DYNAMIC CONSERVATION LAWS AND THE KORTEWEG-De VRIES EQUATION

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Recently we have introduced a geometric method to describe conservation laws more general than Noetherian ones [1,2,3]. In particular in [3] we have given a detailed exposition of this method developping the geometric-differential calculus to build dynamic conserved quantities.

In this communication we shall give an account of some of these results emphasizing their application to the Korteweg-de Vries equation.

Euler-Lagrange operator for lagrangians of any order

1) Lagrangian $f : J\mathcal{D}^k(\mathbf{C}) \to \mathbb{R}$

$J\mathcal{D}^k(\mathbf{C}) = k\text{-}th$ jet-derivative space oven the fiber bundle

$\pi_{\mathbf{C}} : \mathbf{C} \to M$

$M = n$-dimensional manifold.

Fibration structure: $\pi_k : J\mathcal{D}^k(\mathbf{C}) \to M$.

2) Euler-Lagrange operator: $\varepsilon(f) : C^\infty(\mathbf{C}) \to C^\infty(\mathbf{C}, T^*\mathbf{C})$

The intrinsic expression of $\varepsilon(f)$ is given in [3]. For $k = 1$ one has the expression given in [5].

To $\varepsilon(f)$ it is associated a $n+1$ form $\varepsilon[f]$ on $J\mathcal{D}^\infty(\mathbf{C})$ (*Euler-Lagrange form*).

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Cartan form for lagrangians of any order

It is a n-form $\sigma[f]$ on $J\mathcal{D}^{\infty}(C)$ canonically associated to the Lagrangian $f : J\mathcal{D}^{k}(\mathbf{C}) \to \mathbf{R}$. Set $\Omega[f] \equiv f\pi^{*}_{k}\eta$ where η is the volume form on M . Then one has the following *first variational formula:*

$$\pi^{*}_{\infty,k+1}\,\tau\mathcal{L}_{\overset{k}{X}}\,\Omega[f] = \tau(\overset{\infty}{X}\lrcorner\,\varepsilon[f]) + \tau d(\overset{\infty}{X}\lrcorner\,\sigma[f])$$

where: (a) $\overset{s}{X}$ is the s-th prolongation on $J\mathcal{D}^{s}(\mathbf{C})$ of the infinitesimal symmetry X of $\pi_{\mathbf{C}} : \mathbf{C} \to M$; (b) τ is the projection operator

$$\tau : C^{\infty}(\Lambda^{o}_{p}\, J\mathcal{D}^{k}(\mathbf{C})) \to C^{\infty}(\dot{\Lambda}^{o}_{p}(\pi_{k+1}))$$

defined by $(D^{k+1}\gamma)^{*}\tau\omega = (D^{k}\gamma)^{*}\omega$, $\forall\, \gamma \in C^{\infty}(\mathbf{C})$, being $\dot{\Lambda}^{o}_{p}(\pi_{k+1})$ the space of horizontal p-forms on $J\mathcal{D}^{k+1}(\mathbf{C})$ with respect to the fibration $\pi_{k+1} : J\mathcal{D}^{k+1}(\mathbf{C}) \to M$; (c) $\pi_{s,l}$ is the surjection $J\mathcal{D}^{s}(\mathbf{C}) \to J\mathcal{D}^{l}(\mathbf{C})$, $s \geq l$.

If $Dc^{k*}\mathcal{L}_{\overset{k}{X}}\Omega[f] = 0$ for any extremal c of f , then the form

$$\beta \equiv \tau(\overset{\infty}{X}\lrcorner\,\sigma[f]) \in C^{\infty}(\Lambda^{o}_{n-1}\, J\mathcal{D}^{2k}(\mathbf{C}))$$

is a *conservation law*, that is

$$d\{(D^{\infty}c)^{*}\tau(\overset{\infty}{X}\lrcorner\,\sigma[f])\} = 0$$

for every extremal c of f . To β there corresponds the conserved charge

$$Q = \int_{A_t} (D^{\infty}c)^{*}\beta$$

A *dynamic conservation law* of a continuum system $CS = (\mathcal{C}, E_k)$ (see ref. [4]) is given by a dyfferential form

$$\beta \equiv \tau(\overset{\infty}{X} \lrcorner \sigma[\emptyset])$$

where X is an infinitesimal symmetry of CS and $\emptyset$ is a corresponding differential invariant, such that $Ext(\emptyset) \cap \mathcal{S}(E_k) \neq \emptyset$, where $Ext(\emptyset)$ = set of extremals of $\emptyset$ and $\mathcal{S}(E_k)$ = set of solutions of E_k.

As an example let us consider the Korteweg-de Vries equation

(KdV) $$F \equiv u_{xxx} + 6\, u u_x - u_t = 0$$

where $u : \mathbf{R}^2 \to \mathbf{R}$ is a function. (KdV) can be considered as the Euler-Lagrange equation of a suitable Lagrangian of 2-th order $f : J\mathcal{D}^2(\mathbf{C}) \to \mathbf{R}$, $\mathbf{C} \equiv \mathbf{R}^2 \times \mathbf{R}$.

One can see that the vector field $X = 24\, t\partial x - 4\, \partial u : \mathbf{C} \to T\mathbf{C}$ is an infinitesimal symmetry of (KdV) and it is not a Noetherian symmetry. To X there corrisponds the following dynamic conservation law

$$\beta = -4(1 + 6\, tu_x)\, dx - 24(u + t\, u_t)\, dt \;,$$

the corresponding conserved charge is

$$Q = -4 \int (1 + 6\, tu_x)\, dx \;.$$

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