Quasi-linear parabolic equations with degenerate coercivity having a quadratic gradient term

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Abstract.

We study existence and regularity of distributional solutions for possibly degenerate quasi-linear parabolic problems having a first order term which grows quadratically in the gradient. The model problem we refer to is the following

$$\begin{cases} u_t - \operatorname{div}\big(\alpha(u)\nabla u\big) = \beta(u)|\nabla u|^2 + f(x,t), & \text{in } \Omega\times]0,T[; \\ u(x,t) = 0, & \text{on } \partial\Omega\times]0,T[; \\ u(x,0) = u_0(x), & \text{in } \Omega. \end{cases} \tag{1}$$

Here Ω is a bounded open set in $\mathbb{R}^N$, $T>0$. The unknown function $u = u(x,t)$ depends on $x \in \Omega$ and $t \in]0,T[$. The symbol ∇u denotes the gradient of u with respect to x. The real functions α, β are continuous; moreover α is positive, bounded and may vanish at $\pm\infty$. As far as the data are concerned, we require the following assumptions:

$$\int_\Omega \Phi(u_0(x))\,dx < \infty$$

where Φ is a convenient function which is superlinear at $\pm\infty$ and

$$f(x,t) \in L^r\big(0,T;L^q(\Omega)\big) \qquad \text{with} \qquad \frac{1}{r} + \frac{N}{2q} \le 1.$$

We give sufficient conditions on α and β in order to have distributional solutions. We point out that the assumptions on the data do not guarantee in general the boundedness of the solutions; this means that the coercivity of the principal part of the operator can really degenerate. Moreover, a boundedness result is proved when the assumptions on the data are strenghtened.