# Well-Posedness and Stability of Damped Wave Equations with Singular Memory*

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**Abstract.** We study second order integro-differential equations in Hilbert spaces with weakly singular kernels, obtaining uniform estimates in $t$. Then, we apply such estimates to derive the exponential decay at $\infty$ of the energy of solutions.

## 1. Introduction

It is well known that the abstract integro-differential equation

$$\ddot{u}(t) + c_0\dot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)ds = 0 \qquad t \geq 0\,, \tag{1}$$

may be regarded as a model problem for some elastic systems with memory, see [3, 4, 15]. In this paper we are interested in the maximal regularity and asymptotic behaviour of the solutions of the above equation, where $A$ is a positive operator on a Hilbert space $X$, with domain $D(A)$, and $c_0 \geq 0$ is a damping coefficient.

Well-posedness results are known for (1)—in fact, for nonlinear variants of such an equation—when $\beta$ is absolutely continuous (see, e.g.,[12, 2]). So, in this paper, we will focuss our attention on the case of discontinuous kernels that is quite relevant for applications (see, e.g., [14]).

More precisely, we will consider kernels $\beta \in L^1(0,\infty)$ satisfying a standard integral constraint and such that $k(t) := \int_t^\infty \beta(s)ds$ is a kernel of positive type. It is noteworthy that these assumptions imply a commonly accepted form of thermodinamical restriction on the concrete models described by (1), see, e.g., [6, 7]. A typical example covered by our theory is the following:

$$\beta(t) = \frac{1-a}{2}\, e^{-t}\, t^{-a}\, \cos(bt) \qquad t > 0$$

where $0 \leq a < 1$ and $b \in \mathbb{R}$.

Since we impose assumptions of *global* nature for $\beta$, one would also expect *global estimates* to hold for the solutions of (1). In this paper we show that this is indeed the case, obtaining maximal regularity estimates for $u$ uniformly in $t$.

Once well-posedness is established, we turn to study the asymptotic behaviour of the solutions of (1) for $c_0 > 0$. Under a stronger sommability assumption for $\beta$ at $\infty$, we show that the energy of any mild solution decays exponentially at $\infty$. Our approach consists in applying uniform estimates to a perturbed equation, obtained multiplying $u$ by a positive exponential function. For related results on exponential decay the reader is referred to [10, 13], where the case of smooth convolution kernels is considered, and to [8, 11], where the decay of the semigroup associated with (1) is obtained.

The outline of this paper is the following. In section 2 we recall some known facts on integral equations and derive preliminary results. Section 3 is devoted to well-posedness. In particular, we prove existence and regularity results for the resolvent, obtaining uniform estimates in $t$. Finally, in section 4, we show the exponential decay at $\infty$ of the energy of solutions.

*to appear in: Proceedings of the 8th Conference on Control of Distributed Parameter Systems, Graz-Mariatrost

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