,\ell))} \leq C_1$. The values of h_q on the boundary are also uniformly bounded; therefore $C_2 = \|h_q\|_{W^{2,\infty}((0,\ell))} < \infty$ and (i) follows by compactness. In addition, writing

$$h_q(x) \le h_0(q) + C_2 x \quad \forall\, x \in [0,\ell]$$

we obtain

$$-C_1 \le h_q''(\ell) \overset{(11)}{\le} -\int_0^\ell \frac{q}{\kappa(h_0(q)+C_2x)^3}\,dx = \frac{q}{2C_2\kappa}\left[\frac{1}{(h_0(q)+C_2\ell)^2} - \frac{1}{(h_0(q))^2}\right].$$

Therefore, along the subsequence $q \to q_0$ it holds that

$$\frac{q}{h_0^2(q)} \le C_3(1+q)$$

for a suitable constant $C_3 > 0$, which implies (ii) and (iii). □

We first consider the behavior of h_q as $q \downarrow 0$.

Lemma 2.3 *Assume* (4)-(7) *and* (10)*, and let* h_q *be the solution of* (11)*. Then the following holds as* $q \downarrow 0$*:*

$$\begin{aligned}
h_q(x) &\to \tfrac{H_*}{\ell}\, x && \text{in } C^3_{\mathrm{loc}}((0,\ell]) \cap C^2([0,\ell]) \text{ if } B = 0,\\
h_q(x) &\to x\left(ax + \tfrac{H_*}{\ell} - a\ell\right) && \text{in } C^3_{\mathrm{loc}}((0,\ell]) \cap C^1([0,\ell]) \text{ if } B \in (0,\infty),\\
h_q(\ell)h_q''(\ell) &\to -\infty && \text{if } B = \infty,
\end{aligned}$$

where

$$a = \frac{H_*}{2\ell^2}\left(1 - \left(1 + \frac{\ell^3 B^2}{\kappa H_*^2}\right)^{1/2}\right). \tag{13}$$

Proof. If $B = \infty$, Lemma 2.2 (ii) implies that $h_q''(\ell) \to -\infty$ as $q \downarrow 0$, and the assertion follows since $h_q(\ell) \to H_* > 0$ as $q \downarrow 0$. To handle the other two cases we first observe that, since h_q is concave,

$$h_q(x) \ge h_0(q) + \frac{h_1(q) - h_0(q)}{\ell} x =: r_q(x) \quad \text{for all } \; x \in [0,\ell]. \tag{14}$$

Therefore

$$0 \ge h_q''(x) \overset{(11)}{\ge} -\int_0^x \frac{q}{\kappa r_q^3(\xi)}\,d\xi \ge -\frac{\ell}{2\kappa(h_1(q)-h_0(q))}\frac{q}{h_0^2(q)}. \tag{15}$$

If $B = 0$, (15) implies that $h_q'' \to 0$ uniformly in $[0,\ell]$ as $q \downarrow 0$, which immediately yields the conclusion. If $B \in (0,\infty)$, we obtain from (15) that

$$0 \ge h_q''(x) \ge -\frac{\ell B^2}{\kappa H_*}, \quad x \in [0,\ell]$$

for q sufficiently small. Using also Lemma 2.2 (i), given $\overline{x} \in (0, \ell)$ we may extract a subsequence (not relabeled) such that

$$h_q \to p \ \text{ in } \ C^1([0,\ell]) \quad \text{and} \quad h_q''(\overline{x}) \to 2a \le 0 \quad \text{as} \ \ q \downarrow 0. \tag{16}$$

In view of (14), locally in $(0, \ell]$ the functions h_q are uniformly bounded away from zero. Hence, applying standard continuous dependence results to the Cauchy problem

$$\begin{cases} \kappa h''' = -\frac{q}{h^3} \\ h(\overline{x}) = h_q(\overline{x}), \quad h'(\overline{x}) = h_q'(\overline{x}), \quad h''(\overline{x}) = h_q''(\overline{x}), \end{cases}$$

it follows that

$$h_q \to p \ \text{ in } \ C^3_{\text{loc}}((0,\ell]) \quad \text{as} \ \ q \downarrow 0, \tag{17}$$

where p is a non-negative concave parabola satisfying $p(\ell) = H_*$, $p(0) = 0$. Thus

$$p(x) = x\left(ax + \frac{H_*}{\ell} - a\ell\right), \quad a \le 0. \tag{18}$$

To identify the value of a, we note that $p'(x)$ is positive in $[0, x_0]$ for a suitable x_0; hence, by (16), $h_q'(x) > 0$ in $[0, x_0]$ for q sufficiently small and we may write

$$h_q''(x) \overset{(11)}{=} -\int_0^x \frac{q}{\kappa h_q^3(\xi)}\, d\xi = -\int_0^x \frac{q h_q'(\xi)}{\kappa h_q^3(\xi)} \frac{1}{h_q'(\xi)}\, d\xi \quad \text{for} \ \ x \in (0, x_0].$$

Therefore, for $x \in (0, x_0)$ we have

$$\left(\sup_{\xi\in(0,x)} h_q'(\xi)\right)^{-1} \left[\frac{q}{2\kappa h_q^2(\xi)}\right]_0^x \le h_q''(x) \le \left(\inf_{\xi\in(0,x)} h_q'(\xi)\right)^{-1} \left[\frac{q}{2\kappa h_q^2(\xi)}\right]_0^x .$$

Passing to the limit as $q \downarrow 0$, in view of (16), (17), and (18) we obtain

$$-\frac{B^2}{2\kappa}\left(\sup_{\xi\in(0,x)} p'(\xi)\right)^{-1} \le 2a \le -\frac{B^2}{2\kappa}\left(\inf_{\xi\in(0,x)} p'(\xi)\right)^{-1},$$

and the arbitrariness of $x > 0$ together with (18) yields

$$a\left(\frac{H_*}{\ell} - a\ell\right) = -\frac{B^2}{4\kappa}. \tag{19}$$

The negative root of (19) is therefore the only value that a can take. This uniquely determines p and completes the proof of Lemma 2.3. □

The next lemma provides information as $q \uparrow Q$.

Lemma 2.4 *Assume* (4)-(7) *and* (10), *and let h_q be the solution of* (11). *Then*

$$\limsup_{q\to Q^-} h_q(\ell) h_q''(\ell) < 0, \tag{20}$$

and if in addition $h_0(Q) = 0$ *or* $h_1(Q) = 0$*, then*

$$\lim_{q\to Q^-} h_q(\ell)h_q''(\ell) = -\infty. \tag{21}$$

Proof. Let us first observe that

$$\limsup_{q\to Q^-} h_q''(\ell) < 0. \tag{22}$$

Indeed, otherwise along a subsequence $q \uparrow Q$ we would have at the same time

$$h_q(x) \to h_0(Q) + \frac{h_1(Q) - h_0(Q)}{\ell} x \quad \text{in} \quad C^2([0, \ell])$$

and (as a consequence)

$$h_q''' \le -\frac{Q}{\kappa(1 + h_0(Q) + h_1(Q))^3} < 0 \quad \text{in} \quad [0, \ell],$$

which is impossible. If $h_1(Q) > 0$, then (22) implies (20). If instead $h_1(Q) = 0$, let $(0, x_q) \supset (0, \ell]$ be the maximal interval in which h_q can be extended as a positive solution of (11) (note that $x_q < \infty$ since h_q is strictly concave), and denote this extension again by h_q. We claim that

$$x_q \to \ell \quad \text{as} \quad q \uparrow Q. \tag{23}$$

Indeed, it follows from (22) that $h_q''(x) \le -2C_1 < 0$ for all $x \in [\ell, x_q)$ for a suitable constant $C_1 > 0$. Hence

$$0 = h_q(x_q) \le h_1(q) + h_q'(\ell)(x_q - \ell) - C_1(x_q - \ell)^2. \tag{24}$$

In addition, since $h_1(Q) = 0$ and h_q' is decreasing we have

$$\limsup_{q\to Q^-} h_q'(\ell) \le 0. \tag{25}$$

Inequalities (24) and (25) imply (23). We now introduce the functions

$$u_q(y) := \left(\frac{\kappa}{q}\right)^{\frac{1}{4}} h_q(x_q - y),$$

which satisfy

$$\begin{cases} u^3 u''' = 1, \quad u > 0 & \text{in} \quad (0, x_q) \\ u(y) \to 0 & \text{as} \quad y \downarrow 0. \end{cases} \tag{26}$$

It has been proved in [3] that any solution of (26) satisfies

$$u(y) = \frac{2\sqrt{2}}{\sqrt[4]{15}} y^{\frac{3}{4}}(1 + o(1)) \quad \text{as} \quad y \downarrow 0.$$

Therefore, using also (26) we see that

$$u(y)u''(y) \le -C_2 y^{-\frac{1}{2}} \quad \text{for all} \ \ y \in (0,\delta)$$

for suitable constants $\delta > 0$ and $C_2 > 0$. In terms of h_q this reads as

$$h_q(x)h_q''(x) \le -\frac{C_2\sqrt{q}}{2\sqrt{\kappa}}(x_q - x)^{-\frac{1}{2}} \quad \text{for all} \ \ x \in (x_q - \delta, x_q) \ \ \text{and all} \ \ q \in (0, Q).$$

In view of (23), as $q \uparrow Q$ we have $x_q - \delta < \ell < x_q$: therefore

$$\limsup_{q \to Q^-} h_q(\ell)h_q''(\ell) \le -\frac{C_2}{2\sqrt{\kappa}} \limsup_{q \to Q^-} \sqrt{q}(x_q - \ell)^{-\frac{1}{2}} = -\infty,$$

which proves (21) if $h_1(Q) = 0$. Finally, if $h_0(Q) = 0$ and $h_1(Q) > 0$, then by Lemma 2.2 *(iii)* $h_q''(\ell) \to -\infty$ as $q \uparrow Q$, and the proof is complete. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.1 the functions

$$\begin{array}{ll} \mathcal{G} : (0,Q) \to \mathbb{R} & \qquad \mathcal{H} : (0,Q) \to \mathbb{R} \\ \mathcal{G}(q) = -\gamma h_q(\ell) h_q''(\ell) & \qquad \mathcal{H}(q) = h_q'(\ell) \end{array}$$

are continuous in $(0, Q)$. In addition,

$$\mathcal{G} > 0 \quad \text{in} \ \ (0, Q) \tag{27}$$

since h_q are strictly concave for positive q. From Lemma 2.3 and Lemma 2.4 we obtain, respectively,

$$\lim_{q \to 0^+} \mathcal{G}(q) = \begin{cases} 0 & \text{if} \ \ B = 0 \\ 2|a|\gamma H_* & \text{if} \ \ B \in (0,\infty) \\ \infty & \text{if} \ \ B = \infty \end{cases} \tag{28}$$

and

$$\limsup_{q \to Q^-} \mathcal{G}(q) = \begin{cases} C > 0 \text{ if } \ h_0(Q) > 0 \ \text{ and } \ h_1(Q) > 0 \\ \infty \qquad \text{otherwise.} \end{cases} \tag{29}$$

By Lemma 2.3 we also have

$$\lim_{q \to 0^+} \mathcal{H}(q) = \begin{cases} \frac{H_*}{\ell} & \text{if} \ \ B = 0 \\ \frac{H_*}{\ell} + a\ell & \text{if} \ \ B \in (0,\infty). \end{cases}$$

Therefore

$$\overline{q} := \sup \Big\{ \tilde{q} \in (0,Q) : \ h_q'(\ell) > 0 \ \ \forall \, q \in (0,\tilde{q}) \Big\} \tag{30}$$

is well defined if $B = 0$ or if $B > 0$ and $\frac{H_*}{\ell} + a\ell > 0$. Recalling the definition (13) of a, a simple calculation shows that these conditions are equivalent to

$$0 \leq B < \tilde{B}_1 =: \frac{2H_*}{\ell}\sqrt{\frac{2\kappa}{\ell}}.$$

Part (A). To prove the existence of a solution to Problem (I) it suffices to find a point $q_0 \in (0, Q)$ such that $\mathcal{G}(q_0) = G_1(q_0)$. We recall that G_1 is a non-increasing and continuous function such that

$$\lim_{q \to Q^-} G_1(q) = 0 \overset{(29)}{<} \limsup_{q \to Q^-} \mathcal{G}(q).$$

Hence, the existence of q_0 is implied by the condition $\mathcal{G}(0) < G_1(0) = G_*$: in view of (28), this condition is always satisfied if $B = 0$, whereas if $B > 0$ it is equivalent to $-2a\gamma H_* < G_*$. Recalling again (13), a straightforward computation shows that this is true if and only if

$$B < B_0 := \frac{1}{\gamma}\sqrt{\frac{2\kappa}{\ell}}\left(\frac{\ell^2 G_*^2}{2H_*^2} + \gamma G_*\right)^{\frac{1}{2}},$$

which proves (i). If in addition $B < \tilde{B}_1$, then $\overline{q} \in (0, Q]$ is well defined. If $\overline{q} = Q$ then $h_{\overline{q}}$ is automatically nondecreasing, hence (ii) holds for any $\overline{G} > 0$; else, since G_1 is non-increasing, if

$$G_1(\overline{q}) < \overline{G} := \mathcal{G}(\overline{q}) \tag{31}$$

then q_0 can be chosen to be in $(0, \overline{q})$ and (ii) is proved.

Part (B). Let us construct an example of non-existence. Since $B > B_0$, we have $-2a\gamma H_* > G_*$. Hence

$$q_m := \sup\{\tilde{q} \in (0, Q) : \mathcal{G}(q) > G_* \ \forall q \in (0, \tilde{q})\} \tag{32}$$

is well defined in view of (28) (note that q_m depends on G_1 only through G_*). If G_1 is any function satisfying (7), we therefore have $\mathcal{G}(q) > G_* = G_1(0) \geq G(q)$ for all $q \in (0, q_m)$. If in addition $\tilde{q} \in (0, q_m)$ exists such that

$$G_1(\tilde{q}) < g_m := \inf_{q \in (0,Q)} \mathcal{G}(q) \tag{33}$$

(g_m is positive if $B > 0$ in view of (27)–(29)), then $\mathcal{G}(q) > G_1(q)$ for all $q \in (0, Q)$ and Problem (I) has no solution.

To complete the proof, it remains to exhibit the example of non-uniqueness if $B = \infty$. If $Q < \infty$, let

$$G_1(q) = \frac{G_*}{Q}(Q - q).$$

If G_* is sufficiently large, then by (28) and (29) G_1 has at least two intersections with $\mathcal{G}$, i.e. Problem (I) has two solutions. In addition,

$$q_{1M} = \inf\{q' \in (0,Q) : \; f_M(q) < \mathcal{G}(q) \;\; \forall\, q \in (q', Q)\}$$
$$q_{2M} = \sup\{q' \in (0,Q) : \; f_M(q) < \mathcal{G}(q) \;\; \forall\, q \in (0, q')\}$$

are well defined and such that

$$\lim_{G_* \to \infty} q_{1M} = Q, \qquad \lim_{G_* \to \infty} q_{2M} = 0.$$

If $Q = \infty$ the argument is analogous and we omit it. □

3 Quasi-steady states

Let us start with some preliminary results on problem $(\text{II}_{s,\beta})$, which we recall:

$$(\text{II}_{s,\beta}) \begin{cases} (u^3 u''')' = u \\ u(0) = s, \;\; u'(0) = 0, u''(0) = -\beta, \;\; u'''(0) = 0. \end{cases}$$

Observe that, as long as it is defined, any solution of $(\text{II}_{s,\beta})$ satisfies

$$u^3(y) u'''(y) = \int_0^y u(\xi)\, d\xi \tag{34}$$

and

$$u'''' = \frac{1}{u^2} - \frac{3}{u} u' u'''. \tag{35}$$

Lemma 3.1 *For any $s > 0$ and $\beta \geq 0$ there exists a unique solution $u_{s,\beta} \in C^4([0,\infty))$ of $(\text{II}_{s,\beta})$. In addition: ,*

(i) $u_{s,\beta} > 0$ *in* $(0, \infty)$*;*

(ii) *if $\beta > 0$ there exists a unique $a = a_{s,\beta} > 0$ such that $u'_{s,\beta}(a) = 0$;*

(iii) *there exists a unique $L = L_{s,\beta} \geq 0$ such that $u''_{s,\beta}(L) = 0$;*

(iv) *as $(s,\beta) \to (s_0, \beta_0) \in (0,\infty) \times [0,\infty)$, it holds:*

$$u_{s,\beta} \longrightarrow u_{s_0,\beta_0} \quad \textit{in} \;\; C^4_{\text{loc}}([0,\infty)),$$
$$L_{s,\beta} \longrightarrow L_{s_0,\beta_0}.$$

Proof. Fix $s > 0$, $\beta \geq 0$; by standard ODE theory there exists a unique maximal solution $u \in C^4([0,\overline{y}))$, $\overline{y}$ maximal. Note that, taking initial conditions into account, u satisfies

$$u''' \overset{(34)}{>} 0, \quad u'' > -\beta, \quad u' > -\beta y_0 \quad \text{in} \;\; (0, \overline{y}). \tag{36}$$

To prove that $\overline{y} = \infty$ and that u is positive in $(0, \infty)$, it suffices to show that

$$0 < \inf_{(0,y_0)} u \le \sup_{(0,y_0)} u < \infty \quad \text{for any} \ \ y_0 \in (0, \overline{y}] \cap (0, \infty). \tag{37}$$

Indeed, (34) and (37) imply that u''', and therefore also u'' and u', are uniformly bounded in $(0, y_0)$: the conclusion then follows by standard ODE theory noting that the right-hand side of (35) is Lipschitz-continuous on compact subsets of $(u, u', u'', u''') \in (0, \infty) \times \mathbb{R}^3$.

To prove the upper bound in (37), assume by contradiction that $\limsup_{y \to y_0^-} u = \infty$. Since u'' is increasing, we actually have that $u \to \infty$, $u' \to \infty$ and $u'' \to \infty$ as $y \uparrow y_0$. In particular, $u' > 0$ in a left-neighborhood of y_0, which using (35) and (36) implies that $u'''' \le 1/u^2 \le 1$ in a left-neighborhood of y_0: this contradicts $u'' \to \infty$ and proves the upper bound.

By (36), the lower bound in (37) is obvious if $\beta = 0$. Else, assume by contradiction that $y_0 \in (0, \overline{y}] \cap (0, \infty)$ and a sequence $y_n \uparrow y_0$ exist such that $u > 0$ in $[0, y_0)$ and

$$\liminf_{n \to \infty} u(y_n) = 0. \tag{38}$$

For any $y \in (0, y_0)$ and for n sufficiently large, we have

$$u(y) = u(y_n) - \int_y^{y_n} u'(\xi)\, d\xi \overset{(36)}{<} u(y_n) + \beta y_0 (y_n - y),$$

and passing to the limit as $n \to \infty$ we obtain that $u(y) \le \beta y_0 (y_0 - y)$ for all $y \in (0, y_0)$. Then (34) implies that

$$u'''(y) = \frac{1}{u^3(y)} \int_0^y u(\xi)\, d\xi \ge \left(\int_0^{\frac{y_0}{2}} u(\xi)\, d\xi \right) \left(\frac{1}{\beta y_0 (y_0 - y)} \right)^3 \quad \text{for all} \ \ y \in (\frac{y_0}{2}, y_0),$$

which yields $u \to \infty$ as $y \uparrow y_0$, in contradiction with (38); hence the lower bound in (37).

To prove *(ii)* and *(iii)*, let $\beta > 0$ (if $\beta = 0$ then *(iii)* is trivial). Then

$$y_1 = \sup\{y' > 0 : \ u'(y) < 0 \ \ \forall\, y \in (0, y')\} > 0.$$

In $(0, y_1)$ we have $u \le s$, which using (35) and (36) implies that $u'''' \ge s^{-2}$. Hence

$$0 > u'(y) \ge -\beta y + \frac{1}{6s^2} y^3 \quad \text{in} \ \ (0, y_1)$$

which implies that $y_1 < \infty$. Therefore $a_{s,\beta}$ exists, and *(ii)* and *(iii)* follow immediately since u' is convex.

Finally, the locally uniform convergence of solutions in *(iv)* is a consequence of (i) and standard ODE theory. This in turn yields the convergence of the inflection points $L_{s,\beta}$ since they are unique. □

3.1 Problem (II_s)

In this subsection we prove the following result.

Proposition 3.2 *For any $s > 0$ and any $m \in (0, 1/2]$ there exists a unique solution (L_s, u_s) to Problem (II_s). In addition, L_s depends continuously on s.*

Let $s > 0$ be fixed, and define (in view of Lemma 3.1)

$$u_\beta := u_{s,\beta}, \quad L_\beta := L_{s,\beta}, \quad a_\beta := a_{s,\beta}. \tag{39}$$

We recall that, from Lemma 3.1,

$$a_\beta > L_\beta \quad \text{and} \quad u'_\beta < 0 \quad \text{in} \quad (0, a_\beta) \supset (0, L_\beta]. \tag{40}$$

It is convenient to rewrite the last boundary condition in Problem (II_s) as

$$u^{\frac{1}{m}-2}(L) = u(L)u'''(L). \tag{41}$$

Hence, Proposition 3.2 amounts to prove that there exists a unique β such that (41) holds. We shall proceed through lemmas, and the proof of Proposition 3.2 will conclude the section. First of all, note that

$$L_\beta \to 0, \quad u_\beta(L_\beta) \to s \quad \text{and} \quad u'''_\beta(L_\beta) \to 0 \quad \text{as} \quad \beta \downarrow 0.$$

This is an immediate consequence of Lemma 3.1 *(iv)* (since $u''_0 > 0$ in $(0, \infty)$) and in turn implies that

$$u_\beta^{\frac{1}{m}-2}(L_\beta) \to s^{\frac{1}{m}-2} \quad \text{and} \quad u_\beta(L_\beta)u'''_\beta(L_\beta) \to 0 \quad \text{as} \quad \beta \downarrow 0. \tag{42}$$

The next lemma provides information as $\beta \uparrow \infty$.

Lemma 3.3 *Let u_β, L_β be defined by* (39)*; as $\beta \uparrow \infty$, the following holds: ,*

(i) $\beta L_\beta^2 \sim 2s$;

(ii) $\beta^2 u_\beta^2(L_\beta) \sim s/3$;

(iii) $\beta^{-5}\left(u'''_\beta(L_\beta)\right)^2 \sim 24$.

Proof. The functions $v_\beta(z) := u_\beta(z/\sqrt{\beta})$ satisfy

$$\begin{cases} \beta^2(v^3 v''')' = v, \quad z > 0 \\ v(0) = s, \ v'(0) = 0, \ v''(0) = -1, \ v'''(0) = 0. \end{cases}$$

A simple compactness argument shows that

$$v_\beta(z) \to s - \frac{z^2}{2} \quad \text{in} \quad C^4_{\mathrm{loc}}([0, \sqrt{2s})) \quad \text{as} \quad \beta \uparrow \infty. \tag{43}$$

Let $a'_\beta = \sqrt{\beta} a_\beta$ and $L'_\beta = \sqrt{\beta} L_\beta$, so that $v'_\beta(a'_\beta) = 0$ and $v''_\beta(L'_\beta) = 0$.

Proof of (i). By (40) and (43),

$$\liminf_{\beta\to\infty} a'_\beta \geq \liminf_{\beta\to\infty} L'_\beta \geq \sqrt{2s}. \tag{44}$$

On the other hand, we claim that

$$\liminf_{\beta\to\infty} v'_\beta(z) \geq 0 \quad \text{for all} \quad z > \sqrt{2s}, \tag{45}$$

which in turn implies that

$$\limsup_{\beta\to\infty} L'_\beta \leq \limsup_{\beta\to\infty} a'_\beta \leq \sqrt{2s}$$

and together with (44) completes the proof of (i). To see (45), assume by contradiction that $z > \sqrt{2s}$ and a subsequence (not relabeled) exist such that $v'_\beta(z) \leq -C' < 0$. Fix $z_0 \in (0, \sqrt{2s})$. Since v'_β is convex and, by (43), $v'_\beta(z_0) \to -z_0 < 0$ as $\beta \uparrow \infty$, it follows that $v'_\beta \leq -C < 0$ in (z_0, z) for β sufficiently large. In particular

$$v_\beta(z) \leq v_\beta(\sqrt{2s} - \varepsilon) - C(z + \varepsilon - \sqrt{2s}) \quad \text{for all} \quad \varepsilon \in (0, \sqrt{2s} - z_0).$$

Recalling (43) and choosing ε sufficiently small and β sufficiently large, this yields $v_\beta(z) < 0$, a contradiction.

Proof of (ii). It follows from (44) that $L'_\beta > \sqrt{2(s-\varepsilon)}$ for any $\varepsilon > 0$ and β sufficiently large; since v_β is decreasing in $(0, L'_\beta]$, this yields $v_\beta(L'_\beta) < v_\beta(\sqrt{2(s-\varepsilon)}) \to \varepsilon$ as $\beta \uparrow \infty$. Hence

$$v^\star_\beta := v_\beta(L'_\beta) = u_\beta(L_\beta) \to 0 \quad \text{as} \quad \beta \uparrow \infty. \tag{46}$$

By the definition of a'_β, the functions

$$z_\beta(v) : [v_\beta(a'_\beta), s] \to [0, a'_\beta], \quad v_\beta(z_\beta(v)) = v \quad \text{and} \quad \eta_\beta(v) = \left[v'_\beta(z_\beta(v))\right]^2$$

ar well defined and satisfy

$$\begin{cases} \eta''_\beta(v) = -\frac{\phi_\beta(v)}{\sqrt{\eta_\beta(v)}} & v \in (v_\beta(a'_\beta), s) \\ \eta_\beta(s) = 0, \ \ \eta'_\beta(s) = -2 \\ \eta'_\beta(v^\star_\beta) = 0 \end{cases}$$

where

$$\phi_\beta(v) = \frac{2}{\beta^2 v^3} \int_0^{z_\beta(v)} v_\beta(\zeta)\, d\zeta.$$

Furthermore, it follows from (46) and (40) that $v_\beta(a'_\beta) \downarrow 0$ as $\beta \uparrow \infty$, so that by (43)

$$\eta_\beta(v) \to 2(s - v) \quad \text{in} \quad C^1_{\rm loc}((0, s]) \quad \text{as} \quad \beta \uparrow \infty. \tag{47}$$

Let now $v_0 \in (0, s)$. Since $\eta'_\beta(v^\star_\beta) = 0$, we have

$$-\eta'_\beta(v_0) = \int_{v^\star_\beta}^{v_0} \frac{2}{\beta^2 v^3 \sqrt{\eta_\beta(v)}} \left(\int_0^{z_\beta(v)} v_\beta(\zeta)\, d\zeta \right) dv.$$

We use this identity to prove (ii). Note that, since η_β and z_β are decreasing in $[v^\star_\beta, v_0]$,

$$\frac{1}{\sqrt{\eta_\beta(v^\star_\beta)}} \left(\int_0^{z_\beta(v_0)} v_\beta \right) \int_{v^\star_\beta}^{v_0} \frac{2\, dv}{\beta^2 v^3} \le -\eta'_\beta(v_0) \le \frac{1}{\sqrt{\eta_\beta(v_0)}} \left(\int_0^{L'_\beta} v_\beta \right) \int_{v^\star_\beta}^{v_0} \frac{2\, dv}{\beta^2 v^3}. \tag{48}$$

On the upper bound's side of (48), for all $z \in (0, \sqrt{2s})$ and β sufficiently large we have

$$0 \overset{(i)}{\le} \int_0^{L'_\beta} v_\beta(\zeta)\, d\zeta - \int_0^z v_\beta(\zeta)\, d\zeta \le v_\beta(z)(L'_\beta - z)$$

since v_β is decreasing up to L'_β. Therefore, it follows from (43), (i) and the arbitrariness of z that

$$\lim_{\beta\to\infty} \int_0^{L'_\beta} v_\beta(\zeta)\, d\zeta = \int_0^{\sqrt{2s}} \left(s - \frac{\zeta^2}{2} \right) d\zeta = \frac{2}{3} s\sqrt{2s}. \tag{49}$$

Using (47) and (49) into (48), in the limit $\beta \uparrow \infty$ we obtain

$$2 \le \frac{1}{\sqrt{2(s - v_0)}} \left(\frac{2s\sqrt{2s}}{3} \right) \liminf_{\beta\to\infty} \int_{v^\star_\beta}^{v_0} \frac{2\, dv}{\beta^2 v^3} = \frac{2s\sqrt{2s}}{3\sqrt{2(s - v_0)}} \liminf_{\beta\to\infty} \frac{1}{\beta^2 (v^\star_\beta)^2},$$

that is

$$\limsup_{\beta\to\infty} \beta^2 \left(v^\star_\beta \right)^2 \le \frac{s\sqrt{2s}}{3\sqrt{2(s - v_0)}}. \tag{50}$$

On the lower bound's side of (48), we have by (43)

$$\lim_{\beta\to\infty} \left(\int_0^{z_\beta(v_0)} v_\beta \right) = \int_0^{\sqrt{2(s - v_0)}} \left(s - \frac{1}{2} z^2 \right) dz = \left(\frac{2s}{3} + \frac{v_0}{3} \right) \sqrt{2(s - v_0)}. \tag{51}$$

In addition, since η'_β is decreasing we have $0 = \eta'_\beta(v^\star_\beta) \ge \eta'_\beta(v) \ge \eta'_\beta(s) = -2$ for any $v \in [v^\star_\beta, s]$. Therefore, since $\eta_\beta(s) = 0$, for any $v \in (0, s)$ and any β sufficiently large we have

$$-\int_v^s \eta'_\beta(w)\, dw \le \eta_\beta(v^\star_\beta) = -\int_{v^\star_\beta}^s \eta'_\beta(w)\, dw \le 2s.$$

Passing to the limit as $\beta \uparrow \infty$ (using (47)) and $v \downarrow 0$ (in this order) yields

$$\eta_\beta(v^\star_\beta) \to 2s \quad \text{as} \quad \beta \uparrow \infty. \tag{52}$$

Using (51) and (52) into (48) we obtain

$$\liminf_{\beta\to\infty} \beta^2 \left(v^\star_\beta \right)^2 \ge \frac{\sqrt{2(s - v_0)}}{\sqrt{2s}} \left(\frac{s}{3} + \frac{v_0}{6} \right), \tag{53}$$

and (ii) follows from (50), (53) and the arbitrariness of v_0.

Proof of (iii). In view of (49) and *(ii)*, *(iii)* is an immediate consequence of

$$v_\beta'''(L_\beta') = \frac{1}{\beta^2 v_\beta^3(L_\beta')} \int_0^{L_\beta'} v_\beta(z)\, dz.$$

□

The results contained in Lemma 3.3 and in (42) already imply the existence of a solution for Problem (II_s). The next two lemmas provide monotonicity properties which we need in order to prove uniqueness. In view of (40), we can define the inverse function

$$y_\beta(u) : [u_\beta(a_\beta), s] \to [0, a_\beta], \quad u_\beta(y_\beta(u)) = u. \tag{54}$$

Lemma 3.4 *Let u_β, L_β, a_β and y_β be defined by* (39) *and* (54)*. If $\beta_1 < \beta_2$, then ,*

(i) $u_{\beta_1}(a_{\beta_1}) > u_{\beta_2}(a_{\beta_2})$*;*

(ii) $u'_{\beta_1}(y_{\beta_1}(u)) > u'_{\beta_2}(y_{\beta_2}(u))$ *for any* $u \in [u_{\beta_1}(a_{\beta_1}), s)$*;*

(iii) $y_{\beta_1}(u) > y_{\beta_2}(u)$ *for any* $u \in [u_{\beta_1}(a_{\beta_1}), s)$*;*

(iv) $u_{\beta_1}(y) > u_{\beta_2}(y)$ *for any* $y \in (0, a_{\beta_2})$*;*

(v) $u_{\beta_1}(L_{\beta_1}) > u_{\beta_2}(L_{\beta_2})$*.*

Proof. For notational convenience, we let $u_i = u_{\beta_i}$, $a_i = a_{\beta_i}$, $L_i = L_{\beta_i}$ and $y_i = y_{\beta_i}$. As in the previous proof, we define

$$\xi_i(u) = [u_i'(y_i(u)]^2, \quad u \in [u_i(a_i), s],$$

which satisfies

$$\begin{cases} \xi_i''(u) = -\frac{\phi_i(u)}{\sqrt{\xi_i(u)}} & u \in (u_i(a_i), s) \\ \xi_i(s) = 0, \quad \xi_i'(s) = -2\beta \\ \xi_i'(u_i(L_i)) = 0 \end{cases}$$

where

$$\phi_i(u) = \frac{2}{u^3} \int_0^{y_i(u)} u_i(y)\, dy.$$

Since $\beta_1 < \beta_2$, $\xi_1 < \xi_2$ in a left neighborhood of $u = s$. Then it is well defined the infimum

$$u_* = \inf\{u \in (u_0, s) : \ \xi_1 - \xi_2 < 0 \text{ in } (u, s)\}, \quad \text{where} \quad u_0 = \max_i \{u_i(a_i)\}. \tag{55}$$

Noting that

$$y_i(u) = y_i(s) + \int_s^u y_i'(v)\, dv = \int_u^s \frac{dv}{\sqrt{\xi_i(v)}}\, dv, \tag{56}$$

we have $y_1 > y_2$ in (u_*, s), i.e. $u_1 > u_2$ in $(0, y_2(u_*))$. Hence $\phi_1 > \phi_2$ in (u_*, s), which in turn implies that $(\xi_1 - \xi_2)'' < 0$ in (u_*, s). Therefore $u_* = u_0$ and $-\xi_2(u) = u_2'(y_2(u)) <$

$u_1'(y_1(u)) = -\xi_1(u) < 0$ in $[u_0, s)$; recalling the definition of a_i, this means that $u_0 = u_1(a_1)$ and proves (i) and (ii). Using again (56), (55) implies that $y_2 < y_1$ in $[u_0, s)$, i.e. (iii). In turn, this means that $u_2 < u_1$ in $(0, y_2(u_0)]$. After $y_2(u_0)$ u_1 is increasing, hence in fact $u_2 < u_1$ in $(0, a_2)$, i.e. (iv). Finally, if by contradiction $\overline{u} = u_2(L_2) \geq u_1(L_1)$, since $\overline{u} \geq u_1(L_1) > u_1(a_1) = u_0$ we would have

$$(\xi_1 - \xi_2)'(\overline{u}) = 2(u_1 - u_2)''(\overline{u}) = 2u_1''(\overline{u}) \leq 0,$$

which is impossible since $\xi_1 - \xi_2$ is concave (see above) and $(\xi_1 - \xi_2)'(s) > 0$. Hence (v) holds true and the proof is complete. □

Lemma 3.5 *Let $\beta_1 < \beta_2$, and let x_{β_i} denote the inverse functions of u''_{β_i}. Then*

$$\left[\left(u_{\beta_1} u'''_{\beta_1}\right) \circ x_{\beta_1}\right]^2 < \left[\left(u_{\beta_2} u'''_{\beta_2}\right) \circ x_{\beta_2}\right]^2 \quad \text{in } [-\beta_1, 0]; \tag{57}$$

in particular, $u_{\beta_1}(L_{\beta_1}) u'''_{\beta_1}(L_{\beta_1}) < u_{\beta_2}(L_{\beta_2}) u'''_{\beta_2}(L_{\beta_2})$.

Proof. We let for notational convenience

$$u_i := u_{\beta_i}, \quad a_i := a_{\beta_i}, \quad L_i = L_{\beta_i}, \quad y_i := y_{\beta_i}.$$

Since $u_i''' > 0$ in $(0, \infty)$, we can choose u'' as independent variable: the functions

$$x_i(u'') : \; x_i(u_i''(y)) = y, \quad w_i(u'') := \left[u_i(x_i(u'')) u_i'''(x_i(u''))\right]^2,$$

are well defined in $(0, \infty)$, and w_i satisfy

$$\begin{cases} w_i'(u'') = 2 - 4u_i'(x_i(u''))\sqrt{w_i(u'')}, & u'' > -\beta_i \\ w_i(-\beta_i) = 0. \end{cases} \tag{58}$$

We claim that

$$u_2'(x_2(u'')) < u_1'(x_1(u'')) \quad \forall\, u'' \in (-\beta_1, 0]. \tag{59}$$

If (59) holds, then (57) easily follows by comparison from (58), and the particular case in the lemma corresponds to choosing $u'' = 0$.

In order to prove (59), assume for a moment that the following relation holds:

$$u_2(x_2(u'')) \leq u_1(x_1(u'')) \quad \forall\, u'' \in [-\beta_1, 0]. \tag{60}$$

Observing that

$$\begin{aligned} u'' \in (-\beta_1, 0] &\Rightarrow x_1(u'') \in (0, L_1] \\ &\Rightarrow u_1(x_1(u'')) \in [u_1(L_1), s) \subset (u_1(a_1), s), \end{aligned} \tag{61}$$

we can apply Lemma 3.4 *(ii)*, which yields

$$u_1'(x_1(u'')) = u_1'(y_1(u_1(x_1(u'')))) > u_2'(y_2(u_1(x_1(u'')))) \quad \forall\, u'' \in (-\beta_1, 0].$$

Since $\frac{d}{du}(u_2' \circ y_2) = u_2''(y_2(u))/u_2'(y_2(u))$, $u_2' \circ y_2$ is increasing as long as $u_2'' \leq 0$: therefore using (60) we obtain (59). Thus, the rest of the proof will be concerned with verifying (60).

We observe that

$$\frac{d}{du}(u_i \circ x_i) = \frac{u_i'(x_i(u))}{u_i'''(x_i(u))} < 0 \quad \text{in} \quad (-\beta_i, 0) \tag{62}$$

since $u_i''' > 0$ and $u_i' < 0$ as long as $u_i'' < 0$. Therefore

$$u_1(x_1(-\beta_1)) = s = u_2(x_2(-\beta_2)) > u_2(x_2(-\beta_1)) \tag{63}$$

and it follows from Lemma 3.4 *(v)* that

$$u_1(x_1(0)) = u_1(L_1) > u_2(L_2) = u_2(x_2(0)). \tag{64}$$

Suppose by contradiction that $u_1(x_1(u'')) < u_2(x_2(u''))$ at some point $u'' \in (-\beta_1, 0)$. Then (63) and (64) imply that there exist $-\beta_1 < u_{01}'' < u_{02}'' < 0$ such that

$$u_{0j} := u_1(x_1(u_{0j}'')) = u_2(x_2(u_{0j}'')), \quad j = 1, 2$$

and $u_{01} > u_{02}$ in view of (62). We introduce the functions

$$\varphi_i(u) := [u_i''(y_i(u))]^2 .$$

Both φ_1 and φ_2 are well defined in $[u_1(a_1), s]$ in view of Lemma 3.4 *(i)*, and satisfy

$$\varphi_i'(u) = -\frac{2u_i'''(y_i(u))}{u_i'(y_i(u))}\sqrt{\varphi_i(u)}, \quad u \in [u_i(L_i), s)$$

since $u_i'' \leq 0$ in $[0, L_i]$. Note that, in view of (61), (64) and the definition of u_{0j}, we have $[u_{02}, u_{01}] \subset (u_1(L_1), s) \subset (u_2(L_2), s)$ and $\varphi_i(u_{0j}) = (u_{0j}'')^2$, whence

$$\begin{cases} \varphi_i'(u) = -\dfrac{2u_i'''(y_i(u))}{u_i'(y_i(u))}\sqrt{\varphi_i(u)}, \quad u \in [u_{02}, u_{01}] \\ \varphi_i(u_{0j}) = (u_{0j}'')^2, \quad j = 1, 2. \end{cases} \tag{65}$$

We shall now compare φ_1 and φ_2 in order to obtain a contradiction. First we observe that by Lemma 3.4 *(ii)*, *(iii)* and (61)

$$y_1(u) > y_2(u) \quad \text{and} \quad u_1'(y_1(u)) > u_2'(y_2(u)) \quad \text{for all} \quad u \in [u_{02}, u_{01}] \subset (u_1(a_1), s). \tag{66}$$

In addition

$$u_1 > u_2 \quad \text{in} \quad (0, y_2(u)] \subset (0, y_2(u_{02})], \tag{67}$$

which follows from Lemma 3.4 *(iv)* since

$$y_2(u_{02}) = y_2(u_2(x_2(u_{02}''))) = x_2(u_{02}'') < L_2 < a_2.$$

From (34), (66) and (67) we obtain

$$u_1'''(y_1(u)) = \frac{1}{u^3}\int_0^{y_1(u)} u_1(\xi)\,d\xi \geq \frac{1}{u^3}\int_0^{y_2(u)} u_2(\xi)\,d\xi = u_2'''(y_2(u)), \tag{68}$$

and combining (68) with (65) and (66) we conclude by comparison that $\varphi_1(u) > \varphi_2(u)$ in $(u_{02}, u_{01}]$. On the other hand $\varphi_i(u_{01}) = (u_{01}'')^2$ and we have obtained a contradiction. Hence (60) holds and the proof is complete. □

We are now ready to prove Proposition 3.2.

Proof of Proposition 3.2. Let $s > 0$. In view of Lemma 3.1 the functions

$$\mathcal{G}(\beta) := (u_{s,\beta}(L_{s,\beta}))^{\frac{1}{m}-2}, \quad \mathcal{H}(\beta) := u_{s,\beta}(L_{s,\beta})\, u_{s,\beta}'''(L_{s,\beta})$$

are well defined and continuous in $(0,\infty)$. It follows from (8) and Lemma 3.4 *(v)* that $\mathcal{G}$ is decreasing (strictly if $m < 1/2$), and from Lemma 3.5 that $\mathcal{H}$ is strictly increasing. Recalling (42) and Lemma 3.3, it holds

$$\mathcal{G}(\beta) \to s^{\frac{1}{m}-2} \qquad \text{and } \mathcal{H}(\beta) \to 0 \text{ as } \beta \downarrow 0,$$

$$\mathcal{G}(\beta) \to \begin{cases} 0 \text{ if } 0 < m < 1/2 \\ 1 \text{ if } m = 1/2 \end{cases} \qquad \text{and } \mathcal{H}(\beta) \to \infty \text{ as } \beta \uparrow \infty.$$

Hence there exists a unique β_s such that $\mathcal{G}(\beta_s) = \mathcal{H}(\beta_s)$, and u_{s,β_s} is the unique solution of Problem (II_s). Continuous dependence of $L_s := L_{s,\beta_s}$ on s follows from the uniqueness of the solution and from Lemma 3.1 *(iv)*. □

3.2 Proof of Theorem 1.2

It follows from Proposition 3.2 that the solution (L_s, u_s) of Problem (II_s) is unique and that L_s depends continuously on s. Thus the proof of Theorem 1.2 is complete once we have shown that

$$\lim_{s\to 0^+} L_s = 0 \tag{69}$$

and that

$$\lim_{s\to\infty} L_s = \infty. \tag{70}$$

Proof of (69). The functions $v_s(y) = \frac{1}{s}u_s(y)$ satisfy

$$\begin{cases} (v_s^3 v_s''')' = v_s/s^3, \quad y > 0 \\ v_s(0) = 1, \;\; v_s'(0) = v_s'''(0) = 0 \\ v_s''(L_s) = 0, \;\; (v_s(L_s))^{\frac{1-3m}{m}} = s^{\frac{2m-1}{m}} v_s'''(L_s). \end{cases}$$

We have

$$0 = v_s''(L_s) = v_s''(y) + \int_y^{L_s} \frac{1}{s^3 v_s^3(\xi)} \left(\int_0^\xi v_s \right) d\xi.$$

If by contradiction $L_s \geq L > 0$ for a subsequence (not relabeled) $s \downarrow 0$, then

$$v_s''(y) \leq -\int_y^L \frac{1}{s^3 v_s^3(\xi)} \left(\int_0^\xi v_s \right) d\xi.$$

Since $v_s \in [0,1]$ is concave in $(0,L)$, $v_s \geq 1/2$ in $(0,L/2)$. Hence we obtain for $y \in [L/2, L]$

$$v_s''(y) \leq -\int_y^L \frac{1}{s^3} \left(\int_0^{L/2} \frac{1}{2} \right) \leq -\frac{L}{4s^3}(L-y)$$

and therefore $v_s'' \to -\infty$ uniformly in $[L/2, 3L/4]$ as $s \to \infty$. Since $v_s(L/2) \leq 1$ and $v_s'(L/2) \leq 0$, this implies that $v_s(3L/4) \to -\infty$ as $s \to \infty$, a contradiction.

Proof of (70). Let $b_s = |u_s''(0)|\sqrt{s}$. We claim that

$$\lim_{s\to\infty} b_s = \infty. \tag{71}$$

If not, there exists a subsequence (not relabeled) such that

$$b_s \to b^2 \in [0,\infty) \ \text{ as } \ s \to \infty. \tag{72}$$

It is easy to check that the functions $v_s(z) = s^{-1} u_s(s^{\frac{3}{4}} z)$ satisfy

$$\begin{cases} (v_s^3 v_s''')' = v_s, \quad z > 0 \\ v_s(0) = 1,\ v_s'(0) = 0,\ v_s''(0) = -b_s,\ v_s'''(0) = 0 \\ v_s''(L_s') = 0, \ \ (v_s(L_s'))^{\frac{1-3m}{m}} = s^{\frac{7}{4}-\frac{1}{m}} v_s'''(L_s') \end{cases}$$

where $L_s' = s^{-\frac{3}{4}} L_s$. In particular, they coincide with the unique solution v_{1,b_s} of Problem (II_{1,b_s}), and in view of Lemma 3.1 and (72)

$$L_s' \to L_{1,b} \quad \text{and} \quad v_s \to v_{1,b} \ \text{ in } \ C^4_{\mathrm{loc}}([0,\infty)) \quad \text{as} \ \ s \to \infty,$$

from which follows that

$$v_s(L_s') \to v_{1,b}(L_{1,b}) > 0 \ \text{ as } \ s \to \infty. \tag{73}$$

On the other hand, we have

$$v_s'''(L_s') = \frac{1}{v_s^3(L_s')} \int_0^{L_s'} v_s(\xi)\, d\xi \leq \frac{L_s'}{v_s^3(L_s')}$$

and therefore

$$(v_s(L_s'))^{\frac{1}{m}} s^{\frac{1}{m}-\frac{7}{4}} = v_s^3(L_s') v_s'''(L_s') \leq L_s' \to L_{1,b} < \infty \quad \text{as} \ \ s \to \infty;$$

Hence $v_s(L'_s) \to 0$ as $s \to \infty$ in contradiction with (73), and (71) is proved.

Now we are ready to prove (70). Applying Lemma 3.3 with $s = 1$ and $\beta = b_s$, we have that

$$L'_s \sim \sqrt{2} b_s^{-\frac{1}{2}}, \quad v_s(L'_s) \sim \frac{1}{\sqrt{3}} b_s^{-1} \quad \text{and} \quad v'''_s(L'_s) \sim 2\sqrt{6} b_s^{\frac{5}{2}} \quad \text{as} \quad s \to \infty.$$

Hence, for suitable universal constants $C_i > 0$,

$$s^{\frac{1}{m}-\frac{7}{4}} = v'''_s(L'_s)\,(v_s(L'_s))^{3-\frac{1}{m}} \sim C_1 b_s^{-\frac{1}{2}+\frac{1}{m}} \quad \text{as} \quad s \to \infty,$$

that is

$$b_s \sim C_2 s^{\frac{4-7m}{2(2-m)}} \quad \text{as} \quad s \to \infty.$$

Therefore

$$L_s = s^{\frac{3}{4}} L'_s \sim s^{\frac{3}{4}} \sqrt{2} b_s^{-\frac{1}{2}} \sim C_3 s^{\frac{3}{4}-\frac{4-7m}{4(2-m)}} = C_3 s^{\frac{1+2m}{2(2-m)}} \to \infty \quad \text{as} \quad s \to \infty$$

which proves (70) and completes the proof of Theorem 1.2.

A Appendix

By rescaling, Proposition 2.1 follows immediately from the following result:

Proposition A.1 *For any $a_0 > 0$, $a_1 > 0$ and $b > 0$ there exists a unique solution $h \in C^3([0,1])$ of*

$$\begin{cases} h^3 h''' = -b \\ h(0) = a_0, \ h''(0) = 0, \ h(1) = a_1, \end{cases} \tag{A.1}$$

and h depends continuously (in the C^3-norm) on a_0 , a_1 and b as long as they are positive.

Proof. The function

$$G(x,t) = \begin{cases} -\frac{1}{2}(t-1)^2 x & \text{if } 0 \le x \le t \\ \frac{1}{2}[x^2 - (t^2+1)x + t^2] & \text{if } t < x \le 1 \end{cases}$$

is the Green's function of the linear problem

$$\begin{cases} h''' = \psi \in C([0,1]) \\ h(0) = h''(0) = h(1) = 0. \end{cases} \tag{A.2}$$

Indeed, it is easy to check that $\partial_x^3 G(x,t) = \delta(x-t)$ and that $G(0,t) = \partial_{xx}^2 G(0,t) = G(1,t) = 0$ for all $t \in (0,1)$, so that

$$h(x) = \int_0^1 G(x,t)\psi(t)\; dt$$

is a solution of (A.2). Therefore, the problem

$$\begin{cases} h''' = \psi \in C([0,1]) \\ h(0) = a_0, \ h''(0) = 0, \ h(1) = a_1 \end{cases} \tag{A.3}$$

is solved by

$$h(x) = a_0 + (a_1 - a_0)x + \int_0^1 G(x,t)\psi(t)\ dt.$$

The solution of (A.3) is unique since $h''' = 0$ with $h''(0) = 0$ and $h(0) = h(1) = 0$ implies $h = 0$.

Now we apply a fixed point argument. Let

$$S = \{k \in C([0,1]) : \lambda \le h \le \Lambda\},$$

with $0 < \lambda < \Lambda$ to be chosen below, and let $T : S \to C^3([0,1])$ be the map which associates to $k \in S$ the unique solution h of (A.3) with $\psi = -bk^{-3}$, i.e.

$$T(k) = h(x) = a_0 + (a_1 - a_0)x - b\int_0^1 G(x,t)k^{-3}(t)\ dt.$$

Since $G \in W^{2,\infty}$, we have

$$\partial_x^j h(x) = \partial_x^j (a_0 + (a_1 - a_0)x) - b\int_0^1 \partial_x^j G(x,t)k(t)\ dt, \quad j = 0,1,2. \tag{A.4}$$

In particular, since $\partial_x^2 G \ge 0$ and $k > 0$, h is concave and therefore $h \ge \lambda = \min\{a_0, a_1\}$. In addition, noting that $-\frac{1}{8} \le G \le 0$, it holds that

$$h(x) \le \max\{a_0, a_1\} + \frac{b}{8\lambda^3} =: \Lambda.$$

With these choices $T(S) \subseteq S$. In addition, it follows from the representation (A.4) that T is continuous and that $T(S)$ is bounded in $C^1([0,1])$, hence relatively compact in S. This implies the existence of a fixed point $h \in S$, and since $h \ge \lambda > 0$, h is smooth in $[0,1]$.

Uniqueness comes almost for free by monotonicity. Indeed, let h and k be two solutions of (A.1). If $h'(0) = k'(0)$, then they coincide by standard ODE theory. If on the contrary, say, $h'(0) > k'(0)$, then $h > k$ in a right-neighborhood I of $x = 0$, that is $h''' = -bh^{-3} > -bk^{-3} = k'''$ in I, which means that the ordering is preserved all the way down to $x = 1$: this is impossible since $h(1) = k(1) = a_1$. Therefore $h = k$.

Finally, the representation of T implies that h depends continuously (in the $C^0([0,1])$-norm) on the coefficients as long as a_0 and a_1 are positive, and continuity in the C^3-norm follows from (A.1).

□

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