

DYNAMIC CONSERVATION LAWS AND THE KORTEWEG-De VRIES EQUATION

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Recently we have introduced a geometric method to describe conservation laws more general than Noetherian ones [1,2,3]. In particular in [3] we have given a detailed exposition of this method developping the geometric-differential calculus to build dynamic conserved quantities.

In this communication we shall give an account of some of these results emphasizing their application to the Korteweg-de Vries equation.

Euler-Lagrange operator for lagrangians of any order

1) Lagrangian $f : J\mathcal{D}^k(\mathbb{C}) \rightarrow \mathbb{R}$

$J\mathcal{D}^k(\mathbb{C}) = k$ -th jet-derivative space over the fiber bundle

$\pi_{\mathbb{C}} : \mathbb{C} \rightarrow M$

$M = n$ -dimensional manifold.

Fibration structure: $\pi_k : J\mathcal{D}^k(\mathbb{C}) \rightarrow M$.

2) Euler-Lagrange operator: $\epsilon(f) : C^\infty(\mathbb{C}) \rightarrow C^\infty(T^*\mathbb{C})$

The intrinsic expression of $\epsilon(f)$ is given in [3]. For $k=1$ one has the expression given in [5].

To $\epsilon(f)$ it is associated a $n+1$ form $\epsilon[f]$ on $J\mathcal{D}^\infty(\mathbb{C})$ (Euler-Lagrange form).

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Cartan form for lagrangians of any order

It is a n -form $\sigma[f]$ on $J^{\infty}(\mathbb{C})$ canonically associated to the Lagrangian $f : J^{\infty}(\mathbb{C}) \rightarrow \mathbb{R}$. Set $\Omega[f] \equiv f \pi_k^* \eta$ where η is the volume form on M . Then one has the following first variational formula:

$$\pi_{\infty, k+1}^* \tau_{\mathcal{L}}^k \Omega[f] = \tau(\bar{X} \lrcorner \varepsilon[f]) + \tau d(\bar{X} \lrcorner \sigma[f])$$

where: (a) \bar{X} is the s -th prolongation on $J^{\infty}(\mathbb{C})$ of the infinitesimal symmetry X of $\pi_{\mathbb{C}} : \mathbb{C} \rightarrow M$; (b) τ is the projection operator

$$\tau : C^{\infty}(\Lambda_p^0 J^{\infty}(\mathbb{C})) \rightarrow C^{\infty}(\Lambda_p^0(\pi_{k+1}))$$

defined by $(D^{k+1} \gamma)^* \tau \omega = (D^k \gamma)^* \omega$, $\forall \gamma \in C^{\infty}(\mathbb{C})$, being $\Lambda_p^0(\pi_{k+1})$ the space of horizontal p -forms on $J^{\infty}(\mathbb{C})$ with respect to the fibration $\pi_{k+1} : J^{\infty}(\mathbb{C}) \rightarrow M$; (c) $\pi_{s,l}$ is the surjection $J^s(\mathbb{C}) \rightarrow J^l(\mathbb{C})$, $s \geq l$.

If $D_c^k \tau_{\mathcal{L}}^k \Omega[f] = 0$ for any extremal c of f , then the

form

$$\beta \equiv \tau(\bar{X} \lrcorner \sigma[f]) \in C^{\infty}(\Lambda_{n-1}^0 J^{\infty}(\mathbb{C}))$$

is a conservation law, that is

$$d((D^{\infty} c)^* \tau(\bar{X} \lrcorner \sigma[f])) = 0$$

for every extremal c of f . To β there corresponds the conserved charge

$$Q = \int_{A_t} (D^{\infty} c)^* \beta$$

A dynamic conservation law of a continuum system $CS = (\mathcal{C}, E_k)$ (see ref. [4]) is given by a differential form

$$\beta \equiv \tau(\bar{X}) \lrcorner \sigma[\theta]$$

where X is an infinitesimal symmetry of CS and θ is a corresponding differential invariant, such that $Ext(\theta) \cap \mathcal{S}(E_k) \neq \emptyset$, where $Ext(\theta)$ = set of extremals of θ and $\mathcal{S}(E_k)$ = set of solutions of E_k .

As an example let us consider the Korteweg-de Vries equation

$$(KdV) \quad F \equiv u_{xxx} + 6uu_x - u_t = 0$$

where $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function. (KdV) can be considered as the Euler-Lagrange equation of a suitable Lagrangian of 2-th order $f: \mathcal{D}^2(\mathcal{C}) \rightarrow \mathbb{R}$, $\mathcal{C} \equiv \mathbb{R}^2 \times \mathbb{R}$.

One can see that the vector field $X = 24t \partial_x - 4 \partial_u: \mathcal{C} \rightarrow T\mathcal{C}$ is an infinitesimal symmetry of (KdV) and it is not a Noetherian symmetry. To X there corresponds the following dynamic conservation law

$$\beta = -4(1 + 6tu_x) dx - 24(u + tu_t) dt,$$

the corresponding conserved charge is

$$Q = -4 \int (1 + 6tu_x) dx.$$

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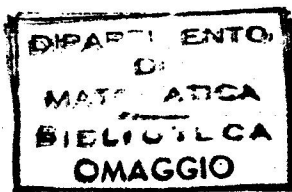
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