EXISTENCE RESULTS FOR PROBLEMS IN FRACTURE MECHANICS

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ABSTRACT. We state new existence results for a minimum problem occurring in a variational model of quasistatic growth for brittle cracks. We assume that the integrand appearing in the crack energy can be depend in a discontinuous way on the spatial variable, hence heterogeneous materials can be considered.

KEYWORDS: Semicontinuity, Existence, Capacity, Fractures.
AMS-MSC: 49J45, 49Q20, 74R10.

1. INTRODUCTION

In this paper we obtain a new existence theorem for a variational problem occurring in fracture mechanics which permits to extend some recent quasistatic evolution results to the case of stratified and heterogeneous materials.
In the last years, many mathematicians considered variational models for the evolution of the fracture process. In [20] Francfort and Marigo presented a model for the quasistatic growth of brittle cracks in elastic materials which is described by an integral functional including very general bulk and crack energy. The crack growth admits a quasistatic evolution, i.e. at each time the equilibrium is obtained by the competition between the elastic energy of the body and the dissipation energy of the fracture process.
In other more recent papers ([12],[13],[14] and [15]) this quasistatic evolution is studied in the framework of nonlinear elasticity. In these papers a precise mathematical formulation of the problem is given in the SBV setting of special functions of bounded variation. The bulk energy of the uncracked part of the body is given by

$$\mathcal{W}(u) := \int_{\Omega\setminus\Gamma} W(x, \nabla u(x))\, dx, \tag{1.1}$$

where Γ is the crack, the function $u : \Omega \setminus \Gamma \to I\!\!R^N$ is the unknown deformation of the body and $W(x,\xi)$ is a quasiconvex function with respect to ξ which describes the material. The energy needed to produce the crack Γ admits the form

$$\mathcal{K}(\Gamma) := \int_{\Gamma} k(x, \nu(x))\, d\mathcal{H}^{N-1}, \tag{1.2}$$

where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and the function k depends on the position x and on the orientation ν. This function describes the "toughness" of the material in different locations and directions, thus including the case of heterogeneous and anisotropic materials. The existence of the quasistatic evolution is obtained by a time discretization and by minimizing the total energy

$$\mathcal{F}(u,\Gamma) := \mathcal{W}(u) + \mathcal{K}(\Gamma)\,, \tag{1.3}$$

at each discretization step.

In all the papers quoted above the existence of minimizers is assured by a compactness theorem due to Ambrosio (see [4] and [7]) and by standard hypotheses which guarantee the lower semicontinuity of $\mathcal{W}$ and $\mathcal{K}$ (see [5] and [6]).
In this paper we prove that this lower semicontinuity still holds under weaker assumptions on the integrand $k(x,\nu)$. More precisely, we allows jumps of the function $k(x,\nu)$ by requiring only a BV dependence on x. More generally, we assume a lower semicontinuity of $k(\cdot,\nu)$ with respect to the C_1 capacity, which is satisfied in particular by the lower approximate limit $k^-(\cdot,\nu)$ of the BV function $k(\cdot,\nu)$.
This setting seems to apply to the case of composite or stratified media, where the energy needed to create the crack may change from point to point in a discontinuous way. More precisely, we consider a fracture energy of the type

$$\mathcal{K}(u) := \int_{J_u} k(x,\nu_u(x))\gamma(|u^+(x)-u^-(x)|)\, d\mathcal{H}^{N-1}\,, \tag{1.4}$$

where $|u^+ - u^-|$ is the difference of the trace of u on both sides of J_u, ν_u is the normal to the jump set J_u and the function γ depends on the material. For $k(x,\nu)=1$ the energy (1.4) was proposed by Barenblatt in [9] (see also [23]), while in [12] and [13] the authors consider the case where $\gamma(s)=1$.
In order to prove our lower semicontinuity result, we use some methods introduced previously in the papers [1], [2], [3], [16] and [17] for general integral functionals defined in BV. More precisely, we approximate from below the functional by linearized functionals, whose continuity is a consequence of a suitable chain rule formula in BV. This formula allows also a dependence on x and it holds under a very weak assumption, as a BV dependence.
The paper contains also a scalar lower semicontinuity result for a functional of the type

$$\mathcal{K}(u) := \int_{J_u} h(x,u^-,u^+,\nu_u)\, d\mathcal{H}^{N-1}\,, \tag{1.5}$$

in which the integrand on the jump set J_u allows a fairly general dependence on its variables.

2. Notation and preliminaries

2.1. **Notation.** Throughout the paper $N>1$ is a fixed integer and the letter c denotes a strictly positive constant, whose value may change from line to line.
Given $x_0 \in I\!R^N$ and $\rho>0$, $B_\rho(x_0)$ denotes the ball in $I\!R^N$ centered in x_0 with radius ρ.
Let Ω be a bounded open subset of $I\!R^N$ with Lipschitz boundary. We denote by $\mathcal{A}(\Omega)$ the family of all open subsets A of Ω and by $\mathcal{B}(\Omega)$ the σ-algebra of all Borel subsets B of Ω.
Set $\mathcal{L}^N$ the Lebesgue measure on $I\!R^N$ and $\mathcal{H}^{N-1}$ the Hausdorff measure of dimension $(N-1)$ on $I\!R^N$.

2.2. **Approximate limits and BV-functions.** If $u \in L^1_{\rm loc}(\Omega; I\!R^M)$ and $x\in\Omega$, the *precise representative of u at x* is defined as the unique value $\widetilde{u}(x) \in I\!R^M$ such that

$$\lim_{\varrho\to 0^+} \frac{1}{\varrho^N} \int_{B_\varrho(x)} |u(y)-\widetilde{u}(x)|\, dx = 0\,. \tag{2.1}$$

The set of points in Ω where the precise representative of x is not defined is called the *approximate singular set* of u and denoted by S_u.

Let $u \in L^1(\Omega; \mathbb{R}^M)$ and $x \in \Omega$. We say that x is an approximate jump point of u if there exists $a, b \in \mathbb{R}^M$ and $\nu \in \mathbb{S}^{N-1}$, such that $a \neq b$ and

$$\lim_{\rho \to 0^+} \fint_{B_\rho^+(x,\nu)} |u(y) - a|\, dy = 0 \qquad \text{and} \qquad \lim_{\rho \to 0^+} \fint_{B_\rho^-(x,\nu)} |u(y) - b|\, dy = 0$$

where $B_\rho^\pm(x,\nu) := \{y \in B_\rho(x) \ : \ \langle y - x, \nu \rangle \gtrless 0\}$. The triplet (a, b, ν) is uniquely determined by previous formulas, up to a permutation of a, b and a change of sign of ν, and it is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The Borel functions u^+ and u^- are called the *upper and lower approximate limit* of u at the point $x \in \Omega$. The set of approximate jump points of u is denoted by J_u.
We recall that the space $BV(\Omega; \mathbb{R}^M)$ of *functions of bounded variation* is defined as the set of all $u \in L^1(\Omega; \mathbb{R}^M)$ whose distributional gradient Du is a bounded Radon measure on Ω with values in the space $\mathbb{M}^{M\times N}$ of $M \times N$ matrices.
We recall the usual decomposition

$$Du = \nabla u\, \mathcal{L}^N + D^c u + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{N-1} \lfloor J_u\,,$$

where ∇u is the Radon-Nikodým derivative of Du with respect to the Lebesgue measure and $D^c u$ is the *Cantor part* of Du. For the sake of simplicity, we denote by $D^s u = D^c u + (u^+ - u^-) \otimes \nu_u\, \mathcal{H}^{N-1} \lfloor J_u$.

We recall that the space $SBV(\Omega; \mathbb{R}^M)$ of special functions of bounded variation is defined as the set of all $u \in \mathrm{BV}(\Omega; \mathbb{R}^M)$ such that $D^s u$ is concentrated on S_u; i.e., $|D^s u|(\Omega \setminus S_u) = 0$.
Let $p > 1$. The space $\mathrm{SBV^p}(\Omega; \mathbb{R}^M)$ is defined as the set of functions $u \in SBV(\Omega; \mathbb{R}^M)$ with $\nabla u \in L^p(\Omega; \mathbb{M}^{M\times N})$ and $\mathcal{H}^{N-1}(S_u) < \infty$. We will say that a sequence $\{u_n\}$ converges to u weakly in $\mathrm{SBV^p}(\Omega; \mathbb{R}^M)$ if $\{u_n\}, u \in \mathrm{SBV^p}(\Omega; \mathbb{R}^M) \cap L^\infty(\Omega; \mathbb{R}^M)$, $u_n(x) \to u(x)$ almost everywhere in Ω, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; \mathbb{M}^{M\times N})$, and $\|u_n\|_\infty$ and $\mathcal{H}^{N-1}(S_{u_n})$ are bounded uniformly with respect to n.
We define $GSBV(\Omega; \mathbb{R}^M)$ the space of *generalized special functions of bounded variation* as the set of all functions $u : \Omega \to \mathbb{R}^M$ such that $\phi(u) \in SBV(\Omega; \mathbb{R}^M)$ for every $\phi \in \mathcal{C}^1(\mathbb{R}^M; \mathbb{R}^M)$ with *supp*$(\nabla\phi) \subset\subset \mathbb{R}^M$.
We define $\mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$ as the set of functions $u \in GSBV(\Omega; \mathbb{R}^M)$ such that $\nabla u \in L^p(\Omega; \mathbb{M}^{M\times N})$ and $\mathcal{H}^{N-1}(S_u) < \infty$. We will say that a sequence $\{u_n\}$ converges to u weakly in $\mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$ if $\{u_n\}, u \in \mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$, $u_n(x) \to u(x)$ almost everywhere in Ω, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; \mathbb{M}^{M\times N})$, and $\mathcal{H}^{N-1}(S_{u_n})$ is bounded uniformly with respect to n.

Finally, we recall now a classical compactness result due to Ambrosio (see [4], [7] and [8, Theorem 4.8]).

Theorem 2.1. *Let $\{u_n\}$ be a sequence in $\mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$ satisfying*

$$\sup_{n \in \mathbb{N}} \left[\|u_n\|_1 + \int_\Omega |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty\,. \tag{2.2}$$

Then there exists a subsequence $\{u_{n_k}\} \subseteq \mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$ weakly converging in $\mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$ to $u \in \mathrm{GSBV^p}(\Omega; \mathbb{R}^M)$, i.e. $u_{n_k}(x) \to u(x)$ for almost every $x \in \Omega$, $\nabla u_{n_k} \rightharpoonup \nabla u$ weakly in $L^P(\Omega; \mathbb{R}^{M\times N})$ and $\mathcal{H}^{N-1}(J_{u_{n_k}})$ is equibounded.

For a general survey on the spaces BV, SBV, $\mathrm{SBV^p}$, $GSBV$ and $\mathrm{GSBV^p}$ functions we refer, for instance, to [8].

2.3. **Capacity.** Given an open set $A \subset I\!\!R^N$, the 1-*capacity* of A is defined by setting

$$C_1(A) := \inf \left\{ \int_{I\!\!R^N} |D\varphi|\, dx \ : \ \varphi \in W^{1,1}(I\!\!R^N), \quad \varphi \geq 1 \quad \mathcal{L}^N\text{−a.e. on } A \right\}.$$

Then, the 1-capacity of an arbitrary set $B \subset I\!\!R^N$ is given by

$$C_1(B) := \inf\{C_1(A) \ : \ A \supseteq B,\ A \text{ open}\}.$$

It is well known that capacities and Hausdorff measure are closely related. In particular, we have that for every Borel set $B \subset I\!\!R^N$

$$C_1(B) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^{N-1}(B) = 0.$$

Definition 2.2. Let $B \subset I\!\!R^N$ be a Borel set with $C_1(B) < +\infty$. Given $\varepsilon > 0$, we call *capacitary ε-quasi-potential* (or simply *capacitary quasi-potential*) of B a function $\varphi_\varepsilon \in W^{1,1}(I\!\!R^N)$, such that $0 \leq \widetilde{\varphi}_\varepsilon \leq 1$ $\mathcal{H}^{N-1}$-a.e. in $I\!\!R^N$, $\widetilde{\varphi}_\varepsilon = 1$ $\mathcal{H}^{N-1}$-a.e. in B and

$$\int_{I\!\!R^N} |D\varphi_\varepsilon|\, dx \leq C_1(B) + \varepsilon.$$

We recall that a function $g : I\!\!R^N \to I\!\!R$ is said C_1-quasi continuous if for every $\varepsilon > 0$ there exists an open set A, with $C_1(A) < \varepsilon$, such that $g|_{A^c}$ is continuous on A^c; C_1-quasi lower semicontinuous and C_1-quasi upper semicontinuous functions are defined similarly.
It is well known that if g is a $W^{1,1}$-function, then its precise representative $\widetilde{g}$ is C_1-quasi continuous (see [19, Sections 9 and 10]). Moreover, to every BV-function g, it is possible to associate a C_1-quasi lower semicontinuous and a C_1-quasi upper semicontinuous representative, as stated by the following theorem (see [10], Theorem 2.5).

Theorem 2.3. *For every function $g \in \mathrm{BV}(\Omega)$, the approximate upper limit g^+ and the approximate lower limit g^- are C_1-quasi upper semicontinuous and C_1-quasi lower semicontinuous, respectively.*

In particular, if B is a Borel subset of $I\!\!R^N$ with finite perimeter, then χ_B^- is C_1-quasi lower semicontinuous and χ_B^+ is C_1-quasi upper semicontinuous.

2.4. **Jointly convex functions.**

Definition 2.4. Let $K \subset I\!\!R^M$ be a compact set and $\phi : K \times K \times I\!\!R^N \to [0, +\infty)$. We say that ϕ is *jointly convex* if there exists a sequence of functions $g_j \in \mathcal{C}(K; I\!\!R^N)$ such that

$$\phi(r,t,\xi) = \sup_{j \in I\!\!N} \langle g_j(r) - g_j(t), \xi \rangle \qquad \forall (r,t,\xi) \in K \times K \times I\!\!R^N. \tag{2.3}$$

Remark 2.5. We recall that a class of jointly convex functions ϕ can be obtained in the following way:

$$\phi(r,t,\nu) = \gamma(|r-t|)\varphi(\nu)$$

where γ is a lower semicontinuous, increasing and subadditive function and φ is convex and even (see Example 5.23 in [8]).

2.5. **Classical approximation results.** We recall an approximation theorem for convex functions due to De Giorgi (see [18]). This result states that any convex function $f : I\!R^\nu \to I\!R$ can be approximated by mean of a sequence of affine functions $a_j + \langle b_j, \xi\rangle$, where

$$a_j := \int_{I\!R^\nu} f(\xi)\big((\nu+1)\alpha_j(\xi) + \langle \nabla\alpha_j(\xi), \xi\rangle\big)d\xi \tag{2.4}$$

$$b_j := -\int_{I\!R^\nu} f(\xi)\nabla\alpha_j(\xi)d\xi, \tag{2.5}$$

with $\alpha_j \in C_0^1(I\!R^\nu)$, $j \in I\!N$, a nonnegative function such that $\int_{I\!R^\nu} \alpha_j(\xi)d\xi = 1$.

Lemma 2.6. *Let $f : I\!R^\nu \to I\!R$ be a convex function and a_j, b_j be defined as in (2.4) and (2.5). Then the following property holds:*

$$f(\xi) = \sup_{j\in I\!N}[a_j + \langle b_j, \xi\rangle] \qquad \forall \xi \in I\!R^\nu .$$

Remark 2.7. The main feature of this approximation is that the coefficients a_j and b_j explicitly depend on f. In particular, when f depends also on (x, s) the explicit formulas (2.4) and (2.5), which can be rewritten as

$$a_j(x,s) := \int_{I\!R^\nu} f(x,s,\xi)\big((\nu+1)\alpha_j(\xi) + \langle \nabla\alpha_j(\xi), \xi\rangle\big)d\xi \tag{2.6}$$

$$b_j(x,s) := -\int_{I\!R^\nu} f(x,s,\xi)\nabla\alpha_j(\xi)d\xi, \tag{2.7}$$

permit to deduce regularity properties of the coefficients, from proper hypotheses satisfied by f. More precisely, if f satisfies some continuity or Lipschitz continuity assumptions with respect to (x, s), then a_j and b_j inherit the same properties, too.

In the next two lemmas, using the same argument as in the proof of Theorem 1.1 in [**?**], we obtain the lower semicontinuity for functionals whose integrands is the supremum of convex functions.

Lemma 2.8. *Let $f, f_j : \Omega \times I\!R \times I\!R^N \to [0, +\infty)$, $j \in I\!N$, be Borel functions, convex and positively 1-homogeneous in the last variable and such that*

$$f(x,s,\xi) = \sup_{j\in I\!N} f_j(x,s,\xi) \qquad \textit{for all } (x,s,\xi) \in (\Omega\setminus N_0)\times I\!R\times I\!R^N ,$$

where $N_0 \subset \Omega$ is a Borel set with $\mathcal{H}^{N-1}(N_0) = 0$. If the functionals $\widetilde{\mathcal{F}}_{f_j}$ defined by

$$\widetilde{\mathcal{F}}_{f_j}(u,\Omega) := \int_{\Omega\cap J_u} \left(\int_{u^-(x)}^{u^+(x)} f_j(x,s,\nu_u)\,ds \right) d\mathcal{H}^{N-1}$$

are weakly lower semicontinuous in $\mathrm{SBV^p}(\Omega)$*, then $\widetilde{\mathcal{F}}_f$, defined in an analogous way, is weakly lower semicontinuous in* $\mathrm{SBV^p}(\Omega)$*, too.*

Lemma 2.9. *Let $k, f_j : \Omega \times I\!R^M \times I\!R^M \times I\!R^N \to [0, +\infty)$, $j \in I\!N$, be Borel functions, convex and positively 1-homogeneous in the last variable and such that*

$$k(x,r,t,\xi) = \sup_{j\in I\!N} f_j(x,r,t,\xi) \qquad \textit{for all } (x,r,t,\xi) \in (\Omega\setminus N_0)\times I\!R^M\times I\!R^M\times I\!R^N ,$$

where $N_0 \subset \Omega$ *is a Borel set with* $\mathcal{H}^{N-1}(N_0) = 0$*. If the functionals* $\mathcal{F}_{f_j}(\cdot, \Omega)$ *defined by*

$$\mathcal{F}_{f_j}(u) := \int_{\Omega \cap J_u} f_j(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{N-1}$$

are weakly lower semicontinuous in $\mathrm{SBV^p}(\Omega)$*, then* $\mathcal{F}_k$*, defined in an analogous way, is weakly lower semicontinuous in* $\mathrm{SBV^p}(\Omega)$*, too.*

2.6. **Chain rule formula.** We consider the two following chain rule formulas. The first one applies to scalar functions $u \in SBV(\Omega)$, while the second one is proved for $u \in SBV(\Omega; \mathbb{R}^M)$.

Proposition 2.10. *Let* $B : \Omega \times \mathbb{R} \to \mathbb{R}^N$ *be a locally bounded Borel function. Assume that*

$$\forall x \in \Omega \text{ the function } B(x, \cdot) \text{ is Lipschitz continuous on } \mathbb{R}\,; \tag{2.8}$$

$$\begin{cases} \forall s \in \mathbb{R} \text{ the function } B(\cdot, s) \text{ belongs to } W^{1,1}(\Omega; \mathbb{R}^N) \text{ and } \exists N_0 \subset \Omega \text{ with} \\ \mathcal{H}^{N-1}(N_0) = 0, \text{ such that } B(\cdot, s) \text{ is approximately continuous in } \Omega \setminus N_0 \ \forall s \in \mathbb{R}\,; \\ \forall K \subset\subset \mathbb{R} \text{ there exists } L = L(K) > 0 \text{ such that} \displaystyle\int_K \left(\int_\Omega |\nabla_x B(x, s)|\, dx \right) ds \le L\,. \end{cases} \tag{2.9}$$

Then for every $u \in SBV(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *and for every* $\psi \in C^1_0(\Omega)$ *we have*

$$\begin{aligned} &\int_\Omega \langle \nabla\psi, B(x, u) \rangle\, dx = \\ &-\int_\Omega \psi \mathrm{div}_x B(x, u)\ dx - \int_\Omega \psi \langle \nabla_s B(x, u), \nabla u \rangle\, dx - \int_{\Omega \cap J_u} \psi \langle B(x, u^+) - B(x, u^-), \nu_u \rangle\, d\mathcal{H}^{N-1}. \end{aligned} \tag{2.10}$$

Proof. See [17, Theorem 1.1 and Lemma 2.2]. □

Remark 2.11. We note that, fixed $u \in SBV(\Omega) \cap L^\infty(\Omega)$, the equality (2.10) holds also if we replace in (2.9) the whole space $\mathbb{R}$ with the bounded ball $B(0, C+1)$, where $C = \|u\|_{L^\infty(\Omega)}$.

Proposition 2.12. *Let* $M > 1$ *and* $B : \Omega \times \mathbb{R}^M \to \mathbb{R}^N$ *be a locally bounded Borel function. Assume that:*

$$\forall x \in \Omega \text{ the function } B(x, \cdot) \text{ belongs to } C^1(\mathbb{R}^M; \mathbb{R}^N)\,; \tag{2.11}$$

$$\begin{cases} \forall s \in \mathbb{R}^M \text{ the function } B(\cdot, s) \text{ belongs to } W^{1,1}(\Omega; \mathbb{R}^N) \text{ and } \exists N_0 \subset \Omega \text{ with} \\ \mathcal{H}^{N-1}(N_0) = 0, \text{ such that } B(\cdot, s) \text{ is approximately continuous in } \Omega \setminus N_0 \ \forall s \in \mathbb{R}^M\,; \\ \forall K \subset\subset \mathbb{R}^M \text{ there exists } L = L(K) > 0 \text{ such that} \displaystyle\int_K \left(\int_\Omega |\nabla_x B(x, s)|\, dx \right) ds \le L\,. \end{cases} \tag{2.12}$$

Then for every $u \in SBV(\Omega; \mathbb{R}^M) \cap L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^M)$ *and for every* $\psi \in C^1_0(\Omega)$ *we have*

(2.13)

$$\int_\Omega \langle \nabla\psi(x), B(x, u(x)) \rangle\, dx = -\int_\Omega \psi(x)\ \mathrm{div}_x B(x, u(x))\ dx$$

$$-\int_\Omega \psi(x)\,\mathrm{tr}\,[\nabla_s B(x,u(x))\cdot\nabla u(x)]\,dx - \int_{\Omega\cap J_u} \psi(x)\,\langle B(x,u^+(x)) - B(x,u^-(x)),\nu_u(x)\rangle\,d\mathcal{H}^{N-1}\,.$$

Proof. It is enough to repeat the same arguments of the scalar version, since the vectorial formula is nothing else that the sum of M terms of the same type as in the scalar case. □

3. Setting of the problem

For every $A \in \mathcal{A}(\Omega)$ and every $u \in \mathrm{GSBV^p}(\Omega;I\!R^M)$, $p > 1$, we set

$$(3.1)\qquad \begin{aligned} G(u,A) &= \int_A W(x,u,\nabla u)\,dx + \int_{A\cap J_u} h(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1}\,,\\ F(u,A) &= G(u,A) + \int_\Omega |u-U_0|^q\,dx\,, \end{aligned}$$

where $q \geq 1$, $W : \Omega\times I\!R^M \times I\!M^{M\times N} \to [0,+\infty)$ is a Carathéodory function, $k : \Omega\times I\!R^M \times I\!R^M \times I\!R^N \to [0,+\infty)$ is a Borel function and $U_0 : \Omega \to I\!R^M$ belongs to $L^\infty(\Omega;I\!R^M)$.
Our aim is to prove a lower semicontinuity theorem for this functional, along sequences $\{u_n\} \in \mathrm{GSBV^p}(\Omega;I\!R^M)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and $\|\nabla u_n\|_p$, $\mathcal{H}^{N-1}(J_{u_n})$ are uniformly bounded with respect to $n \in I\!N$.
In the following we assume that the function k satisfies

$$(3.2)\qquad h(x,r,t,\cdot)\quad \text{is convex in } I\!R^N\ \forall (x,r,t)\in\Omega\times I\!R^M\times I\!R^M\,;$$

$$(3.3)\qquad h(x,r,t,\cdot)\quad \text{is positively 1-homogeneous in } I\!R^N\ \forall (x,r,t)\in\Omega\times I\!R^M\times I\!R^M\,;$$

$$(3.4)\qquad h(x,r,t,\nu) = h(x,t,r,-\nu)\quad \forall (x,r,t,\nu)\in\Omega\times I\!R^M\times I\!R^M\times \mathbb{S}^{N-1}\,;$$

$$(3.5)\qquad h(x,\cdot,\cdot,\nu)\quad \text{is lower semicontinuous } \forall (x,\nu)\in\Omega\times \mathbb{S}^{N-1}\,.$$

As the lower semicontinuity of the last term in F is trivial, the result can be obtained proving, for every $A \in \mathcal{A}(\Omega)$, the lower semicontinuity of the two functionals

$$(3.6)\qquad (u,A)\mapsto \int_A W(x,u,\nabla u)\,dx \qquad \text{and} \qquad (u,A)\mapsto \int_{A\cap J_u} h(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1}$$

separately. Since, under suitable hypotheses, the lower semicontinuity of the first one is a well known fact in the literature, we focus our attention on the lower semicontinuity of the second one. To this purpose, we consider different cases and for each of them we add further conditions on h.

4. Semicontinuity in the scalar case

In this section, we treat the scalar case $M = 1$ and we assume that

$$(4.1)\qquad h_f(x,r,t,\nu) = \frac{1}{t-r}\int_r^t f(x,s,(t-r)\nu)\,ds\,,$$

where $f : \Omega\times I\!R\times I\!R^N \to [0,+\infty)$ is a locally bounded Borel function which satisfies

$$(4.2)\qquad f(x,s,\cdot)\quad \text{is convex and positively 1-homogeneous } \forall (x,s)\in\Omega\times I\!R\,;$$

$$(4.3)\qquad |f(x,s_1,\xi) - f(x,s_2,\xi)| \leq \Lambda|\xi|\omega(s_1-s_2)\quad \forall (x,s_1,\xi),(x,s_2,\xi)\in\Omega\times I\!R\times I\!R^N\,,$$

where $\Lambda > 0$ and $\omega : \mathbb{R} \to \mathbb{R}$ is a continuous function with $\omega(0) = 0$.

Proposition 4.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ be a locally bounded Borel function satisfying (4.2) and (4.3). Assume that $f(\cdot, s, \xi) \in W^{1,1}(\Omega)$ for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and that there exists a Borel set $N_0 \subset \Omega$, with $\mathcal{H}^{N-1}(N_0) = 0$, such that $f(\cdot, s, \xi)$ is approximately continuous in $\Omega \setminus N_0$, for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Finally, we assume that*

$$\tag{4.4} \begin{cases} \text{for every bounded set } K \subset\subset \mathbb{R} \times \mathbb{R}^N \,, \text{ there exists } L = L(K) > 0 \text{ such that} \\ \displaystyle\int_K \left(\int_\Omega |\nabla_x f(x, s, \xi)| \, dx \right) ds \, d\xi \le L \,. \end{cases}$$

Then, for every $\{u_n\} \subseteq \mathrm{SBV^p}(\Omega)$ and $u \in \mathrm{SBV^p}(\Omega)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and

$$\sup_{n \in \mathbb{N}} \left[\|u_n\|_\infty + \int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty \,,$$

we have

$$\tag{4.5} \int_{\Omega \cap J_u} h_f(x, u^-, u^+, \nu_u) \, d\mathcal{H}^{N-1} \le \liminf_{n \to +\infty} \int_{\Omega \cap J_{u_n}} h_f(x, u_n^-, u_n^+, \nu_{u_n}) \, d\mathcal{H}^{N-1} \,.$$

Proof. By Lemma 2.6 and Remark 2.7, for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and for every $(s, \nu) \in \mathbb{R} \times \mathbb{S}^{N-1}$, we have that

$$\tag{4.6} f(x, s, \nu) = \sup_{j \in \mathbb{N}} [\langle b_j(x, s), \nu \rangle]^+ = \sup_{j \in \mathbb{N}} \sup \left\{ \langle b_j(x, s), \nu \rangle \, \psi(x) \; : \; \psi \in \mathcal{C}_0^1(\Omega), \; 0 \le \psi \le 1 \right\} \,.$$

Hence, by Lemma 2.8, it is enough to prove the continuity of the functional

$$\tag{4.7} u \mapsto \int_{\Omega \cap J_u} \psi(x) \left[\int_{u^-(x)}^{u^+(x)} \langle b(x, s), \nu_u \rangle \, ds \right] d\mathcal{H}^{N-1} \,,$$

where, for the sake of simplicity, we omit the subscript index j. In order to achieve this goal, we note that

$$\int_{\Omega \cap J_u} \psi(x) \Big\langle \int_{u^-(x)}^{u^+(x)} b(x, s) \, ds, \, \nu_u(x) \Big\rangle \, d\mathcal{H}^{N-1} = \int_{\Omega \cap J_u} \psi(x) \, \langle B(x, u^+(x)) - B(x, u^-(x)), \nu_u(x) \rangle \, d\mathcal{H}^{N-1} \,,$$

where we define $B : \Omega \times \mathbb{R} \to \mathbb{R}^N$ as $B(x, s) = \int_0^s b(x, t) \, dt$. We remark that, by our assumptions, the function B satisfies all the hypotheses of Proposition 2.10; hence we obtain

$$\int_{\Omega \cap J_u} \psi(x) \Big\langle \int_{u^-(x)}^{u^+(x)} b(x, s) \, ds, \, \nu_u(x) \Big\rangle \, d\mathcal{H}^{N-1} = - \int_\Omega \langle \nabla \psi(x), B(x, u(x)) \rangle \, dx$$

$$- \int_\Omega \psi(x) \, \mathrm{div}_x B(x, u(x)) \, dx - \int_\Omega \psi(x) \, \langle \nabla_s B(x, u(x)), \nabla u(x) \rangle \, dx \,.$$

Now, it is sufficient to prove the continuity of each integral in the second term of the previous equality; i.e.,

$$\int_\Omega \langle \nabla\psi(x), B(x,u(x))\rangle \, dx = \lim_{n\to+\infty} \int_\Omega \langle \nabla\psi(x), B(x,u_n(x))\rangle \, dx \,; \tag{4.8}$$

$$\int_\Omega \psi(x)\, \mathrm{div}_x B(x,u(x)) \, dx = \lim_{n\to+\infty} \int_\Omega \psi(x)\, \mathrm{div}_x B(x,u_n(x)) \, dx \,; \tag{4.9}$$

$$\int_\Omega \psi(x)\, \langle \nabla_s B(x,u(x)), \nabla u(x)\rangle \, dx = \lim_{n\to+\infty} \int_\Omega \psi(x)\, \langle \nabla_s B(x,u_n(x)), \nabla u_n(x)\rangle \, dx \,. \tag{4.10}$$

Since $B(x,\cdot)$ and $\mathrm{div}_x B(x,\cdot)$ are Lipschitz continuous functions, $\{u_n\}$ converges pointwise almost everywhere to u and it is equibounded in $L^\infty(\Omega)$, then equalities (4.8) and (4.9) hold. In order to prove equality (4.10), we observe that $\nabla_s B(x,u_n) \to \nabla_s B(x,u)$ strongly in $L^{p'}(\Omega; I\!R^N)$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; I\!R^N)$. This concludes the proof. □

Remark 4.2. Repeating the previous proof and taking into account Remark 2.11, it is not difficult to check that, if we replace in all the assumptions of Proposition 4.1 the whole space $I\!R$ with $B(0, C+1)$, the inequality

$$\int_{\Omega\cap J_u} h_f(x,u^-,u^+,\nu_u)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} h_f(x,u_n^-,u_n^+,\nu_{u_n})\, d\mathcal{H}^{N-1} \tag{4.11}$$

holds for every $\{u_n\} \subseteq \mathrm{SBV^p}(\Omega)$ and $u \in \mathrm{SBV^p}(\Omega)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and

$$\sup_{n\in I\!N} \|u_n\|_{L^\infty(\Omega)} \le C \qquad \text{and} \qquad \sup_{n\in I\!N} \left[\int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty \,.$$

Theorem 4.3. *Let $f : \Omega \times I\!R \times I\!R^N \to [0,+\infty)$ be a locally bounded Borel function satisfying (4.2) and (4.3). Assume that*

$$f(\cdot,s,\xi) \quad \text{is } C_1\text{-quasi lower semicontinuous for every } (s,\xi) \in I\!R \times I\!R^N \,; \tag{4.12}$$

$$f(x,s,\xi) > 0 \quad \text{for all } (x,s,\xi) \in (\Omega\setminus N_0) \times I\!R \times (I\!R^N \setminus \{0\}), \text{ where } \mathcal{H}^{N-1}(N_0) = 0. \tag{4.13}$$

Then, for every $\{u_n\} \subseteq \mathrm{GSBV^p}(\Omega)$ and $u \in \mathrm{GSBV^p}(\Omega)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and

$$\sup_{n\in I\!N} \left[\int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty \,, \tag{4.14}$$

we have

$$\int_{\Omega\cap J_u} h_f(x,u^-,u^+,\nu_u)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} h_f(x,u_n^-,u_n^+,\nu_{u_n})\, d\mathcal{H}^{N-1} \,. \tag{4.15}$$

Proof. We follow some ideas as in the proof of Theorem 3.6 in [3]. Notice that since f is locally bounded in $\Omega \times I\!R \times I\!R^N$ and positively 1-homogeneous with respect to ξ, for any open set $\Omega' \times \mathcal{O} \subset\subset \Omega \times I\!R$, there exists a constant $\Lambda' = \Lambda'(\Omega', \mathcal{O})$ such that

$$0 \le f(x,s,\xi) \le \Lambda' |\xi| \qquad \text{for all } (x,s,\xi) \in \Omega' \times \mathcal{O} \times I\!R^N \,. \tag{4.16}$$

Condition (4.16), together with the convexity of f with respect to ξ immediately yields that

$$|f(x,s,\xi_1)-f(x,s,\xi_2)|\le c\Lambda'|\xi_1-\xi_2| \qquad \text{for all } (x,s,\xi_1),(x,s,\xi_2)\in\Omega'\times\mathcal{O}\times I\!R^N\,, \tag{4.17}$$

for some constant $c>0$. Let us now fix h and two dense sequences $\{s_i\}\subseteq\mathcal{O}$ and $\{\xi_j\}\subseteq I\!R^N$. For all i,j there exists an open set $A_{i,j,h}\subset\Omega$, $A_{i,j,h}\supset N_0$, with $C_1(A_{i,j,h})<1/(h2^{i+j})$, such that $f(\cdot,s_i,\xi_j)$ is lower semicontinuous in $\Omega\setminus A_{i,j,h}$. Setting $A_h=\cup_{i,j}A_{i,j,h}$ we obtain that A_h is open, $C_1(A_h)\le 1/h$ and we may assume that $\{A_h\}$ is a decreasing sequence. Making use of (4.3) and (4.17), one easily gets that f is lower semicontinuous in $(\Omega'\setminus A_h)\times\mathcal{O}\times I\!R^N$.
We claim that, given h and $(x_0,s_0)\in(\Omega'\setminus A_h)\times\mathcal{O}$, for all $\varepsilon>0$ there exists $\delta>0$ such that

$$f(x_0,s_0,\xi)\le(1+\varepsilon)f(x,s,\xi) \tag{4.18}$$

for all $(x,s,\xi)\in(\Omega'\setminus A_h)\times\mathcal{O}\times I\!R^N$ such that $|x-x_0|+|s-s_0|<\delta$.
To prove this, we argue by contradiction, assuming that there exist $(x_0,s_0)\in(\Omega'\setminus A_h)\times\mathcal{O}$ and $\varepsilon_0>0$ such that there exist three sequences $\{x_k\}\subset\Omega'\setminus A_h$, $\{s_k\}\subset\mathcal{O}$, with $|x_k-x_0|+|s_k-s_0|<1/k$, and $\{\xi_k\}\subset I\!R^N$ such that

$$f(x_0,s_0,\xi_k)>(1+\varepsilon_0)f(x_k,s_k,\xi_k)\,. \tag{4.19}$$

Clearly, by the positive 1-homogeneity of $f(x,s,\cdot)$, we may assume that $|\xi_k|=1$, for every $k\in I\!N$; hence, up to a subsequence, there exists $\xi_0\in\mathbb{S}^{N-1}$ such that $\xi_k\to\xi_0$. Then, passing to the limit when $k\to+\infty$ in (4.19) and using the lower semicontinuity of f and the continuity of $f(x_0,s_0,\cdot)$, we get that

$$f(x_0,s_0,\xi_0)=\lim_{k\to+\infty}f(x_0,s_0,\xi_k)\ge(1+\varepsilon_0)\liminf_{k\to+\infty}f(x_k,s_k,\xi_k)\ge(1+\varepsilon_0)f(x_0,s_0,\xi_0)\,.$$

Hence, $f(x_0,s_0,\xi_0)=0$, which is a contradiction since $x_0\in\Omega\setminus N_0$. This proves the claim; i.e., (4.18) holds.
Then, by Lemma 3.3. of [3], there exist $\{a_j^h\}\subset C_0^\infty(I\!R^N\times I\!R)$ and $\{\psi_j^h\}\subset C^\infty(I\!R^N)$ such that, for all $j\in I\!N$, $0\le a_j^h\le 1$, ψ_j^h is a convex and even function satisfying

$$f(x,s,\xi)=\sup_{j\in I\!N}a_j^h(x,s)\psi_j^h(\xi) \qquad \text{for all } (x,s,\xi)\in(\Omega'\setminus A_h)\times\mathcal{O}\times I\!R^N\,. \tag{4.20}$$

Moreover, since $f(x,s,\cdot)$ is positively 1-homogeneous, it is possible to assume that ψ_j^h is positively 1-homogeneous, too. Indeed,

$$f(x,s,\xi)=\sup_{t>0}\frac{f(x,s,t\xi)}{t}=\sup_{t>0}\sup_{j\in I\!N}a_j^h(x,s)\frac{\psi_j^h(t\xi)}{t}=\sup_{j\in I\!N}a_j^h(x,s)\sup_{t>0}\frac{\psi_j^h(t\xi)}{t}$$

and $\xi\mapsto\frac{\psi_j^h(t\xi)}{t}$ is positively 1-homogeneous. Therefore, by (4.16), we have also

$$0\le\psi_j^h(\xi)\le\Lambda'|\xi| \qquad \text{for all } \xi\in I\!R^N\,. \tag{4.21}$$

Now we follow the proof of Theorem 3.4 in [3]. Let $\varphi_h\in W^{1,1}(I\!R^N)$ be a capacitary quasi-potential of A_h. More precisely, let us assume that there exists a Borel set $N_h\subset I\!R^N$, with $C_1(N_h)=\mathcal{H}^{N-1}(N_h)=0$, such that $0\le\widetilde{\varphi}_h(x)\le 1$ for every $x\in I\!R^N\setminus N_h$, $\widetilde{\varphi}_h=1$ on $A_h\setminus N_h$ and

$$\int_{I\!R^N}|\nabla\widetilde{\varphi}_h|\,dx\le C_1(A_h)+\frac{1}{h}<\frac{2}{h}\,.$$

Set, for all $j \in \mathbb{N}$, $\widetilde{\alpha}_j^h(x,s) = \max\{a_j^h(x,s) - \widetilde{\varphi}_h(x), 0\}$ for all $(x,s) \in (\Omega' \setminus N_h) \times \mathcal{O}$, $\widetilde{\alpha}_j^h(x,s) = 0$, otherwise. We have that, since $\widetilde{\varphi}_h(x) \geq 0$,

$$(4.22)\quad 0 \leq \widetilde{\alpha}_j^h(x,s) \leq 1, \quad a_j^h(x,s) \geq \widetilde{\alpha}_j^h(x,s) \geq a_j^h(x) - \widetilde{\varphi}_h(x) \quad \text{for all } (x,s) \in (\Omega' \setminus N_h) \times \mathcal{O}\,.$$

Moreover, setting $N_0 = \cup_h N_h$, $C_1(N_0) = \mathcal{H}^{N-1}(N_0) = 0$ and, for every $h, j \in \mathbb{N}$, we have that

$$f(x,s,\xi) \geq \widetilde{\alpha}_j^h(x,s)\psi_j^h(\xi) \qquad \text{for all } (x,s,\xi) \in (\Omega' \setminus N_0) \times \mathcal{O} \times \mathbb{R}^N\,. \tag{4.23}$$

In fact, if $x \in (\Omega' \setminus A_h) \setminus N_0$, (4.23) follows from (4.20) and (4.22), while, if $x \in A_h \setminus N_0$, inequality (4.23) holds since $\widetilde{\varphi}_h(x) = 1$, hence $\widetilde{\alpha}_j^h(x,s) = 0$. Finally, we set for all $h, j \in \mathbb{N}$

$$g_j^h(x,s,\xi) = \widetilde{\alpha}_j^h(x,s)\psi_j^h(\xi), \qquad g_h(x,s,\xi) = \sup_{j \in \mathbb{N}} g_j^h(x,s,\xi), \qquad f_h(x,s,\xi) = \sup_{j \in \mathbb{N}} a_j^h(x,s)\psi_j^h(\xi)$$

for all $(x,s,\xi) \in \Omega' \times \mathcal{O} \times \mathbb{R}^N$.
Now, we firstly assume that $u_n, u \in \mathrm{SBV^p}(\Omega)$ with $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and

$$\sup_{n \in \mathbb{N}} \left[\|u_n\|_\infty + \int_\Omega |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty\,. \tag{4.24}$$

In particular, by (4.24) there exists $C > 0$ such that $\sup \|u_n\|_\infty \leq C$ and $\|u\|_\infty \leq C$. We choose $\mathcal{O} = B(0, C+1) \subset \mathbb{R}$, in the previous construction. Notice that each function g_j^h satisfies the assumptions of Proposition 4.1 with $\mathbb{R}$ replaced by $\mathcal{O}$ (see Remark 4.2). Therefore, we have

$$\mathcal{F}_{g_j^h}(u,\Omega') := \int_{\Omega' \cap J_u} \left(\int_{u^-(x)}^{u^+(x)} g_j^h(x,s,\nu_u)\, ds \right) d\mathcal{H}^{N-1} \leq \liminf_{n \to +\infty} \int_{\Omega' \cap J_{u_n}} \left(\int_{u_n^-(x)}^{u_n^+(x)} g_j^h(x,s,\nu_{u_n})\, ds \right) d\mathcal{H}^{N-1}\,;$$

hence by Lemma 2.8 the same is true for the functionals $\mathcal{F}_{g_h}(\cdot,\Omega')$, defined, for any $h \in \mathbb{N}$, in an analogous way as before.
To prove (4.15), we fix $h \in \mathbb{N}$ and set

$$\psi_h(\xi) = \sup_{j \in \mathbb{N}} \psi_j^h(\xi) \qquad \text{for all } \xi \in \mathbb{R}^N\,.$$

From (4.23), (4.22) and (4.20), we then get that

$$\liminf_{n \to +\infty} \int_{\Omega' \cap J_{u_n}} h_f(x, u_n^-, u_n^+, \nu_{u_n})\, d\mathcal{H}^{N-1} = \liminf_{n \to +\infty} \int_{\Omega' \cap J_{u_n}} \left(\int_{u_n^-(x)}^{u_n^+(x)} f(x,s,\nu_{u_n})\, ds \right) d\mathcal{H}^{N-1}$$

$$\geq \liminf_{n \to +\infty} \mathcal{F}_{g_h}(u_n,\Omega') \geq \mathcal{F}_{g_h}(u,\Omega') \geq \mathcal{F}_{f_h}(u,\Omega') - \int_{\Omega' \cap J_u} \big(u^+(x) - u^-(x)\big)\widetilde{\varphi}_h(x)\psi_h(\nu_u)\, d\mathcal{H}^{N-1}$$

$$\geq \int_{(\Omega' \setminus A_h) \cap J_u} \left(\int_{u_n^-(x)}^{u_n^+(x)} f(x,s,\nu_u)\, ds \right) d\mathcal{H}^{N-1} - \int_{\Omega' \cap J_u} \big(u^+(x) - u^-(x)\big)\widetilde{\varphi}_h(x)\psi_h(\nu_u)\, d\mathcal{H}^{N-1}\,.$$

Thus, recalling (4.21), we obtain

$$
\begin{aligned}
&\liminf_{n\to+\infty} \int_{\Omega'\cap J_{u_n}} h_f(x,u_n^-,u_n^+,\nu_{u_n})\,d\mathcal{H}^{N-1} \\
&\geq \int_{(\Omega'\setminus A_h)\cap J_u} h_f(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} - \Lambda' \int_{\Omega'\cap J_u} \big(u^+(x)-u^-(x)\big)\widetilde{\varphi}_h(x)\,d\mathcal{H}^{N-1}\,.
\end{aligned}
\tag{4.25}
$$

Since $\widetilde{\varphi}_h \to 0$ strongly in $W^{1,1}(\mathbb{R}^N)$ as $h\to\infty$, we have that, up to a subsequence, $\widetilde{\varphi}_h(x)\to 0$ for $\mathcal{H}^{N-1}$-almost every $x\in\mathbb{R}^N$ (see Proposition 1.2 in [11]). Therefore, letting $h\to+\infty$ in (4.25) and recalling that $A_{h+1}\subset A_h$ for all h and that $\mathcal{H}^{N-1}(\cap_h A_h)=0$, from the Dominated Convergence Theorem we get (4.15) in Ω'. Hence, letting $\Omega'\nearrow\Omega$, the thesis is achieved for $u_n, u\in \mathrm{SBV^p}(\Omega)$ satisfying (4.24).
Finally, by a standard truncation argument, we obtain that inequality (4.15) holds in $\mathrm{GSBV^p}$.
□

Remark 4.4. We note that, as a consequence of previous theorem, the assert in Proposition 4.1 holds also if we omit condition (4.4), up to assume (4.13).

Theorem 4.5. *Let $W:\Omega\times\mathbb{R}\times\mathbb{R}^N\to[0,+\infty)$ be a Carathéodory function such that $W(x,s,\cdot)$ is a convex function for every $(x,s)\in\Omega\times\mathbb{R}$, satisfying*

$$
|W(x,s_1,\xi)-W(x,s_2,\xi)|\leq \Lambda(1+|\xi|^p)\omega(s_1-s_2)\qquad \forall (x,s_1,\xi),(x,s_2,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^N\,,
$$

where $\Lambda>0$ and $\omega:\mathbb{R}\to\mathbb{R}$ is a continuous function with $\omega(0)=0$. Let $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to[0,+\infty)$ be a locally bounded Borel function satisfying (4.2), (4.3), (4.12) and (4.13), and $h_f:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\to[0,+\infty)$ be the function defined in (4.1). Let $\{u_n\}\subseteq\mathrm{GSBV^p}(\Omega)$ and $u\in\mathrm{GSBV^p}(\Omega)$ such that $u_n(x)\to u(x)$ for almost every $x\in\Omega$ and

$$
\sup_{n\in\mathbb{N}}\left[\int_\Omega |\nabla u_n|^p\,dx + \mathcal{H}^{N-1}(J_{u_n})\right] < +\infty\,. \tag{4.26}
$$

Then we have

$$
\begin{aligned}
&\int_\Omega W(x,u,\nabla u)\,dx + \int_{\Omega\cap J_u} h_f(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} \leq \\
&\leq \liminf_{n\to+\infty}\left[\int_\Omega W(x,u_n,\nabla u_n)\,dx + \int_{\Omega\cap J_{u_n}} h_f(x,u_n^-,u_n^+,\nu_{u_n})\,d\mathcal{H}^{N-1}\right].
\end{aligned}
$$

Proof. It is enough to prove that

$$
\int_\Omega W(x,u,\nabla u)\,dx \leq \liminf_{n\to+\infty}\int_\Omega W(x,u_n,\nabla u_n)\,dx\,, \tag{4.27}
$$

$$
\int_{\Omega\cap J_u} h_f(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{N-1} \leq \liminf_{n\to+\infty}\int_{\Omega\cap J_{u_n}} h_f(x,u_n^-,u_n^+,\nu_{u_n})\,d\mathcal{H}^{N-1}\,. \tag{4.28}
$$

Inequality (4.27) follows by [8, Theorem 5.8] (see also [24]), while inequality (4.28) follows by Theorem 4.3. □

5. Semicontinuity in the vectorial case

In the following we will consider the vectorial case $M > 1$.

Proposition 5.1. *Let $a : \Omega \to [0, +\infty)$ be a locally bounded function belonging to $W^{1,1}(\Omega)$ and coinciding with its precise representative and $\phi : \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{S}^{N-1} \to [0, +\infty)$ be a jointly convex function. Then, for every $\{u_n\} \subseteq \mathrm{SBV}^p(\Omega; \mathbb{R}^M)$ and $u \in \mathrm{SBV}^p(\Omega; \mathbb{R}^M)$ such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and*

$$\sup_{n\in\mathbb{N}} \left[\|u_n\|_\infty + \int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty \,,$$

we have

$$\int_{\Omega\cap J_u} a(x)\phi(u^-, u^+, \nu_u) \, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} a(x)\phi(u_n^-, u_n^+, \nu_{u_n}) \, d\mathcal{H}^{N-1} \,. \tag{5.1}$$

Proof. We follow the outline of the proof of Theorem 5.22 in [8], where however we need the chain rule formula proved in Proposition 2.12, which permits to have an explicit dependence on the spatial variable x.
Let

$$C := \sup_{n\in\mathbb{N}} \left[\|u_n\|_\infty + \int_\Omega |\nabla u_n|^p \, dx + \mathcal{H}^{N-1}(J_{u_n}) \right]$$

and $\overline{B}(0, C) \subset \mathbb{R}^M$ be the closed ball of radius C, centered at the origin. By the definition of jointly convex functions and taking into account that ϕ is nonnegative, there exists a sequence of functions $g_j \in \mathcal{C}(\overline{B}(0, C); \mathbb{R}^N)$ (which then can be extended to functions in $\mathcal{C}_0(\mathbb{R}^M; \mathbb{R}^N)$) such that

$$\phi(r, t, \nu) = \sup_{j\in\mathbb{N}} \left[\langle g_j(r) - g_j(t), \nu\rangle\right]^+ = \sup_{j\in\mathbb{N}} \sup\left\{\langle g_j(u^+) - g_j(u^-), \nu_u\rangle \, \psi : \psi \in C_0^1(\Omega) \,, 0 \le \psi \le 1\right\} .$$

By Lemma 2.9, it is enough to prove the lower semicontinuity for functionals of the type

$$u \to \int_{\Omega\cap J_u} \left[\langle a(x)\big(g(u^+) - g(u^-)\big), \nu_u\rangle\right] \psi(x) \, d\mathcal{H}^{N-1} .$$

It is not restrictive to assume that $g \in \mathcal{C}_0^\infty(\mathbb{R}^M; \mathbb{R}^N)$, since the general case can be obtained by standard approximations as in [8, proof of Theorem 5.22]. By using the chain rule formula (2.13) with $B(x, s) = a(x)g(s)$, we have

$$\int_{\Omega\cap J_u} a(x)\langle g(u^+) - g(u^-), \nu_u\rangle \, \psi \, d\mathcal{H}^{N-1} = -\int_\Omega a(x)\langle \nabla\psi(x), g(u(x))\rangle \, dx$$

$$- \int_\Omega \psi(x) \, \langle \nabla a(x), g(u(x))\rangle \, dx - \int_\Omega \psi(x) \, a(x) \, \mathrm{tr} \left[\nabla \mathrm{g}(\mathrm{u}(\mathrm{x})) \cdot \nabla \mathrm{u}(\mathrm{x})\right] dx \,.$$

Now, it is sufficient to prove the continuity of each integral in the second term of the previous equality; i.e.,

$$\int_\Omega a(x)\langle \nabla\psi(x), g(u(x))\rangle \, dx = \lim_{n\to+\infty} \int_\Omega a(x)\langle \nabla\psi(x), g(u_n(x))\rangle \, dx \,; \tag{5.2}$$

$$\int_\Omega \psi(x)\,\langle \nabla a(x), g(u(x))\rangle\, dx = \lim_{n\to+\infty} \int_\Omega \psi(x)\,\langle \nabla a(x), g(u_n(x))\rangle\, dx\,; \tag{5.3}$$

$$\int_\Omega \psi(x)\, a(x)\, \mathrm{tr}\,[\nabla \mathrm{g}(\mathrm{u}(\mathrm{x}))\cdot \nabla \mathrm{u}(\mathrm{x})]\, dx = \lim_{n\to+\infty} \int_\Omega \psi(x)\, a(x)\, \mathrm{tr}\,[\nabla \mathrm{g}(\mathrm{u_n}(\mathrm{x}))\cdot \nabla \mathrm{u_n}(\mathrm{x})]\, dx\,. \tag{5.4}$$

Since g is continuous, $\{u_n\}$ converges pointwise almost everywhere to u and it is equibounded in $L^\infty(\Omega)$, taking into account that $a\nabla\psi$ and $\psi\nabla a$ belong to $L^1(\Omega;I\!R^N)$, equalities (5.2) and (5.3) hold. In order to prove equality (5.4), we observe that $a\,\psi \in L^\infty(\Omega)$, $\nabla g(u_n) \to \nabla g(u)$ strongly in $L^{p'}(\Omega; I\!M^{N\times M})$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^p(\Omega; I\!M^{M\times N})$. This concludes the proof. □

Proposition 5.2. *Let $a:\Omega\to[0,+\infty)$ be a locally bounded function belonging to $W^{1,1}(\Omega)$ and coinciding with its precise representative, $\gamma:[0,+\infty)\to[0,+\infty)$ be a lower semicontinuous, increasing and subadditive function such that $\gamma(0)=0$, and $\varphi: I\!R^N\to[0,+\infty)$ be a convex, even and positively 1-homogeneous function. Then, for every $\{u_n\}\subseteq \mathrm{GSBV^p}(\Omega;I\!R^M)$ and $u\in \mathrm{GSBV^p}(\Omega;I\!R^M)$ such that $u_n(x)\to u(x)$ for almost every $x\in\Omega$ and*

$$\sup_{n\in I\!N}\left[\int_\Omega |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n})\right] < +\infty\,, \tag{5.5}$$

we have

$$\int_{\Omega\cap J_u} a(x)\gamma(|u^+-u^-|)\varphi(\nu_u)\, d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} a(x)\gamma(|u_n^+-u_n^-|)\varphi(\nu_{u_n})\, d\mathcal{H}^{N-1}\,. \tag{5.6}$$

Proof. Firstly we assume that

$$\sup_{n\in I\!N}\left[\|u_n\|_\infty + \int_\Omega |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n})\right] < +\infty\,.$$

By Remark 2.5, the function $\phi(r,t,\nu)=\gamma(|r-t|)\varphi(\nu)$ is jointly convex; moreover, $\{u_n\}, u \in \mathrm{GSBV^p}(\Omega;I\!R^M)\cap L^\infty(\Omega;I\!R^M)\subset \mathrm{SBV^p}(\Omega;I\!R^M)$; hence, by Proposition 5.1 the thesis follows. The general case $u_n, u\in \mathrm{GSBV^p}(\Omega;I\!R^M)$ satisfying (5.5) can be obtained as in the proof of Theorem 3.7 in [5]. □

We extend now that last result to a more general case, pointed out that this is the most significative case in the framework of quasi-static models in mechanics of fractures (see [**?**]).

Theorem 5.3. *Let $k:\Omega\times I\!R^N\to[0,+\infty)$ be a locally bounded Borel function satisfying*

$$k(\cdot,\xi) \quad \text{is } C_1\text{-quasi lower semicontinuous for every } \xi\in I\!R^N\,; \tag{5.7}$$

$$k(x,\cdot) \quad \text{is convex and positively 1-homogeneous in } I\!R^N\ \forall x\in\Omega\,; \tag{5.8}$$

$$k(x,\xi)=k(x,-\xi) \quad \forall (x,\xi)\in\Omega\times I\!R^N\,; \tag{5.9}$$

$$k(x,\xi)>0 \quad \text{for all } (x,\xi)\in(\Omega\setminus N_0)\times(I\!R^N\setminus\{0\}),\ \text{where } \mathcal{H}^{N-1}(N_0)=0. \tag{5.10}$$

Let $\gamma:[0,+\infty)\to[0,+\infty)$ be a locally bounded, lower semicontinuous, increasing and subadditive function such that $\gamma(0)=0$. Then, for every $\{u_n\}\subseteq \mathrm{GSBV^p}(\Omega;I\!R^M)$ and $u\in\mathrm{GSBV^p}(\Omega;I\!R^M)$

such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and

$$\sup_{n\in I\!N}\left[\int_\Omega |\nabla u_n|^p\,dx + \mathcal{H}^{N-1}(J_{u_n})\right] < +\infty\,,$$

we have

$$\int_{\Omega\cap J_u} \gamma(|u^+ - u^-|)k(x,\nu_u)\,d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} \gamma(|u_n^+ - u_n^-|)k(x,\nu_{u_n})\,d\mathcal{H}^{N-1}\,. \tag{5.11}$$

Proof. As in the proof of Theorem 4.3, we may assume that, for every $\Omega' \subset\subset \Omega$, there exists $\Lambda' > 0$ such that

$$0 \le k(x,\xi) \le \Lambda'|\xi| \qquad \forall (x,\xi) \in \Omega' \times I\!R^N\,,$$

and that, for every $h \in I\!N$, there exists an open set $A_h \subset \Omega'$, independent of ξ, with $\mathcal{C}_1(A_h) \le 1/h$, such that $k(\cdot,\xi)|_{\Omega'\setminus A_h}$ is lower semicontinuous on $\Omega' \setminus A_h$ and $A_{h+1} \subseteq A_h$. Moreover, fixed $h \in I\!N$, with the same construction made there, we obtain that, for every $j \in I\!N$, there exist $\{a_j^h\} \subset C_0^\infty(I\!R^N)$ and $\{\psi_j^h\} \subset C^\infty(I\!R^N)$ such that $0 \le a_j^h \le 1$ and ψ_j^h is a convex, even and positively 1-homogeneous function satisfying

$$k(x,\xi) = \sup_{j\in I\!N} a_j^h(x)\psi_j^h(\xi) \qquad \text{for all } (x,\xi) \in (\Omega' \setminus A_h) \times I\!R^N\,, \tag{5.12}$$

$$0 \le \psi_j^h(\xi) \le \Lambda'|\xi| \qquad \text{for all } \xi \in I\!R^N\,. \tag{5.13}$$

On the other hand, there exists a set $N_0 \subset \Omega'$, with $C_1(N_0) = \mathcal{H}^{N-1}(N_0) = 0$, such that setting

$$\widetilde{\alpha}_j^h(x) = \begin{cases} \max\{a_j^h(x) - \widetilde{\varphi}_h(x), 0\} & \forall x \in \Omega' \setminus N_0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\widetilde{\varphi}_h$ is the capacitary quasi-potential of A_h, we have

$$0 \le \widetilde{\alpha}_j^h(x) \le 1, \quad a_j^h(x) \ge \widetilde{\alpha}_j^h(x) \ge a_j^h(x) - \widetilde{\varphi}_h(x) \quad \text{for all } x \in \Omega' \setminus N_0 \tag{5.14}$$

and

$$k(x,\xi) \ge \widetilde{\alpha}_j^h(x)\psi_j^h(\xi) \qquad \text{for all } (x,\xi) \in (\Omega' \setminus N_0) \times I\!R^N\,. \tag{5.15}$$

Finally, we set for all $h, j \in I\!N$

$$g_j^h(x,\xi) = \widetilde{\alpha}_j^h(x)\psi_j^h(\xi), \qquad g^h(x,\xi) = \sup_{j\in I\!N} g_j^h(x,\xi), \qquad k_h(x,\xi) = \sup_{j\in I\!N} a_j^h(x)\psi_j^h(\xi)$$

for all $(x,\xi) \in \Omega' \times I\!R^N$. Notice that the functions $\widetilde{\alpha}_j^h, \gamma, \psi_j^h$ satisfy the assumptions of a, γ, φ in Proposition 5.2; therefore, the functionals $\mathcal{F}_j^h$ defined by

$$\mathcal{F}_j^h(u,\Omega') := \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\,g_j^h(x,\nu_u)\,d\mathcal{H}^{N-1}$$

satisfy the inequality (5.6), with Ω replaced by Ω'. Hence by Lemma 2.9 the same is true for the functionals $\mathcal{F}^h$, defined by

$$\mathcal{F}^h(u,\Omega') := \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\,g^h(x,\nu_u)\,d\mathcal{H}^{N-1}\,,$$

for any $h \in I\!N$.

To prove (5.11), we fix $h \in I\!N$ and set

$$\psi_h(\xi) = \sup_{j \in I\!N} \psi_j^h(\xi) \qquad \text{for all } \xi \in I\!R^N .$$

From (5.15), (5.14), (5.12) and (5.13), we get that

$$\begin{aligned} &\liminf_{n\to+\infty} \int_{\Omega'\cap J_{u_n}} \gamma(|u_n^+ - u_n^-|) k(x,\nu_{u_n})\, d\mathcal{H}^{N-1} \\ &\geq \int_{(\Omega'\setminus A_h)\cap J_u} \gamma(|u^+ - u^-|) k(x,\nu_u)\, d\mathcal{H}^{N-1} - \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\widetilde{\varphi}_h(x)\psi_h(\nu_u)\, d\mathcal{H}^{N-1} \\ &\geq \int_{(\Omega'\setminus A_h)\cap J_u} \gamma(|u^+ - u^-|) k(x,\nu_u)\, d\mathcal{H}^{N-1} - \Lambda \int_{\Omega'\cap J_u} \gamma(|u^+ - u^-|)\widetilde{\varphi}_h(x)\, d\mathcal{H}^{N-1} . \end{aligned} \tag{5.16}$$

Since $\widetilde{\varphi}_h \to 0$ strongly in $W^{1,1}(I\!R^N)$ as $h \to \infty$, we have that, up to a subsequence, $\widetilde{\varphi}_h(x) \to 0$ for $\mathcal{H}^{N-1}$-almost every $x \in I\!R^N$ (see Proposition 1.2 in [11]). Therefore, letting $h \to +\infty$ in (5.16), recalling that $A_{h+1} \subset A_h$ for all h and that $\mathcal{H}^{N-1}(\cap_h A_h) = 0$ and taking into account that γ is locally bounded, from the Dominated Convergence Theorem we get the thesis in Ω' for $u \in \mathrm{GSBV^p}(\Omega; I\!R^M) \cap L^\infty(\Omega; I\!R^M) \subset \mathrm{SBV^p}(\Omega; I\!R^M)$. Finally, inequality (5.11) holds, letting $\Omega' \nearrow \Omega$. The general case $u \in \mathrm{GSBV^p}(\Omega; I\!R^M)$ can be obtained as in the proof of Theorem 3.7 in [5]. □

As a consequence of previous proposition, we obtain the following result.

Corollary 5.4. *Let $k : \Omega \times I\!R^N \to [0,+\infty)$ be a locally bounded function satisfying all the assumptions of Theorem 5.3. Then, for every $\{u_n\} \subseteq \mathrm{GSBV^p}(\Omega; I\!R^M)$ and $u \in \mathrm{GSBV^p}(\Omega; I\!R^M)$ such that such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and*

$$\sup_{n \in I\!N} \left[\int_\Omega |\nabla u_n|^p\, dx + \mathcal{H}^{N-1}(J_{u_n}) \right] < +\infty ,$$

we have

$$\int_{\Omega\cap J_u} k(x,\nu_u)\, d\mathcal{H}^{N-1} \leq \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} k(x,\nu_{u_n})\, d\mathcal{H}^{N-1} . \tag{5.17}$$

Proof. It is enough to consider the function $\phi(x,r,t,\nu) = \gamma(|t-r|)k(x,\nu)$, where $\gamma(0) = 0$ and $\gamma(s) = 1$ for $s > 0$. Hence the conclusion follows by Theorem 5.3. □

Corollary 5.5. *Let $k : \Omega \times I\!R^N \to [0,+\infty)$ be a locally bounded Borel function such that, for every $\xi \in I\!R^N$, $k(\cdot,\xi) \in BV(\Omega)$, it coincides with its approximate lower limit and satisfies* (5.8)–(5.10)*. Then the same conclusion of Corollary 5.4 holds.*

Proof. It is a direct consequence of Theorem 2.3 and Corollary 5.4. □

In the first part of this section we investigated the lower semicontinuity of the surface integral in the functional G defined in (3.1). In the following, we study the complete functional G, i.e. we take into account also the volume integral, considering both the quasiconvex and the polyconvex case.

5.1. **The quasiconvex case.** In this subsection we prove a lower semicontinuity result for integral functionals where the volume term has a quasiconvex integrand.

Theorem 5.6. *Let $W : \Omega \times I\!R^M \times M\!I^{M\times N} \to [0,+\infty)$ be a Carathéodory function satisfying*

$$W(x,s,\cdot) \quad \text{is quasiconvex on } M\!I^{M\times N} \qquad \forall (x,s) \in \Omega \times I\!R^M\,, \tag{5.18}$$

$$a_0|\xi|^p - b_0(x) \le W(x,s,\xi) \le a_1|\xi|^p + b_1(x) \qquad \forall (x,s,\xi) \in \Omega \times I\!R^M \times M\!I^{M\times N}\,, \tag{5.19}$$

for some constants $a_0, a_1 > 0$ and some nonnegative function $b_0, b_1 \in L^1(\Omega)$. Let $k : \Omega \times I\!R^N \to [0,+\infty)$ and $\gamma : [0,+\infty) \to [0,+\infty)$ be as in Theorem 5.3. Let $\{u_n\} \subseteq \mathrm{GSBV^p}(\Omega; I\!R^M)$ and $u \in \mathrm{GSBV^p}(\Omega; I\!R^M)$ be such that $u_n(x) \to u(x)$ for almost every $x \in \Omega$ and $\mathcal{H}^{N-1}(J_{u_n}) \le C$, for a positive constant C and for every $n \in I\!N$. Then, if G is the functional defined in (3.1)*, with $h(x,r,t,\nu) = \gamma(|t-r|)k(x,\nu)$, we have*

$$G(u) \le \liminf_{n\to+\infty} G(u_n)\,.$$

Proof. It is enough to prove

$$\int_\Omega W(x,u,\nabla u)\,dx \le \liminf_{n\to+\infty} \int_\Omega W(x,u_n,\nabla u_n)\,dx\,, \tag{5.20}$$

$$\int_{\Omega\cap J_u} \gamma(|u^+ - u^-|)\,k(x,\nu_u)\,d\mathcal{H}^{N-1} \le \liminf_{n\to+\infty} \int_{\Omega\cap J_{u_n}} \gamma(|u_n^+ - u_n^-|)\,k(x,\nu_{u_n})\,d\mathcal{H}^{N-1}\,. \tag{5.21}$$

Inequality (5.20) follows by [8, Theorem 5.29] (see also [6] and [25]). Inequality (5.21) follows by Theorem 5.3. □

By using the previous compactness and lower semicontinuity results (Theorems 2.1 and 5.6) and the standard Direct Method of Calculus of Variations, we obtain the following existence theorem.

Theorem 5.7. *Let $W : \Omega \times I\!R^M \times M\!I^{M\times N} \to [0,+\infty)$ be a Carathéodory function satisfying the assumptions of Theorem 5.6. Let $k : \Omega \times I\!R^N \to [0,+\infty)$ and $\gamma : [0,+\infty) \to [0,+\infty)$ be as in Theorem 5.3. Moreover, assume that there exists $\lambda > 0$ such that*

$$k(x,\xi) \ge \lambda|\xi| \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega \text{ and } \forall \xi \in I\!R^N,$$
$$\gamma(t) \ge \lambda \qquad \text{for every } t > 0\,.$$

Let F be the functional defined in (3.1)*, with $h(x,r,t,\xi) = \gamma(|r-t|)k(x,\xi)$. Then the minimum problem*

$$\min\{F(u) : u \in \mathrm{GSBV^p}(\Omega; I\!R^M)\}$$

has at least one solution.

5.2. **The polyconvex case.** In this subsection, we obtain a result analogous to the one in the previous subsection also for the polyconvex case. To this purpose, we recall the following theorem (see [22]).

Theorem 5.8. *Let $W : \Omega \times I\!R^M \times M\!I^{M\times N} \to [0,+\infty)$ be a polyconvex function in the last variable, satisfying*

$$W(x,u,\xi) \ge \sum_{k=1}^{M\wedge N} \beta_k |adj_k \xi|^{p_k} \qquad (x,s,\xi) \in \Omega \times I\!R^M \times M\!I^{M\times N}\,,$$

where $\beta_k > 0$ *for every* $k = 1, \dots, M \wedge N$*, and the exponents* p_k *satisfy the following inequalities*

$$p_1 \geq 2\,, \qquad p_k \geq \frac{p_1}{p_1 - 1} \quad \text{if } k = 2, \dots, M \wedge N - 1, \qquad p_{M\wedge N} > 1\,.$$

Then, if $\{u_n\} \subseteq \mathrm{GSBV}(\Omega; I\!R^M)$ *and* $u \in \mathrm{GSBV}(\Omega; I\!R^M)$ *is such that* $u_n \to u$ *strongly in* $L^1(\Omega; I\!R^M)$*, we have*

$$\int_\Omega W(x, u, \nabla u)\, dx \leq \liminf_{n\to+\infty} \int_\Omega W(x, u_n, \nabla u_n)\, dx\,.$$

Theorem 5.9. *Let* $W : \Omega \times I\!R^M \times I\!M^{M\times N} \to [0, +\infty)$ *be a Carathéodory function satisfying all the assumptions of Theorem 5.8. Let* $k : \Omega \times I\!R^N \to [0, +\infty)$ *and* $\gamma : [0, +\infty) \to [0, +\infty)$ *be as in Theorem 5.3. Let* $\{u_n\} \subseteq \mathrm{GSBV}(\Omega; I\!R^M)$ *and* $u \in \mathrm{GSBV}(\Omega; I\!R^M)$ *be such that* $u_n \to u$ *strongly in* $L^1(\Omega; I\!R^M)$ *and* $\mathcal{H}^{N-1}(J_{u_n}) \leq C$*, for a positive constant* C *and for every* $n \in I\!N$*. Then, if* G *is the functional defined in* (3.1)*, with* $h(x, r, t, \nu) = \gamma(|t - r|)k(x, \nu)$*, we have*

$$G(u) \leq \liminf_{n\to+\infty} G(u_n)\,.$$

An existence result in GSBV, analogous to Theorem 5.7, holds when $W(x, s, \cdot)$ is polyconvex and satisfies the assumptions of the previous theorem.

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