Homogenization and concentration of capacity in the rod outer segment with incisures

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Abstract
We present a quantitative model of the spatio-temporal dynamics of second messengers mediating phototransduction in retinal rods. The spatial domain (the rod outer segment) has a quite complex geometry, involving different “thin” domains, whose thickness is three orders of magnitude smaller than the other dimensions.

The model relies on a “pointwise” application of first principles leading to a system of evolution equations set in such a structured geometry.

Then, exploiting an idea first presented in [2], the diffusion problem is reduced to one with a simpler geometry, still preserving the essential features of the original one. This is achieved by an homogenization and concentration limit. However, here we take into account for the first time

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the presence of “incisures”, which are important for phototransduction, and introduce new mathematical features mainly in the concentration limit.


Keywords: homogenization, signal transduction, concentration of capacity, disconnected structure, reticular structure.

1 Introduction

Figure 1 is a schematic rendering of the anatomy of a rod outer segment (ROS) in the retina of vertebrates. It consists of a right circular cylinder, of height and diameter of the order of 10 micrometers, containing a parallel stack of equispaced, equal, thin circular discs with an incisure on them.

The distance between any two such discs is of the order of 10 nanometers and, the thickness of the cylindrical layer between the stack and the lateral boundary of the ROS is likewise of the order of 10 nanometers.

The number of incisures per disc can range from 1 in mice and humans (as in Figures 1 and 2), to 18 in tiger salamander. They start from the edge of the disc, with a width of the order of 10 nanometers and run, roughly radially, toward the center in a spike–like manner. They could be as long as the radius of the cross section of the ROS (toad) or less than one-half the radius (rat). The discs making up the stack inside the ROS, are held in a stable position by a scaffold of filaments and, incisures in individual discs are aligned with each other, identifying a blade-like region as in the left Figure 3 (see [11]).

Phototransduction is the process by which photons of light are transformed, by a biochemical cascade taking place in the ROS and its boundary, into electrical pulses that mediate vision. Each of the discs in the stack is a photon–capturing device. The conversion of light energy into an electrical pulse, is mediated by the diffusion of second messengers cGMP (Cyclic Guanosin Monophosphate) and Ca$^{2+}$ (Calcium ions) within the cytosol, i.e., that part of the ROS not occupied by the stack of discs. For an introduction to phototransduction we refer to [15].

In this work we present a quantitative model of the spatio-temporal dynamics of second messengers in retinal rod photoreceptors, that takes into account the geometric complexity of the ROS, whose main points are:

i. a periodic arrangement of interdiscal spaces;

ii. an outer shell, surrounding the stack of discs;

iii. a blade-like region generated by the incisures.
equations set in such a structured geometry (\cite{4}), as in infinitely thin and by obtaining a limiting set of evolution equations from this arrangement (i) and a concentration limit within the region (ii) and (iii).

Exploiting an idea first presented in \cite{2}, the diffusion problem is reduced to one with a simpler geometry, still preserving the essential features of the original one. This is achieved by an homogenization limit within the periodic arrangement (i) and a concentration limit within the regions (ii) and (iii).

The mathematical content of this contribution is in computing the homogenized and concentrated limit of such a family of evolution problems.

2 Geometry of the Rod Outer Segment

2.1 The Incises

An incisure of size $\varepsilon$ cut on a disc $D_R$ of radius $R$, is a spike-like region $V_{\varepsilon}$, as in Figure 2. In the coordinate system of the figure, the incisure has vertex at

Figure 1: Left: Cross section of the Rod Outer Segment and stack of discs with an incisure.

All these regions are thin, in the sense that their thickness is three orders of magnitude smaller than the other dimensions. First the physics of the phenomenon is modeled by a system of evolution equations set in the interior of the cytosol with flux terms prescribed on each of the faces of the layered discs as well as on the remaining parts of the boundary of the ROS. The model consists of a “pointwise” application of first principles and leads to a system of evolution equations set in such a structured geometry (\cite{4}), as in § 3. Its physical meaning and information are extracted by regarding the layers making up the cytosol as infinitely thin and by obtaining a limiting set of evolution equations from this process.

Exploiting an idea first presented in \cite{2}, the diffusion problem is reduced here to one with a simpler geometry, still preserving the essential features of the original one. This is achieved by an homogenization limit within the periodic arrangement (i) and a concentration limit within the regions (ii) and (iii).

The mathematical content of this contribution is in computing the homogenized and concentrated limit of such a family of evolution problems.
some $0 \leq r_0 < R$ and is given by
\[ \mathcal{V}_\varepsilon = \{ r_0 < x_1, |x_2| < \beta_\varepsilon h(x_1 - r_0) \} \cap D_R \]  
(2.1)
where $\beta_\varepsilon$ is a given positive constant, and $h$ is a convex, non-negative function defined and continuous in $[0, R - r_0]$, and satisfying
\[ h \in C^1(0, R - r_0); \quad h(0) = 0; \quad h(y) > 0; \quad 0 \leq h'(y) \leq C, \]
for all $y \in (0, R - r_0)$, for a given positive constant $C$. Moreover
\[ \sqrt{\frac{h'}{h^{1+s}}} \in L^{1+\tau}(0, R - r_0), \quad \text{for some } s, \tau > 0. \]  
(2.3)

![Figure 2: Incisure with “vertex” at $r_0$](image)

Examples of such incisures include
\[ \mathcal{V}_\varepsilon = \{ r_0 < x_1, |x_2| < \beta_\varepsilon (x_1 - r_0)\alpha \} \cap D_R \quad \text{for some } 1 \leq \alpha. \]  
(2.4)

The incisure cuts an arc on the rim of $D_R$, as in Figure 2. For $a \geq 1$ fixed, $\beta_\varepsilon$ is chosen so that the length of such an arc is $a \varepsilon$. For example if $\mathcal{V}_\varepsilon$ is a circular sector ($r_0 = 0$ and $\alpha = 1$ in (2.4)), then $\beta_\varepsilon = \tan(a \varepsilon / 2R)$. The parameter $\varepsilon$ is taken as the “size” of the incisure and, for $\varepsilon \ll 1$, the 2-dimensional measure $|\mathcal{V}_\varepsilon|$ of the incisure, does not exceed the order of $\varepsilon$.

For $r_0 < r < R$ consider the portion of the circle centered at the origin with radius $r$, cut by the incisure, and let $2\theta_\varepsilon(r)$ be the corresponding angle. Therefore the length of such an arc is $2r \theta_\varepsilon(r)$. It can be easily seen that under our assumptions we have $\theta_{\varepsilon, r} \geq 0$. Denote by $\theta_{\varepsilon, \max} = \theta_\varepsilon(R)$ the largest angle defined by $\theta_\varepsilon$, so that $2R \theta_{\varepsilon, \max} = a \varepsilon$, and
\[ \gamma^{-1} \varepsilon \leq \beta_\varepsilon = \frac{R \sin \theta_{\varepsilon, \max}}{h(R \cos \theta_{\varepsilon, \max} - r_0)} \leq \gamma \varepsilon \]  
(2.5)
for a constant $\gamma$ depending upon $R$, $a$ and $r_0$, and independent of $\varepsilon$. 

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Lemma 2.1 Let $V_\varepsilon$ be the incisure described in (2.1). Then for $r > r_o$, 
\[ 0 \leq r\theta_{\varepsilon,r}(r) \leq \gamma \varepsilon, \]
for a constant $\gamma$ independent of $\varepsilon$.

Proof: Write the polar representation of the curve delimiting the incisure as
\[ r \sin \theta = \beta \varepsilon h(r \cos \theta - r_o) \quad \text{for} \quad r \cos \theta > r_o. \] (2.6)
Taking the derivative with respect to $r$ of both sides, proves the Lemma. ■

The function $r \rightarrow \theta_{\varepsilon}(r)$ which is defined for $r \geq r_o$ will be regarded as defined in $(0, R)$ by setting it to be zero for $r \in [0, r_o]$. Thus
\[ \theta_{\varepsilon}(r) = \begin{cases} 0 & \text{for} \ 0 \leq r \leq r_o; \\ \theta_{\varepsilon}(r) & \text{for} \ r_o < r < R. \end{cases} \] (2.7)

2.2 The Incised ROS

Introduce coordinates $x = (x_1, x_2)$ and $x = (\pi, z)$, as in Figure 3, and denote by $C_{\varepsilon}$ and $B_{\varepsilon}$, the cylindrical domains
\[ C_{\varepsilon} \overset{\text{def}}{=} D_R - V_\varepsilon \times \{0 < z < \varepsilon\}; \quad B_{\varepsilon} \overset{\text{def}}{=} V_\varepsilon \times \{0 < z < H\} \] (2.8)
for a given positive number $H$. In what follows $\varepsilon$ is a parameter that will be let go to zero. Therefore $C_{\varepsilon}$ is a thin cylinder with sizable cross section $D_R - V_\varepsilon$, whereas $B_{\varepsilon}$ looks like a knife blade: it is a right cylinder of height $H$ and whose cross section is a thin incisure.

Figure 1 represents the longitudinal cross section of a right circular cylinder $\Omega_{\varepsilon}$, of height $H$, and of transversal cross-section a disc $D_{R+\sigma_\varepsilon}$ where $R$, $H$ and $\sigma$ are fixed positive numbers and $\varepsilon$ is a small positive parameter.

\[ \Omega_{\varepsilon} = D_{R+\sigma_\varepsilon} \times \{0 < z < H\}; \quad \Omega = D_R \times \{0 < z < H\}. \]
The cylinder $\Omega$ is included in $\Omega_{\varepsilon}$, is coaxial with it, and it is formally obtained from $\Omega_{\varepsilon}$ for $\varepsilon = 0$. The cylinder $\Omega$ houses a vertical stack of thin, equipspaced, cylindrical domains $C_{\varepsilon,j}$, for $j = 1, 2, \ldots, n$, each congruent to $C_{\varepsilon}$. They are thin in the sense that their thickness $\varepsilon \ll H$. Their mutual distance is $\nu \varepsilon$, where $\nu$ is a fixed positive number. The first, $C_{\varepsilon,1}$, has distance $\frac{1}{2} \nu \varepsilon$ from the lower face of the cylinder $\Omega_\varepsilon$ and the last one $C_{\varepsilon,n}$, has distance $\frac{1}{2} \nu \varepsilon$ from the upper face of $\Omega_\varepsilon$. The indicated geometry implies that the ratio of the volume occupied by the $C_{\varepsilon,j}$ to the volume of $\Omega - B_{\varepsilon}$, is
\[ \frac{\text{vol} \left( \bigcup_{j=1}^n C_{\varepsilon,j} \right)}{\text{vol}(\Omega - B_{\varepsilon})} = \mu_o \quad \text{where} \quad \mu_o \overset{\text{def}}{=} \frac{1}{1 + \nu}. \] (2.9)
The gap between $\Omega_{\varepsilon}$ and $\Omega$ is the outer shell
\[ S_{\varepsilon} = \{D_{R+\sigma_\varepsilon} - \overline{D}_R\} \times \{0 < z < H\}. \]
Figure 3: Left: Geometrical description of the rod outer segment. Right: The domain obtained in the "limit" as $\varepsilon \to 0$.

Figure 3 shows $\Omega_\varepsilon \cap \partial B_{\varepsilon}$. The spaces between two contiguous $C_{\varepsilon,j}$ and $C_{\varepsilon,j+1}$ and within $\Omega \setminus B_{\varepsilon}$, are the interdiscal spaces. For $j = 1, 2, \ldots, (n-1)$ these are equal cylinders of cross section $(D_R - V_\varepsilon)$ and height $v \varepsilon$. We label them by $I_j$, for $j = 0, 1, 2, \ldots, n$, by defining $I_0$ as the space between the lower face $\{z = 0\}$ of $\Omega \setminus B_{\varepsilon}$ and the lower face of $C_{\varepsilon,1}$, and $I_n$ as the space between the upper face of $C_{\varepsilon,n}$ and the upper face $\{z = H\}$ of $\Omega \setminus B_{\varepsilon}$. The upper and lower faces of the interdiscal spaces $I_j$ are denoted by $\partial I_j^\pm$. We also denote by $L_j$ the lateral surface of the cylinders $C_{\varepsilon,j}$, and by $\Lambda_j$ the lateral surface of the interdiscal spaces $I_j$.

For each of the either three-dimensional or two-dimensional domains introduced, consider the corresponding space–time cylindrical domain over a time interval $(0, T)$, for a fixed $T > 0$; for example for $\Omega$ and $S_\varepsilon$,

$$\Omega_T = \Omega \times [0, T], \quad S_{\varepsilon,T} = S_\varepsilon \times [0, T]$$

and similarly for the remaining ones.

3 The Family of $\varepsilon$–Problems

We will compute the limit $\varepsilon \to 0$ of the solution of the heat equation set in the domain $\Omega_\varepsilon$ from which the cylindrical domains $C_{\varepsilon,j}$ have been removed, with non–linear variational data on the faces $\partial I_j^\pm$, and when the mass is concentrated
in the outer shell $S_\varepsilon$ and on the blade–like domain $B_\varepsilon$. Set,

$$a_\varepsilon(x) = \begin{cases} 1 & \text{for } x \in \bigcup_{j=0}^{n} I_j; \\ \frac{\varepsilon_o}{\varepsilon} & \text{for } x \in S_\varepsilon; \\ \frac{\theta_\varepsilon(x)}{\theta_\varepsilon(r)} & \text{for } x \in B_\varepsilon \end{cases}$$

(3.1)

and

$$\tilde{\Omega}_\varepsilon = \Omega_\varepsilon - \bigcup_{j=1}^{n} C_j = \bigcup_{j=0}^{n} I_j \cup S_\varepsilon \cup B_\varepsilon \quad (3.2)$$

where $\varepsilon_o \in (0, 1)$ is fixed and $\varepsilon \in (0, \varepsilon_o]$. From (2.6), for all $0 < \varepsilon \leq \varepsilon_o$,

$$\theta_\varepsilon(r) = \sin^{-1} \left( \frac{\beta_\varepsilon h(r \cos \theta_\varepsilon - r_o)}{r} \right) \quad (3.3)$$

Therefore,

$$\theta_\varepsilon(r) \leq \frac{2\beta_\varepsilon}{r} h(r \cos \theta_\varepsilon - r_o) \quad \text{and} \quad \theta_\varepsilon(r) \geq \frac{\beta_\varepsilon}{r} h(r \cos \theta_\varepsilon - r_o).$$

Since $h$ is an increasing function of its argument, and $\theta_{\varepsilon_o}(r) \geq \theta_{\varepsilon}(r)$

$$h(r \cos \theta_{\varepsilon_o}(r) - r_o) \leq h(r \cos \theta_{\varepsilon}(r) - r_o).$$

Therefore

$$1 \leq \frac{\theta_{\varepsilon_o}(r)}{\theta_{\varepsilon}(r)} \leq 2 \frac{\beta_\varepsilon}{\beta_{\varepsilon_o}} \frac{h(r \cos \theta_{\varepsilon_o} - r_o)}{h(r \cos \theta_{\varepsilon} - r_o)} \leq \frac{\varepsilon_o}{\varepsilon}. \quad (3.4)$$

Consider the family of problems,

$$\begin{cases} u_\varepsilon \in C(0, T; L^2(\tilde{\Omega}_\varepsilon)) \cap L^2 \left( 0, T; W^{1,2}(\tilde{\Omega}_\varepsilon) \right); \\ \frac{\partial}{\partial t} u_\varepsilon - \text{div} a_\varepsilon(x) \nabla u_\varepsilon = 0 \quad \text{weakly in } \tilde{\Omega}_\varepsilon, T, \quad (3.5) \end{cases}$$

with the variational and initial data,

$$\begin{cases} \nabla u_\varepsilon \cdot n = -\frac{1}{2} \varepsilon \nu (u_\varepsilon - f) & \text{on } \partial I_j^+, j = 0, \ldots, (n-1); \\ \partial I_j^-, j = 1, \ldots, n; \\ \nabla u_\varepsilon \cdot n = 0 & \text{on } \partial I_j^-, j = 1, \ldots, n; \\ z = 0 \text{ and } z = H \\ \nabla u_\varepsilon \cdot n = 0 & \text{on } L_j, j = 1, \ldots, n; \\ \nabla u_\varepsilon \cdot n = 0 & \text{on } |r| = R + \sigma \varepsilon \end{cases}$$

$$u_\varepsilon(\cdot, 0) = u_o \quad \text{in } \tilde{\Omega}_\varepsilon$$

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where \( n \) is the unit exterior normal to the indicated surfaces. The initial datum 
\( u_0 \) is a given positive constant. The function \( f \) is the restriction to the indicated 
surfaces of a non-negative, bounded function with bounded gradient, defined in \( \mathbb{R}^3 \times \mathbb{R} \). The set of constants \( \{ \sigma, \nu, H, \varepsilon_0, R, T, u_0, \sup f, \sup |\nabla x f|, \) 
\( h(R - r_0), \sup h', a \} \) are the data, and we say that a constant \( \gamma \) depends only 
on the data if it can be determined apriori only in terms of these quantities and it is independent of \( \varepsilon \). The formulation (3.4) implies that \( u_2 \) is continuous 
from within each of the interdiscal spaces \( I_j \) into \( S_\varepsilon \) and into \( B_\varepsilon \), through the 
cylindrical surfaces \( A_j \), except possibly at the tip of the blade \( B_\varepsilon \).

![Figure 4: Left: Transversal Cross section of the ROS. Right: Limit of such a cross section as \( \varepsilon \to 0 \).](image)

### 3.1 Homogenization, Concentrated Capacity and Motivation

For \( \varepsilon = \varepsilon_0 \) the problem (3.4) is the heat equation in \( \tilde{\Omega}_\varepsilon \). For \( 0 < \varepsilon < \varepsilon_0 \) the mass of \( u_2 \) in the outer shell \( S_\varepsilon \) and on the longitudinal blade-like region \( B_\varepsilon \), 
is concentrated. Roughly speaking, the mass is divided by the local thickness of 
these regions to account for a local shrinkage of \( S_\varepsilon \cup B_\varepsilon \) of the same order. We 
will let \( \varepsilon \to 0 \) and \( n \to \infty \) in such a way that (2.9) continues to hold; this way 
the ratio between the volume occupied by the cylindrical domains \( C_{\varepsilon, j} \) and the 
volume of \( \Omega \), remains \( \mu_0 + O(\varepsilon) \).

The geometry of \( \tilde{\Omega}_\varepsilon \) exhibits two orders of thin compartments, available to 
the diffusion, i.e., the interdiscal spaces \( I_j \), and the region \( S_\varepsilon \cup B_\varepsilon \) surrounding 
them. In the limit the disc with incisure \( D_R - V_\varepsilon \) tends formally to the incised 
disc \( D_R - V \), as in Figure 4, where \( V \) is the segment

\[
V = \{ r_0 < x_1 < R \} \times \{ x_2 = 0 \}.
\]
The cylindrical, blade–like domains $B_ε$ tend formally to the rectangle 

$$B = V \times \{0 < z < H\}$$

and the domains $Ω − B_ε$ tend formally to incised cylinder $Ω − B$. The outer shell $S_ε$ tends formally to $S$, the lateral boundary of $Ω$. The problems in (3.4)–(3.5) tend, in a sense to be made precise, to:

i. A family of 2–dimensional, transversal diffusion processes, parametrized with $z ∈ (0, H)$, taking place on the incised discs $\{D_R − V\} × \{z\}$ (§ 4.1). We refer to this as the interior diffusion.

ii. A boundary diffusion, by the Laplace–Beltrami operator, taking place on the limiting surface $S$ (§ 4.3).

iii. A diffusion process, taking place on the longitudinal, limiting rectangle $B$ (§ 4.2).

Moreover,

iv. The exterior fluxes of the transversal 2–dimensional diffusions on the incised discs $\{D_R − V\} × \{z\}$, as well as the exterior fluxes of the diffusion on the limiting rectangular blade $B$, serve as source terms in the boundary surface diffusion on the limiting outer shell $S$ and the limiting rectangle $B$ (§ 4.2, § 4.3).

v. The trace on $S ∪ B$ of the solution of interior diffusion coincides with the solution of the boundary diffusion on $S ∪ B$ (§ 4.3).

As indicated in § 1, this problem of homogenization–concentration is motivated by the diffusion of the second messengers cGMP (Cyclic Guanosin Monophosphate) and $Ca^{2+}$ (Calcium ions) in the cytosol of a rod outer segment in visual transduction. A rod outer segment in the retina of vertebrates, looks like $Ω_ε$. The cytosol is the region $Ω_ε$ from which the cylindrical domains $C_{ε_o,j}$ have been removed. Such cylindrical domains $C_{ε_o,j}$ are called discs in the biological literature and we will do so in what follows.

The diffusion equation (3.4)–(3.5) should be replaced by a system (for cGMP and for Calcium) with somewhat more structured boundary conditions. The homogenized limiting process is suggested by the actual physical dimensions of $R$, $H$, and $ε_o$. The form of the fluxes in (3.5) is also generated by the physical problem, where the function $f$ is a non–linear function of Calcium. Finally the problem starts from dark equilibrium, where cGMP is uniformly distributed in the rod outer segment. This motivates the assumption that $u_o$ is constant. The results are not affected by this assumption and $u_o$ could be taken as the restriction to $\tilde{Ω}_ε$ of a smooth non–negative function defined in $R^3$.

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1For the Salamander $H ≈ 22μm$, $R ≈ 5.5μm$, $ε_o ≈ 14nm$, $νε_o ≈ 14nm$, $σε_o ≈ 15nm$, $n_o ≈ 1,000$; see [14]. The discs in the ROS of the Salamander exhibit up to 18 incisures, each or largest width of the order of 10 nanometers. We refer to the review article [15] for a detailed description of the rod anatomy.
We have chosen to present the main mathematical ideas in the context of a single equation and a single incisure per disc. We postpone to § 10 a description of the visual transduction cascade generated by a photon captured by a disc $C_{ε,j}$. There we discuss its mathematical setting and compute the corresponding homogenized–concentrated limits, still for a single incisure per disc. The case of multiple incisures per disc, is a straightforward extension of the ideas and resulted presented here.

We refer to [3] for further comments on the unusual nature of such homogenization problems in layered media and their mathematical significance.

4 The Homogenized–Concentrated Limit

As $ε → 0$ the family of problems (3.4)–(3.5) tends, in a sense to be made precise, to a problem involving three limiting functions

\[
\begin{align*}
&u \text{ defined in } Ω − B, \text{ called the interior limit;} \\
&u_S \text{ defined in } S, \text{ called the limit on the outer shell;} \\
&u_B \text{ defined in } B, \text{ called the limit in the incisure.}
\end{align*}
\]

4.1 The Interior Limit

\[
\begin{align*}
&u \in C(0, T; L^2(Ω)) \cap L^2(0, T; L^2(0, H; W^{1,2}(D_R))) \\
&u_t − \Delta_x u = −(u − f) \text{ weakly in } Ω − B 
\end{align*}
\]

where $\Delta_x$ denotes the Laplacian with respect to the transversal variable $x$ only.

These are diffusion processes, parametrized with $z ∈ (0, H)$, taking place on the incised disc $D_R − V$. Also, the homogenized limit transforms the boundary fluxes in (3.5) into source terms holding in $Ω − B$.

4.2 The Limit in the Incisure

Denote by $r$ and $z$ the coordinates in the limiting blade $B$ and for $0 < δ ≪ 1$ set

\[
V(δ) = \{r_o + δ < r < R\} \quad \text{and} \quad B(δ) = V(δ) \times \{0 < z < H\}.
\]

The restrictions of $\{u_ε\}$ to $B_ε$ converge to a function $u_B$ defined in $B_T$ and satisfying

\[
u_B \in C(0, T; L^2(B(δ))) \cap L^2(0, T; W^{1,2}(B(δ))) \cap C(B_T)
\]
for all $0 < δ ≪ 1$. Moreover

\[
\sqrt{2r} \theta \in u_B, \sqrt{2r} \theta \in \nabla u_B \in L^2(B_T).
\]

This function is related to the interior limit $u$ as follows. First by virtue of (4.1) the function $x → u(x, z, t)$ has a trace on $V$ for a.e., $z ∈ (0, H)$, and such traces are in $L^2_{loc}(B_T)$ Then,

\[
u_B(r, z, t) = u(r, z, t)|_{B_T} \text{ in } L^2_{loc}(B_T).
\]
Next, denoting by $u_{x_2}^\pm$ the $x_2$-derivative of the interior limit $u$ from either side of $B$,

$$r\theta_{x_2}(r) u_{B,t} - \text{div} \{ r\theta_{x_2}(r) \nabla u_B \} = (1 - \mu_o) \frac{u_{x_2}^+ - u_{x_2}^-}{2}$$  \hspace{1cm} (4.6)$$

weakly in $B_T$ (here $\nabla$ and div are relative to $B$). Moreover,

$$u_{B,z}(r,0,t) = u_{B,z}(r,H,t) = 0$$  \hspace{1cm} (4.7)$$

for all $t \in (0,T)$ and $r \in (r_o,R)$. The regularity requirement (4.1) is not sufficient to insure that $u_0$ has a trace in $L^1(B_T)$.

### 4.3 The Limit in the Outer Shell

Denote coordinates on the limit surface $S$ by $\theta \in (-\pi,\pi)$ and $z \in (0,H)$. The restrictions of $\{u_\varepsilon\}$ to the outer shell $S_\varepsilon$ converge to a function $u_S$ defined in $S_T$ and satisfying

$$u_S \in C(0,T;L^2(S)) \cap L^2(0,T;W^{1,2}(S)) \cap C(S_T).$$  \hspace{1cm} (4.8)$$

These functions are related to the interior limit $u$ and the limit $u_B$ in the incisure, as follows. First by virtue of (4.1) the function $x \mapsto u(x,z,t)$ has a trace on $\partial D_R$, for a.e., $z \in (0,H)$, and such traces are in $L^2(S_T)$. Then,

$$u_S(\theta,z,t) = u(x,z,t) \big|_{\partial D_R} \quad \text{in} \quad L^2(S_T).$$  \hspace{1cm} (4.9)$$

Moreover

$$u_S(0,z,t) = u_B(R,z,t) \quad \text{in} \quad L^2(S_T \cap \{ |\theta| = 0 \})$$  \hspace{1cm} (4.10)$$

Next, denoting by $\Delta_S$ the Laplace–Beltrami operator on the limiting surface $S$, and by $\rho$ the radial variable on $D_R$,

$$u_{S,t} - \Delta_S u_S = -\frac{(1 - \mu_o)}{\sigma_\varepsilon_o} u_{\rho} \big|_{\partial D_R - \{ \theta = 0 \}}$$

$$- \delta_{\{\theta = 0\}} \frac{2R\theta_{\varepsilon_o}(R)}{\sigma_\varepsilon_o} u_{B,r} \big|_{r = R}$$

weakly in $S_T$, where $\delta_{\{\theta = 0\}}$ is the Dirac mass concentrated on the line $\{ \theta = 0 \}$ traced on the outer shell by the limiting incisure $B$. Moreover

$$u_{S,z}(\theta,0,t) = u_{S,z}(\theta,H,t) = 0$$  \hspace{1cm} (4.12)$$

for all $t \in (0,T)$ and $\theta \in (-\pi,\pi)$. The regularity requirements (4.1) is not sufficient to insure that $u_\rho$ has a trace in $L^1(S_T \cap \{ |\theta| > \delta \})$. Likewise the regularity class (4.4) and the equation (4.6) are not sufficient to insure that $u_{B,r}$ has a value for $r = R$. 
4.4 Weak Form of the Homogenized Limit

The following is the equivalent weak form of (4.1)–(4.12). The functions \( u, u_B, u_S \) are in the indicated regularity classes and

\[
(1 - \mu_0) \left\{ \int_{\Omega - B_T} \left\{ -u \varphi_t + \nabla u \cdot \nabla \varphi + (u - f) \varphi \right\} dx dt 
- \int_{\Omega - B} u_0 \varphi(x, 0) dx \right\}_{\text{interior}} 
+ \sigma \varepsilon_o \left\{ \int_{S} \left\{ -u_S \varphi_t + \nabla u_S \cdot \nabla \varphi \right\} d\eta dt 
- \int_{S} u_0 \varphi(x, 0) dx \right\}_{\text{outer shell}} 
+ \left\{ \int_{B_T} 2r \theta_{\varepsilon_o}(r) \left\{ -u_B \varphi_t + \nabla u_B \cdot \nabla \varphi \right\} dr dz dt 
- \int_{B_T} 2r \theta_{\varepsilon_o}(r) u_0 \varphi(x, 0) dr dz \right\}_{\text{incisure}} = 0
\]

(4.13)

valid for all testing functions \( \varphi \in W^{1,2}(\Omega_T) \) whose traces on \( S_T \) and \( B_T(\delta) \) satisfy

\[
\varphi \big|_{S_T} \in W^{1,2}(S_T); \quad \varphi \big|_{B_T} \in W^{1,2}(B_T(\delta)) \quad \text{for all } 0 < \delta \ll 1.
\]

Moreover

\[
\sqrt{r} \theta_{\varepsilon_o}(r) \left( \varphi \big|_{B_T}, \nabla \varphi \big|_{B_T} \right) \in L^2(\Omega_T).
\]

**Remark 4.1** The homogenized–concentrated limit “remembers” the shape of the initial incisure through the function \( r \rightarrow \theta_{\varepsilon_o}(r) \). Indeed \( 2r \theta_{\varepsilon_o}(r) \) is precisely the thickness of the physical incisure.

**Remark 4.2** The equation (4.6) satisfied by \( u_B \), holds on the interval \((r_o, R)\). A condition for the values of \( u_B \) at \( r = R \) is provided by (4.5). However, due to the degeneracy of the coefficient \( r \theta_{\varepsilon_o}(r) \) at \( r_o \), the limiting process does not identify the value of \( u_B \) for \( r = r_o \). It turns out that (4.6) is still well defined in the class (4.4). Such a class quantifies the interplay between the possible singularity of \( u_B \) near \( r_o \) and the vanishing of the coefficient \( r \theta_{\varepsilon_o}(r) \). According to the results of [19] and [9], the requirements (4.4) insure that the problem in the limiting blade \( B \) is well posed with no further specifications; prescribing the values of \( u_B \) at \( r = r_o \) would make the data over-determined, and in general incompatible.
5 Compactness

Proposition 5.1 Let $u_\varepsilon$ be a solution of (3.4)–(3.5) and denote by $\gamma$ a constant depending only upon the data and independent of $\varepsilon$.

\begin{equation}
0 \leq u_\varepsilon(x,t) \leq \gamma \quad \text{for all} \quad (x,t) \in \tilde{\Omega}_\varepsilon,T
\end{equation}

\begin{equation}
\sup_{0 \leq t \leq T} \left\| \sqrt{a_\varepsilon} u_\varepsilon(\cdot,t) \right\|_{2,\tilde{\Omega}_\varepsilon} + \left\| \sqrt{a_\varepsilon} \nabla u_\varepsilon \right\|_{2,\tilde{\Omega}_\varepsilon,T} \leq \gamma
\end{equation}

\begin{equation}
\int_0^{T-h} \int_{\tilde{\Omega}_\varepsilon} a_\varepsilon [u_\varepsilon(t+h) - u_\varepsilon(t)]^2 \, dx \, dt \leq \gamma h \quad \text{for all} \quad h \in (0,T).
\end{equation}

Since $u_{\varepsilon,z} = 0$ for $z = 0$ and $z = H$, by a periodic even reflection, (3.4)–(3.5) can be regarded as set in the infinite cylinder $D_{R+\sigma_\varepsilon} \times \mathbb{R}$ from which one removes a periodically layered sequence of equal discs $\{C_j\}$. By the same token, after redefining of $I_o$ and $I_n$, the interdiscal spaces $\{I_j\}$ form a periodically layered sequence of equal cylinders.

Consider the incised limiting disc $D_R - \mathcal{V}$ and for $0 < \delta \ll 1$ set

\begin{equation}
D(\delta) = \{ \mathbf{r} \in (D_R - \mathcal{V}) \mid \text{dist} \{ \mathbf{r}, \partial(D_R - \mathcal{V}) \} > \delta \}.
\end{equation}

For a fixed interdiscal space $I_j$ let $\{ z = \zeta_{2j} \}$ and $\{ z = \zeta_{2j+1} \}$ be the planes containing the faces $\partial I_j^\pm$ and set,

\begin{equation}
I(\delta) = D(\delta) \times (\zeta_{2j}, \zeta_{2j+1}); \quad I(\delta,T) = I(\delta) \times (0,T).
\end{equation}

Proposition 5.2 There exists a constant $\gamma$ independent of $\varepsilon$ and $\delta$, such that

\begin{equation}
\sup_{I(\delta,T)} \left\{ |\nabla u_\varepsilon| + |u_\varepsilon, t| \right\} \leq \frac{\gamma}{\delta^2}.
\end{equation}

Proposition 5.3 Fix two distinct interdiscal spaces $I_i$ and $I_j$ and let $h$ be such that $I_j = (0,0,h) + I_i$. For a fixed $\delta \in (0,1)$ let $I(\delta,T)$ be defined as in (5.5). There exists a constant $\alpha \in (0,1)$ depending only upon the data and independent of $\varepsilon, h, \delta$ and a positive constant $\gamma(\delta)$ depending only upon the data and independent of $h$ and $\varepsilon$, such that

\begin{equation}
\sup_{I(\delta,T)} \left| u_\varepsilon(\mathbf{r}, z+h, t) - u_\varepsilon(\mathbf{r}, z, t) \right| \leq \gamma |h|^\alpha.
\end{equation}

The proof of Propositions 5.1 and 5.2, is almost identical to the proof of the analogous Proposition 3.1 and 3.2 of [3], to which we refer for details. The proof of Proposition 5.3 is postponed to §8–§9. Next we compute the homogenized–concentrated limit of (3.4)–(3.5), by assuming these results.
6 The Interior Limit

In writing the weak formulation of (3.4)–(3.5) within $\tilde{\Omega}_{\varepsilon,T}$, fix $\delta \in (0, 1)$ and take testing functions $\varphi \in C^\infty(\Omega_T)$, such that $\varphi(\cdot, 0) = 0$, $\varphi(\cdot, T) = 0$, and such that

$$\varphi \to \varphi(x, z, t) \in C^\infty(D(\delta)) \text{ for all } z \in (0, H), t \in (0, T].$$

(6.1)

If $\varepsilon$ is sufficiently small, $a_\varepsilon = 1$ within $\tilde{\Omega}_\varepsilon \cap \{\text{supp}(\varphi)\}$. Therefore, taking into account the variational boundary data (3.5),

$$\text{(Int)}_{\varepsilon} \overset{\text{def}}{=} \sum_{j = 0}^{n} \int_{I_j} \left\{ - u_\varepsilon \varphi_t - u_\varepsilon \Delta x \varphi + u_\varepsilon z \varphi_z \right\} dx dt$$

(6.2)

$$+ \frac{1}{2} \nu \varepsilon \sum_{j = 1}^{n} \int_{\partial I_j} (u_\varepsilon - f) \varphi d\Gamma dt$$

$$+ \frac{1}{2} \nu \varepsilon \sum_{j = 0}^{n-1} \int_{\partial I_j^+} (u_\varepsilon - f) \varphi d\Gamma dt.$$

By a minor variant of the arguments of [3] there exists a function

$$u \in C(\mathbb{R}^N \times \mathbb{R}) \cap L^2(0, T; W^{1,2}(\Omega))$$

(6.3)

such that, by possibly passing to a subsequence, relabelled with $\varepsilon$,

$$\lim_{\varepsilon \to 0} \text{(Int)}_{\varepsilon} = (1 - \mu_o) \int_{\Omega_T} \left\{ - u \varphi_t + \nabla \varphi \cdot \nabla u + (u - f) \varphi \right\} dx dt.$$  

(6.4)

7 The Global Limit

In the weak formulation of (3.4)–(3.5) we now take testing functions $\varphi \in C^1(\mathbb{R}^3 \times \mathbb{R})$, vanishing for $t = T$. Write down the weak formulation and divide the various resulting integrals into the domains where the coefficients $a_\varepsilon$ are constant, i.e.,

$$\left\{ \sum_{j = 0}^{n} \int_{I_j} \left\{ - u_\varepsilon \varphi_t + \nabla u_\varepsilon \cdot \nabla \varphi \right\} dx dt - \int_{I_j} u_o \varphi(x, 0) dx \right\}$$

$$+ \frac{1}{2} \mu_\varepsilon \sum_{j = 1}^{n-1} \int_{\partial I_j} (u_\varepsilon - f) \varphi d\Gamma dt + \frac{1}{2} \nu \varepsilon \sum_{j = 1}^{n} \int_{\partial I_j} (u_\varepsilon - f) \varphi d\Gamma dt$$

$$+ \frac{\varepsilon}{\varepsilon} \left\{ \int_{S_\varepsilon} \left\{ - u_\varepsilon \varphi_t + \nabla u_\varepsilon \cdot \nabla \varphi \right\} dx dt - \int_{S_\varepsilon} u_o \varphi(x, 0) dx \right\}$$

$$+ \left\{ \int_{B_\varepsilon} \frac{\theta_\varepsilon(r)}{\theta_\varepsilon(r)} \left\{ - u_\varepsilon \varphi_t + \nabla u_\varepsilon \cdot \nabla \varphi \right\} dx dt$$

$$- \int_{B_\varepsilon} \frac{\theta_\varepsilon(r)}{\theta_\varepsilon(r)} u_o \varphi(x, 0) dx \right\} = 0.$$

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Parametrize the limiting outer shell $S$ by its cylindrical coordinates $\theta \in [0, 2\pi)$ and $z \in (0, H)$ and denote be $\nabla_S$ the gradient on $S$ formally given by
\[
\nabla_S = \left( \frac{1}{R} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right).
\]
By the same arguments of [3], there exists a function
\[
u_S \in C(\mathbb{R}^N \times \mathbb{R}) \cap L^2(0, T; W^{1, 2}(S))
\]
such that
\[
\lim_{\varepsilon \to 0} \varepsilon \left\{ \int \int_{S_{\varepsilon, t}} \left\{ -u\varphi + \nabla u \cdot \nabla \varphi \right\} dxdt - \int_{S_{\varepsilon, t}} u_0\varphi(x, 0) dx \right\} \text{outer shell}
\]
\[
= \sigma \varepsilon \left\{ \int \int_{S_T} \left\{ -u_S\varphi + \nabla u_S \cdot \nabla \varphi \right\} dydt - \int_{S} u_0\varphi(x, 0) d\eta \right\} \text{outer shell}
\]
where $d\eta$ is the surface measure on $S$. Moreover the restriction $u|_S$ of the interior limit $u$ on $S$ equals $u_S$.

Letting $\varepsilon \to 0$ in the global weak formulation and using (6.4) and (7.2) gives
\[
(1 - \mu_o) \left\{ \int \int_{\Omega_T - B_{\varepsilon}} \left\{ -u\varphi + \nabla u \cdot \nabla \varphi + (u - f)\varphi \right\} dxdt
\]
\[- \int_{\Omega_B} u_0\varphi(x, 0) dx \right\} \text{interior}
\]
\[
+ \sigma \varepsilon \left\{ \int \int_{S_T} \left\{ -u_S\varphi + \nabla u_S \cdot \nabla \varphi \right\} dydt - \int_{S} u_0\varphi(x, 0) d\eta \right\} \text{outer shell}
\]
\[
+ \lim_{\varepsilon \to 0} \left\{ \int \int_{B_{\varepsilon, t}} \frac{\theta_\varepsilon(r)}{\theta_\varepsilon(r)} \left\{ -u\varphi + \nabla u \cdot \nabla \varphi \right\} dxdt
\]
\[- \int_{B_{\varepsilon}} \frac{\theta_\varepsilon(r)}{\theta_\varepsilon(r)} u_0\varphi(x, 0) dx \right\} \text{incisure} = 0.
\]

Let $D(\delta)$ be the domain introduced in (5.4) and denote by $\Omega(\delta)$ the cylinder
\[
\Omega(\delta) = \{ DR - D(\delta) \} \times \{ 0 < z < H \}.
\]
To compute the last limit, write the various integrals in cylindrical coordinates, with pole at the origin and vector radius $r_o < r < R$ and set
\[
(r, z, t) \rightarrow \Pi(\varepsilon, r, z, t) \overset{\text{def}}{=} \frac{1}{2 \theta_\varepsilon(r)} \int_{-\theta_\varepsilon(r)}^{\theta_\varepsilon(r)} u_{\varepsilon}(r, \theta, z, t) d\theta.
\]
The testing function is chosen as \( \varphi \in C^1(\mathbb{R}^N \times \mathbb{R}) \), and independent of \( \theta \) in \( \Omega(\delta) \). For such a choice,

\[
\begin{align*}
\int_0^H \int_{r_o}^R \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} -u_{\varepsilon,\theta,\varphi(t)} \varphi \, d\theta \, dr \, dz &= \int \int -2r\theta_{\varepsilon}(r) \overline{u}_{\varepsilon,\varphi} \, dr \, dz \\
\int_0^H \int_{r_o}^R \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} u_{\varepsilon,\varphi,\varphi} \, d\theta \, dr \, dz &= \int \int 2r\theta_{\varepsilon}(r) \overline{u}_{\varepsilon,\varphi} \, dr \, dz \\
\int_0^H \int_{r_o}^R \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} u_{\varepsilon,\varphi}(x,0) \, d\theta \, dr \, dz &= \int \int 2r\theta_{\varepsilon}(r) u_{\varepsilon,\varphi}(x,0) \, dr \, dz
\end{align*}
\]

Keeping in mind that \( \varphi \) is independent of \( \theta \) in a neighborhood of the incisure, we transform and estimate the integral involving \( u_{\varepsilon,r} \) as follows.

\[
\int \frac{1}{2} \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} u_{\varepsilon,r} \, d\theta = \overline{u}_{\varepsilon,r} \tag{7.6}
\]

The last term in braces is majorized by

\[
\int \theta_{\varepsilon}(r) \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} u_{\varepsilon,\theta,\varphi} \, d\theta \, dr \, dz = \int \int 2r\theta_{\varepsilon}(r) \overline{u}_{\varepsilon,\varphi} \, dr \, dz + J_z.
\]

From this

\[
\int_0^H \int_{r_o}^R \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} u_{\varepsilon,\varphi} \, d\theta \, dr \, dz = \int \int 2r\theta_{\varepsilon}(r) \overline{u}_{\varepsilon,\varphi} \, dr \, dz + J_z.
\]

The last term \( J_z \) is majorized by making use of the energy estimates (5.2) and Lemma 2.1. Since \( \theta_{\varepsilon,r}(r) \geq 0 \)

\[
\begin{align*}
\int_0^T |J_z| \, dt &\leq 2 \sup |\varphi_t| \int_0^T \int_0^H \int_{r_o}^R \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \theta_{\varepsilon,r}(r) \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} \left| u_{\varepsilon,\theta} \right| \, d\theta \, dr \, dz \, dt \\
&\leq \gamma \int_0^T \int_0^H \int_{r_o}^R \frac{\theta_{\varepsilon}(r)}{\theta_{\varepsilon}(r)} \theta_{\varepsilon,r}(r) \left( \int_{-\theta_{\varepsilon}(r)}^{\theta_{\varepsilon}(r)} u_{\varepsilon,\theta,\varphi}^2 \, d\theta \right)^{1/2} \, dr \, dz \, dt \\
&\leq \gamma' \left( \int_{r_o}^R r^2 \theta_{\varepsilon,r}(r) \, dr \right)^{1/2} \leq \gamma'' \sqrt{O(\varepsilon)} \left( \int_{r_o}^R \theta_{\varepsilon,r}(r) \, dr \right)^{1/2} = O(\varepsilon).
\end{align*}
\]

Similar arguments, starting from (7.6) give

\[
\int \int \theta_{\varepsilon}(r) \left( \overline{u}_{\varepsilon,\varphi}^2 + \overline{u}_{\varepsilon,\varphi,\varphi}^2 \right) \, dr \, dz \, dt \leq \gamma \quad \text{uniformly in } \varepsilon. \quad \tag{7.7}
\]

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Moreover one verifies that the averages $\bar{u}_\varepsilon$ share the same time–regularity as in
(5.3). For $0 < \delta \ll 1$ let $B(\delta)$ be defined as in (4.2). The previous remarks and
estimates imply that
$$\bar{u}_\varepsilon \in C(0, T; L^2(B)) \cap L^2(0, T; W^{1,2}(B(\delta)))$$
(7.8)
uniformly in $\varepsilon$ for every $0 < \delta \ll 1$ fixed. Moreover
$$\sqrt{\theta_{\varepsilon_0}(r)} \nabla \bar{u}_\varepsilon \in L^2(B_T) \quad \text{uniformly in } \varepsilon.$$  (7.9)

This implies that there exists a function $u_B$ in the same regularity classes as
(7.8)–(7.9) such that $\{u_\varepsilon\} \to u_B$ strongly in $L^2(B_T)$ and moreover,
$$\sqrt{\theta_{\varepsilon_0}(r)} \nabla u_\varepsilon \to \sqrt{\theta_{\varepsilon_0}(r)} \nabla u_B \quad \text{weakly in } L^2(B_T).$$

Letting now $\varepsilon \to 0$ in (7.3) establishes the weak formulation (4.13) for all smooth
testing functions $\varphi$ that are independent of $\theta$ in a neighborhood of the limiting
incisure $B$. The next Lemma establishes that these functions are dense in
$W^{1,2}(\mathbb{R}^3 \times \mathbb{R})$ thereby completing the limiting process.

**Lemma 7.1** Let $\varphi \in W^{1,2}(\Omega_T) \cap W^{1,2}(S_T) \cap W^{1,2}(B_T) =: X$. For every $\varepsilon > 0$
there exists $\varphi_\varepsilon \in C^1(\mathbb{R}^3 \times \mathbb{R})$, independent of $\theta$ in a neighborhood of $B_T$, such
that $\|\varphi - \varphi_\varepsilon\|_X < \varepsilon$.

**Proof:** It suffices to assume $\varphi$ dependent only on the transversal variables $\overline{x}$
on the square $Q = \max\{|x_1|, |x_2|\} < 2$, with the segment defining the limiting incisure, being replaced by $\{|x_1| < 1\}$. It suffices also to assume $\varphi$ Lipschitz continuous in $\overline{Q}$ with some Lipschitz constant $L$. Let $Q_\varepsilon$ be the rectangle
$$Q_\varepsilon = \{|x_1| < 1 + \varepsilon\} \times \{|x_2| < \varepsilon\}$$
and define
$$\tilde{\varphi}_\varepsilon(x_1, x_2) = \begin{cases} \varphi(x_1, 0) & \text{in } Q_\varepsilon \\ \varphi(x_1, x_2) & \text{in } Q - \overline{Q}_2\varepsilon \end{cases}$$

This function is Lipschitz continuous, with Lipschitz constant not to exceed $2L$.
The Kirzbraun–Pucci extension $\varphi_\varepsilon$ is defined in the whole $Q$ with the same
Lipschitz modulus of continuity (see [8]). One verifies that such an extension
approximates $\varphi$ in the suitable Sobolev norm, up to redefining $\varepsilon$.

### 8 Proof of Proposition 5.3–Part I

Having fixed $I_i$ and $I_j$ and $h$ as in the statement, set
$$\overline{\varphi}(\overline{x}, z, t) = u_\varepsilon(\overline{x}, z, t) - u_\varepsilon(\overline{x}, z + h, t)$$
$$\overline{f}(\overline{x}, z, t) = f(\overline{x}, z, t) - f(\overline{x}, z + h, t) \quad \text{in } I_i.$$  (8.1)
Then $\Phi$ satisfies the boundary value problem in $I_{i,T}$,
\[
\begin{align*}
\Phi_t - \Delta \Phi &= 0, \\
\Phi(x,z,0) &= 0, \\
\Phi_x(x,\zeta_{2i+1},t) &= -\frac{1}{2} \nu \varepsilon \Phi(x,\zeta_{2i+1},t), \\
\Phi_x(x,\zeta_{2i},t) &= \frac{1}{2} \nu \varepsilon \Phi(x,\zeta_{2i},t).
\end{align*}
\] (8.2)

This boundary value problems does not contain the values of $\Phi$ on the lateral boundary $\Lambda_i$ of $I_i$. While such an information is not directly available, it turns out that to establish the Hölder estimate in (5.7) it suffices to have only an estimate of such boundary values in their $L^1(\Lambda_{i,T})$–average.

### 8.1 Estimating the integral of $\Phi$ on $\Lambda_{i,T}$

Subdivide $\Lambda$ into its portion touching the incisure $B_\varepsilon$ and the remaining one, i.e.,
\[
\Lambda_{i}^{\text{inc}} = \Lambda_i \cap \{|\theta| < \theta_\varepsilon(R)\}, \quad \Lambda_{i}^{\text{out-shell}} = \Lambda_i - \Lambda_{i}^{\text{inc}}.
\]

**Lemma 8.1** There exists a constant $\gamma$ depending only upon the data and independent of $\varepsilon$ and $h$, such that,
\[
\frac{1}{|\Lambda_{i,T}^{\text{out-shell}}|} \iint_{\Lambda_{i,T}^{\text{out-shell}}} |\Phi(x,z,t)| \, d\eta dt \leq \gamma \sqrt{|h|}, \quad (8.3)
\]
\[
\iint_{\Lambda_{i,T}^{\text{inc}}} \beta_\varepsilon h(x_1 - r_0) |\Phi|^2 \, d\eta dt \leq \gamma \varepsilon (\varepsilon h + \varepsilon^2), \quad (8.4)
\]

where $d\eta$ is the surface measure on $\Lambda_i$.

**Proof:** The index $i$ being fixed, we drop it in this proof for the sake of notational simplicity.

The portion of $\Lambda$ facing the outer shell is a connected portion of a right cylindrical surface. Thus by essentially the same arguments of [3], the estimate (8.3) follows.

For the estimate on $\Lambda_{i}^{\text{inc}}$, let
\[
\Lambda^+ = \Lambda_{i}^{\text{inc}} \cap \{x_2 > 0\}; \quad \Lambda^- = \Lambda_{i}^{\text{inc}} \cap \{x_2 < 0\};
\]
\[
\mathcal{V}_\varepsilon^+ = \mathcal{V}_\varepsilon \cap \{x_2 > 0\}; \quad \mathcal{V}_\varepsilon^- = \mathcal{V}_\varepsilon \cap \{x_2 < 0\};
\]
and let $x_2 = x_2(x_1) = \beta_\varepsilon h(x_1 - r_0)$ be the Cartesian representation of the curve delimiting the incisure $\mathcal{V}_\varepsilon^+$ as in (2.1). Then for all $r_0 < x_1 < R \cos \theta_\varepsilon(R)$ and all $-x_2(x_1) < y < 0$
\[
f(x_1, x_2(x_1), z + h) - f(x_1, x_2(x_1), z) = f(x_1, y, z + h) - f(x_1, y, z)
\]
\[
+ \int_{y}^{x_2(x_1)} [f_x(x_1, s, z + h) - f_x(x_1, s, z)] \, ds
\]
for a smooth function of its arguments. From this,

\[
|f(x_1, x_2(x_1), z + h) - f(x_1, x_2(x_1), z)|^2 \\
\leq 2 |f(x_1, y, z + h) - f(x_1, y, z)|^2 \\
+ 2 \left| \int_{x_2(x_1)}^{x_2(x_1 + h)} [f_{x_2}(x, s, z + h) - f_{x_2}(x, s, z)] \, ds \right|^2
\]

For a fixed \( x_1 \in (r_0, R \cos \theta_z(R)) \), integrate both sides in \( dy \) for \( y \in (-x_2(x_1), 0) \) to get

\[
x_2(x_1) |f(x_1, x_2(x_1), z + h) - f(x_1, x_2(x_1), z)|^2 \\
\leq 2 \int_{-x_2(x_1)}^{0} |f(x_1, y, z + h) - f(x_1, y, z)|^2 \, dy \\
+ 2 \int_{-x_2(x_1)}^{0} \left( \int_{y}^{x_2(x_1)} [f_{x_2}(x, s, z + h) - f_{x_2}(x, s, z)] \, ds \right)^2 \, dy
\]

Now integrate over the upper curve \( x_2 = x_2(x_1) \), for the elemental arc \( d\ell = \sqrt{1 + x_2'^2} \, dx_1 \), and then in \( dz \) for \( z \in (\zeta_2, \zeta_{2+1}) \). This gives,

\[
\int_{\Lambda^+} x_2(x_1) |f(\bar{x}, z + h) - f(\bar{x}, z)|^2 \, d\ell \, dz \\
\leq \gamma \int_{\zeta_2}^{\zeta_{2+1}} \int_{r_0}^{R \cos \theta_z(R)} \int_{-x_2(x_1)}^{0} |f(x_1, y, z + h) - f(x_1, y, z)|^2 \, dy \, dx_1 \, dz \\
+ \gamma \int_{\zeta_2}^{\zeta_{2+1}} \int_{r_0}^{R \cos \theta_z(R)} x_2(x_1) \\
\left( \int_{-x_2(x_1)}^{x_2(x_1)} \left( |f_{x_2}(x, y, z + h)| + |f_{x_2}(x, y, z)| \right) \, dy \right)^2 \, dx_1 \, dz.
\]

We estimate further

\[
\int_{\Lambda^+} x_2(x_1) |f(\bar{x}, z + h) - f(\bar{x}, z)|^2 \, d\ell \, dz \\
\leq \gamma \int_{\zeta_2}^{\zeta_{2+1}} \int_{V_x} \int_{z}^{z+h} |f(\bar{x}, z + h) - f(\bar{x}, z)|^2 \, d\bar{x} \, dz \\
+ \gamma \int_{\zeta_2}^{\zeta_{2+1}} \int_{V_x} x_2^2(x_1) \left( |\nabla f(\bar{x}, z + h)|^2 + |\nabla f(\bar{x}, z)|^2 \right) \, d\bar{x} \, dz \\
\leq \gamma h \int_{\zeta_2}^{\zeta_{2+1}} \int_{z}^{z+h} \int_{V_x} |f_z(\bar{x}, \zeta)|^2 \, d\bar{x} \, d\zeta \, dz \\
+ \gamma \int_{\zeta_2}^{\zeta_{2+1}} \int_{V_x} \left( |\nabla f(\bar{x}, z + h)|^2 + |\nabla f(\bar{x}, z)|^2 \right) \, d\bar{x} \, dz.
\]
Repeat the same estimates for $\Lambda^-$, and write the sum of the resulting inequalities for $f$ replaced by $u_\epsilon$. Using the notation in the first of (8.1), recalling that $u_\epsilon$ can be regarded as periodic in $z$ and integrating also in $dt$ over $(0, T)$ gives

$$\int_{\Lambda^inc} x_2(x_1)|u|^2\,d\eta\,dt \leq \gamma h \int_0^T \int_{\zeta_{2i}+1}^{\zeta_{2i+1}} \int_Z |u_\epsilon(\overline{\tau}, \zeta)|^2 d\overline{\tau} d\zeta\,dz\,dt + \gamma \epsilon^2 \int_0^T \int_{\zeta_{2i}+1}^{\zeta_{2i+1}} \int_{V_\epsilon} |\nabla u_\epsilon|^2\,dz\,dt = I_1 + I_2,$$

where $d\eta$ is the surface measure on $\Lambda^inc$. These integrals are majorized by making use of the energy estimates (5.2), and by observing that

$$\frac{\theta_\epsilon(r)}{\theta_{\epsilon\nu}(r)} \leq \gamma(\epsilon_\nu) \epsilon,$$

which follows from the polar representation (2.6). Therefore

$$I_1 \leq \gamma \epsilon^2 h \|\sqrt{a_\epsilon} \nabla u_\epsilon\|_{L^2(I_{\nu\epsilon})}^2 \quad \text{and} \quad I_2 \leq \gamma \epsilon^3 \|\sqrt{a_\epsilon} \nabla u_\epsilon\|_{L^2(I_{\nu\epsilon})}^2.$$

Combining these estimates, (8.4) follows.

9 Proof of Proposition 5.3. Part II

To proceed, in (8.2) perform the change of variables $z \to (z - \zeta_{2i})$ and continue to denote by $z$ the transformed variables and by $\overline{\tau}$ the transformed function. The domain $I_i$ is mapped into $I = (D_R - V_\epsilon) \times \{0 < z < \nu\epsilon\}$, and (8.2) continues to hold in $I_T$ with the same boundary conditions. Denote by $\Lambda$ the lateral boundary of $I$ which is the transformed of $\Lambda_i$. Fix a non–negative function $\xi_\lambda \in C_0^\infty(I)$ and consider the boundary value problem,

$$\begin{aligned}
\phi \in W^{1,2}(0, T; L^2(I)) \cap L^2(0, T; W^{1,2}(I)); \\
\phi_t + \Delta \phi = -\xi_\lambda; \\
\phi(x, T) = 0; \\
\phi(\overline{\tau}, z, t) \big|_{\Lambda} = 0; \\
\phi_z(\overline{\tau}, 0, t) = \frac{1}{2} \nu \epsilon \phi(\overline{\tau}, 0, t); \\
\phi_z(\overline{\tau}, \nu\epsilon, t) = -\frac{1}{2} \nu \epsilon \phi(\overline{\tau}, \nu\epsilon, t).
\end{aligned}
$$

The solution of (9.1) is non–negative and it can be constructed by the Galerkin procedure. A basis for the Galerkin procedure can be constructed since, for all $0 < \epsilon \ll 1$, the incisure has Newtonian capacity bounded below by the capacity of a rectangle, which in $R^3$ is positive. This gives $\phi_t \in L^2(I_T)$, with upper bounds depending upon the $L^2(I_T)$–norm of $\xi_\lambda$.

Multiply (9.1) by $\overline{\tau}$ and integrate by parts the Laplacian of $\phi$ over $I_T$. Next take $\phi$ as testing function in the weak formulation of (8.2). Adding the resulting
inequalities,

\[ \left| \int_T \int_{I_T} \xi_{\lambda} \partial u dr dt \right| = - \int_0^T \int_{\Lambda} \nabla \varphi \cdot n \partial u dr dt \]

\[ - \frac{1}{2} \nu \int_0^T \int_{D_R - V_{\varepsilon}} \{ (\varphi \overline{T})(x,0,t) + (\varphi \overline{T})(x,\nu \varepsilon,t) \} d\overline{T} dt \]

\[ \leq \left| \int_0^T \int_{\Lambda} \nabla \varphi \cdot n \partial u dr dt \right| \]

\[ + \gamma \varepsilon \int_0^T \int_{D_R - V_{\varepsilon}} \{ \varphi \overline{T}(x,0,t) + \varphi \overline{T}(x,\nu \varepsilon,t) \} d\overline{T} dt \]

where \( n \) is the outward unit normal to \( \Lambda \). These calculations are rigorous and can be justified by local regularization. The continuity of \( u_{\varepsilon} \) will result from estimating the right hand side of (9.2) independent of \( \varepsilon, h \) and the choice of the \( \xi_{\lambda} \in C_0^\infty(T) \). This is preceded by some estimations of the function \( \varphi \).

### 9.1 Estimating \( \varphi \) Away from the Support of \( \xi_{\lambda} \)

For \( 0 < \delta < 1 \) let \( D(\delta) \) be defined as in (5.4) and set also,

\[ D_{\varepsilon}(\delta) = \{ x \in (D_R - V_{\varepsilon}) \mid \text{dist} \{ x; \partial(D_R - V_{\varepsilon}) \} > \delta \} \]

\[ I_{\varepsilon}(\delta) = D_{\varepsilon}(\delta) \times \{ 0 < z < \nu \varepsilon \}; \quad I_{\varepsilon}(\delta,T) = I_{\varepsilon}(\delta) \times (0,T); \]

\[ D^\varepsilon(\delta) = D_R - D_{\varepsilon}(\delta); \]

\[ I^\varepsilon(\delta) = D^\varepsilon(\delta) \times \{ 0 < z < \nu \varepsilon \}; \quad I^\varepsilon(\delta,T) = I^\varepsilon(\delta) \times (0,T). \]

By these definitions \( D_{\varepsilon}(\delta) \subset (D_R - V_{\varepsilon}) \), and the incisure \( V_{\varepsilon} \) is contained in \( D^\varepsilon(\delta) \), provided \( 0 < \varepsilon < \delta \) is sufficiently small. Select \( \xi_{\lambda} \in C_0^\infty(I_T) \) as an approximation of the identity at some fixed point \( (x_o,z_o,t_o) \in I(4\delta,T) \). The kernels \( \xi_{\lambda} \) satisfy,

\[ \int_{I_T} \xi_{\lambda} dx dt = 1 \quad \text{for all } \lambda > 0, \quad \text{supp} \xi_{\lambda} \subset B_{\lambda}(x_o,z_o,t_o), \]

and

\[ \lim_{\lambda \to 0} \int_{I_T} \xi_{\lambda} \psi(x,z,t) dx dt = \psi(x_o,z_o,t_o) \]

for all continuous functions \( \psi \) defined in \( I_T \).

**Proposition 9.1** Let \( 0 < \varepsilon < \delta \) be so small that \( D(\delta) \subset (D_R - V_{\varepsilon}) \), and \( V_{\varepsilon} \subset D^\varepsilon(\delta) \). There exists a constant \( \gamma \) depending only upon the data and independent of \( \varepsilon, \lambda, \delta \) such that

\[ \sup_{I^\varepsilon(\delta,T)} \varphi \leq \frac{\gamma}{\delta^2} \frac{1}{\varepsilon} \]

Moreover

\[ 0 \leq -\nabla \varphi \cdot n \bigg|_{\Lambda \text{out-shell}} \leq \frac{\gamma}{\delta^2 |\ln(1 - \delta)|} \frac{1}{\varepsilon} \]
and
\[
0 \leq - (\nabla \varphi \cdot \mathbf{n}) (x_1, x_2) \bigg|_{\operatorname{Inc}} \leq \tilde{\gamma}(\delta) \frac{\gamma}{\delta^2 \varepsilon} \sqrt{\frac{h'(x_1 - r_o)}{h(x_1 - r_o)}} \tag{9.5}
\]
for all \( r_o \leq x_1 < R \cos \theta_{\varepsilon, \text{max}} \). Here \( \tilde{\gamma}(\delta) \) is a constant dependent on \( \delta \) but independent of \( \varepsilon \) and \( \delta \), and \( h(\cdot) \) is the function entering in the Cartesian representation of the upper boundary of the incisure as in (2.1).

**Proof:** The first two of these are established by a minor variant of the arguments in [3], to which we refer for details. To prove (9.5) fix \( x_1 \in (r_o, R \cos \theta_{\varepsilon, \text{max}}) \) and let \( (x_1, \pm x_2(x_1)) \) be the corresponding points on the upper and lower curves bounding the incisure \( \mathcal{V}_\varepsilon \), symmetric with respect to the horizontal line \( x_2 = 0 \).

From such points draw the tangent lines to this curve and let \( \xi_1 \) be their common intercept. By direct calculation
\[
(x_1 - \xi_1) = \frac{h(x_1 - r_o)}{h'(x_1 - r_o)}
\]
and
\[
\rho(x_1) \overset{\text{def}}{=} \text{dist} \{ (x_1, \pm x_2(x_1)) ; (\xi_1, 0) \} = \frac{h(x_1 - r_o)}{h'(x_1 - r_o)} \sqrt{1 + \beta^2 h^2(x_1 - r_o)}.
\]

For fixed \( x_1 \in (r_o, R \cos \theta_{\varepsilon, \text{max}}) \), introduce a polar coordinate system with pole at \( (\xi_1, 0) \), shifted to be the origin, and polar coordinates \( (\rho, \theta) \) and set
\[
\theta_o = \arctan \beta \epsilon h'(x_1 - r_o).
\]

Consider the function
\[
\psi_o(\rho, \theta) = \rho \frac{\gamma}{2 \pi - 2 \theta_o} \sin \frac{\pi \theta - \pi \theta_o}{2 \pi - 2 \theta_o} \text{ and } \psi(\rho, \theta) = \tilde{\gamma}(\delta) \frac{\gamma}{\delta^2 \varepsilon} \psi_o,
\]
where \( \gamma \) is the same constant appearing in (9.3) and \( \tilde{\gamma}(\delta) \) is chosen so that \( \tilde{\gamma}(\delta) \psi_o \geq 1 \) on the portion of the boundary of \( D'_\varepsilon(\delta) \) interior to \( D_R \). One verifies that \( \psi \) is harmonic and positive, in the sector
\[
\Sigma(x_1) \overset{\text{def}}{=} \{ \theta_o < \theta < 2\pi - \theta_o \}
\]
and it vanishes for \( \theta = \theta_o \) and \( \theta = 2\pi - \theta_o \). Moreover \( \psi_z = 0 \) on \( z = 0, \nu \varepsilon \), and by the indicated choice of \( \tilde{\gamma}(\delta) \) and (9.3), it exceeds \( \varphi \) in the boundary of \( \mathcal{I}_\varepsilon(\delta) \).

Therefore \( \psi \) is a majorant of \( \varphi \) on such a set, and both vanish at \( (x_1, \pm x_2(x_1)) \).

Therefore denoting by \( \mathbf{n} \) the unit normal to the boundary of \( \partial \mathcal{V}_\varepsilon \) at \( (x_1, x_2(x_1)) \)
\[
0 \leq - (\nabla \varphi \cdot \mathbf{n}) \leq - (\nabla \psi \cdot \mathbf{n}) = \frac{1}{\rho(x_1)} \frac{\partial \psi}{\partial \theta} \]
\[
\leq \tilde{\gamma}(\delta) \frac{\gamma}{\delta^2 \varepsilon} \sqrt{\frac{h'(x_1 - r_o)}{h(x_1 - r_o)}}
\]

\[\blacksquare\]
9.2 Proof of Proposition 5.3 Concluded

Reasoning as in [3], one shows that the last term in (9.2) is bounded above by \( \gamma h \), and that

\[
- \int_0^T \int_{\Lambda_i^{out-shell}} \| \varphi \| \nabla \varphi \cdot \mathbf{n} \, d\eta dt \leq \gamma \sqrt{h}.
\]

It is only left to estimate a similar integral on \( \Lambda_i^{inc} \).

For \( p \geq 2 \), and \( x_2(x_1) = \beta \varepsilon h(x_1 - r_o) \),

\[
- \int_{\Lambda_i^{inc}} \| \varphi \| \nabla \varphi \cdot \mathbf{n} \, d\eta dt \leq \frac{\gamma(\delta) \| u \|_{\infty}}{\varepsilon} \left( \int_{\Lambda_i^{inc}} x_2(x_1) \| u \|^2 \, d\eta dt \right)^{\frac{p-1}{p}}
\]

\[
\leq \frac{\gamma(\delta, \| u \|_{\infty}, T)}{\varepsilon^{\frac{p-1}{p}}} \left( \int_{\Lambda_i^{inc}} x_2(x_1) \frac{1}{h} \left( \frac{h'}{h} \right)^{\frac{p}{p-1}} dx_1 \right)^{\frac{p-1}{p}}
\]

Note that the integral above is finite owing to assumption (2.3), at least if \( p \) is chosen large enough.

With these estimates at hand, the proof is now concluded as in [3].

10 Applications to Visual Transduction

The membrane on the lateral boundary of the rod outer segment contains ionic channels. These are kept open by the presence of cGMP (Cyclic Guanosin Monophosphate), allowing the influx of Calcium ions \( \text{Ca}^{2+} \). When cGMP is depleted the channels close, thereby causing a drop in membrane current. Vision is mediated by these variations of ionic current on the boundary of the rods ([15]). In turn, the depletion of cGMP is caused by a biochemical cascade triggered by the capture of a photon by one of the discs, say \( C_j \).

Both cGMP and \( \text{Ca}^{2+} \) can diffuse within the cytosol. They cannot penetrate the discs, although cGMP can be depleted or generated by sources located on the faces of the discs.

Denote by \( u_{\varepsilon_o} \) and \( v_{\varepsilon_o} \) the volumic, dimensionless concentrations of cGMP and \( \text{Ca}^{2+} \) in the cytosol and rescale lengths and times so that the various parameters, for example \( R, \varepsilon_o \) are all dimensionless. The functions \( u_{\varepsilon_o} \) and \( v_{\varepsilon_o} \)
satisfy the diffusion equations,

\[ u_{\varepsilon_o} \varepsilon_o, t - D_u \Delta u_{\varepsilon_o} = 0 \quad \text{in } \tilde{\Omega}_{\varepsilon_o, T} \]
\[ v_{\varepsilon_o} \varepsilon_o, t - D_v \Delta v_{\varepsilon_o} = 0 \]

where \( D_u \) and \( D_v \) are the cytosolic diffusion coefficients of cGMP and Ca\(^{2+} \) respectively. Their non-linear coupling occurs through their fluxes on the faces \( \partial I_j^\pm \subset \Omega_{\varepsilon_o} \)

\[ D_u u_{\varepsilon_o, z} = -\frac{1}{2} \nu_{\varepsilon_o} \left\{ \gamma_o u_{\varepsilon_o} - f(v_{\varepsilon_o}) \right\} - \chi_{\{z = z_*\}} u_{\varepsilon_o} f_1(v_{\varepsilon_o}, x, t) \quad \text{on } \partial I_j^+; \]
\[ D_u u_{\varepsilon_o, z} = \frac{1}{2} \nu_{\varepsilon_o} \left\{ \gamma_o u_{\varepsilon_o} - f(v_{\varepsilon_o}) \right\} \quad \text{on } \partial I_j^- \quad (10.1) \]

where \( \gamma_o \) is a given positive constant and \( f \) and \( f_1 \) are given, positive, bounded, smooth functions of their arguments. The coordinate \( z = z_* \) is that of the face \( \partial I_j^+ \) hit by the photon. The characteristic function \( \chi_{\{z = z_*\}} \) permits one to account for the depletion sources, localized at \( z_* \) and due to the action of the photon. Moreover cGMP does not penetrate the discs \( C_j \) through their lateral boundaries \( L_j \), nor it can exit the boundary of the rod, i.e.,

\[ \nabla u_{\varepsilon_o} \cdot n = 0 \quad \text{on } \left\{ \begin{array}{l} L_j / \bar{n} = R + \sigma_{\varepsilon_o} \\ z = 0 \\ z = H. \end{array} \right. \quad (10.2) \]

Calcium \( v_{\varepsilon_o} \) does not penetrate the discs \( C_j \) nor outflows the rod through its bottom \( z = 0 \) or top \( z = H \), i.e.,

\[ \nabla v_{\varepsilon_o} \cdot n = 0 \quad \text{on } \left\{ \begin{array}{l} L_j / \partial I_j^\pm \\ z = 0 \\ z = H. \end{array} \right. \quad (10.3) \]

However it can flow through the lateral boundary of the rod,

\[ D_v \nabla v_{\varepsilon_o} \cdot n = -g_1(v_{\varepsilon_o}) + g_2(u_{\varepsilon_o}) \quad \text{on } \bar{n} = R + \sigma_{\varepsilon_o} \quad (10.4) \]

for given positive, bounded, smooth functions \( g_1(\cdot) \) and \( g_2(\cdot) \). Here \( n \) is the outward unit exterior normal to \( \Omega_{\varepsilon_o} \) at the indicated surfaces. The phenomenon starts at time \( t = 0 \) from a dark equilibrium, when the system is in a constant steady state, i.e., \( u_{\varepsilon_o} = u_o \) and \( v_{\varepsilon_o} = v_o \) for two given positive constants \( u_o \) and \( v_o \). A complete description of the phototransduction cascade, as well as a detailed derivation of the model above is in [15, 4].

The values \( \varepsilon_o \ll R \) suggests the homogenization process described in the previous Sections. Here by letting \( \varepsilon \to 0 \) along a suitable sequence the face
\( z = z_* \) is kept at a constant \( z \) level. Thus \( j_* = j_*(\varepsilon) \) but \( \zeta_{2j_*+1} = z_* \) for all \( \varepsilon \leq \varepsilon_o \).

Consider the interdiscal space \( I_{j_*} \) whose face \( \partial I_{j_*}^+ \) is hit by the photon. This and the incisure are the only physical compartments where cGMP can flow from the outer shell to the depletion sites activated by the photon. To keep the spatial localization of the activation site, the width of \( I_{j_*} \) is sent to zero by a capacity concentration similar to the one in the outer shell, and in the incisure.

The homogenized–concentrated limit has the same structure as that seen above, except that concentrating the mass on \( I_{j_*} \) gives rise to a further transversal 2-dimensional diffusion on \( D_R - V \) at the level \( z_* \) both for cGMP and Calcium. Their exterior fluxes at \( z_* \) serve as sources in the boundary and incisure diffusion processes, localized through a Dirac mass at \( z_* \).

### 10.1 The Approximating Problems

The functions in play are \( u_{\varepsilon} \) and \( v_{\varepsilon} \) defined in \( \tilde{\Omega}_{\varepsilon,T} \) and representing dimensionless approximations of cGMP and Ca\(^{2+} \). Set,

\[
a_{\varepsilon}(x) = \begin{cases} 
1, & \text{for } x \in \bigcup_{i \neq j} I_j \\
\frac{\varepsilon_o}{\varepsilon}, & \text{for } x \in (I_{j_*} - B_{\varepsilon}) \cup S_{\varepsilon} \\
\frac{\theta_{\varepsilon_o}(r)}{\theta_{\varepsilon}(r)}, & \text{for } x \in B_{\varepsilon}.
\end{cases}
\]

The families \( \{u_{\varepsilon}\} \) and \( \{v_{\varepsilon}\} \) satisfy

\[
u_{\varepsilon} u_{\varepsilon}, v_{\varepsilon} \in C(0, T; L^2(\tilde{\Omega}_{\varepsilon})) \cap L^2(0, T; W^{1,2}(\tilde{\Omega}_{\varepsilon}))
\]

and

\[
a_{\varepsilon}(x) \frac{\partial}{\partial t} u_{\varepsilon} - D_u \text{div} a_{\varepsilon}(x) \nabla u_{\varepsilon} = 0 \quad \text{weakly in } \tilde{\Omega}_{\varepsilon,T}.
\]

\[
a_{\varepsilon}(x) \frac{\partial}{\partial t} v_{\varepsilon} - D_v \text{div} a_{\varepsilon}(x) \nabla v_{\varepsilon} = 0
\]

The boundary conditions completing these equations are given as follows. The flux conditions (10.2) and (10.3) translate to similar homogeneous conditions for \( u_{\varepsilon} \) and \( v_{\varepsilon} \), whereas (10.1) are rescaled to

\[
\frac{\varepsilon_o}{\varepsilon} D_u u_{\varepsilon,z} = \frac{1}{2} \nu_{\varepsilon_o} \{ \gamma_o u_{\varepsilon} - f(v_{\varepsilon}) \} \quad \text{on } \partial I_{j_*}^+
\]

\[
- \chi_{(z = z_*)} u_{\varepsilon} f_1(v_{\varepsilon}, x, t)
\]

\[
\frac{\varepsilon_o}{\varepsilon} D_v v_{\varepsilon,z} = \frac{1}{2} \nu_{\varepsilon_o} \{ \gamma_o u_{\varepsilon} - f(v_{\varepsilon}) \} \quad \text{on } \partial I_{j_*}^-
\]

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Finally, (10.4) becomes
\[ \frac{\varepsilon}{\varepsilon} D_{v} \nabla v_{e} \cdot n = -g_{1}(v_{e}) + g_{2}(u_{e}) \quad \text{on } |\mathbf{r}| = R + \sigma \varepsilon. \]
Here \( n \) is the unit exterior normal to \( \tilde{\Omega}_{\varepsilon} \) on the indicated surfaces. The initial conditions are those of dark equilibrium, i.e., \( u_{e}(\cdot, 0) = u_{o} \) and \( v_{e}(\cdot, 0) = v_{o} \) for two given positive constants \( u_{o}, v_{o} \).

10.2 The Homogenized–Concentrated Limit

As \( \varepsilon \to 0 \) the sets \( \tilde{\Omega}_{\varepsilon,T}, S_{\varepsilon,T}, B_{\varepsilon,T} \) tend formally to \( \Omega_{T}, S_{T}, B_{T} \). The interdiscal space \( I_{j} \) tends formally to the “activated” disc \( D_{*} = D_{R} \times \{ z_{*} \} \). As \( \varepsilon \to 0 \), the \( (u_{e}, v_{e}) \) generates four pairs of functions

- \( u, v \) defined in \( \Omega_{T} - B_{T} \), called the interior limit
- \( u_{*}, v_{*} \) defined in \( D_{*} - V_{T} \), called the limit on the activated level \( z_{*} \)
- \( u_{S}, v_{S} \) defined in \( S_{T} \), called the limit in the outer shell
- \( u_{B}, v_{B} \) defined in \( B_{T} \), called the limit in the incisure

10.2.1 The Interior Limit

The interior limit \( (u, v) \) satisfies
\[ u, v \in C(0, T; L^{2}(\Omega)) \cap L^{2}(0, T; L^{2}(0, H; W^{1,2}(D_{R}))) \] (10.6)
and
\[ u_{t} - D_{u} \Delta u = -\{ \gamma_{o} u - f(v) \} \quad \text{weakly in } \Omega_{T} - B_{T}. \]
\[ v_{t} - D_{v} \Delta v = 0 \]
This is a family of diffusion processes in \( D_{R} - V \), parametrized with \( z \in \langle 0, H \rangle \). The boundary fluxes in the first of (10.5) are transformed into sources in \( \Omega \).

10.2.2 The Limit on the Activated Level \( z_{*} \)

The limit \( (u_{*}, v_{*}) \) on the activated level \( z_{*} \) satisfies
\[ u_{*}, v_{*} \in C(0, T; L^{2}(D_{*})) \cap L^{2}(0, T; W^{1,2}(D_{*})) \]
and
\[ u_{*,t} - D_{u} \Delta u_{*} = \{ \gamma_{o} u_{*} - f(v_{*}) \} - \frac{1}{\nu_{\varepsilon}} u_{*} f_{1}(v_{*}, \mathbf{r}, z_{*}, t) \]
\[ v_{*,t} - D_{v} \Delta v_{*} = 0. \]
weakly in \( \{ D_{*} - V \}_{T} \). Thus at the activated level \( z_{*} \) the concentrated limit is a diffusion equation holding on the incised disc and with sources inherited from the fluxes in the first of (10.5).
10.2.3 The Limit in the Incisure

The restrictions of \( \{ u_\varepsilon \}, \{ v_\varepsilon \} \) to \( \mathcal{B}_\varepsilon \) converge to functions \( u_\varepsilon, v_\varepsilon \) defined in \( \mathcal{B}_T \) and satisfying

\[
2r\varepsilon_o(r) \{ u_\varepsilon, v_\varepsilon, \nabla_B u_\varepsilon, \nabla_B v_\varepsilon \} \in L^2(\mathcal{B}_T).
\]

These functions are related to the interior limit by the trace equalities

\[
u_\varepsilon \bigg|_{z=z_*} = u_* \bigg|_{\mathcal{B}_T} \quad \text{and} \quad v_\varepsilon \bigg|_{z=z_*} = v_* \bigg|_{\mathcal{B}_T}
\]

Moreover

\[
u_\varepsilon \bigg|_{z=z_*} = u_* \bigg|_{\mathcal{B}_T} \quad \text{and} \quad v_\varepsilon \bigg|_{z=z_*} = v_* \bigg|_{\mathcal{B}_T}
\]

Let \( \mathcal{V}(\delta) \) be defined as in (4.2). Since \( z \to u_\varepsilon, v_\varepsilon \) are continuous in \( L^2(\mathcal{V}(\delta)_T) \) for all \( 0 < \delta \ll 1 \), these local trace equalities imply that

\[
u \bigg|_{z=z_*} = u_* \bigg|_{\mathcal{B}_T} \quad \text{and} \quad v \bigg|_{z=z_*} = v_* \bigg|_{\mathcal{B}_T}
\]

Therefore, while the interior limit \( u_*(\vec{x}, z_*, t) \) might differ from the limit \( u_*(\vec{x}, t) \) outside the limiting incisure \( \mathcal{V} \), they coincide on \( \mathcal{V}(\delta) \) for all \( 0 < \delta \ll 1 \). A similar statement holds for \( v_*, v_* \).

Next, denote by \( u_{x_2}^+ \) and \( v_{x_2}^+ \) the \( x_2 \)-derivative of the interior limits \( u, v \) from either side of \( \mathcal{B} \). Denote likewise by \( u_{x_2}^+, v_{x_2}^+ \) the \( x_2 \)-derivative of the limits \( u_*, v_* \) on the activated disc \( D_* \), from either side of \( \mathcal{V} \). Then there holds in a weak sense in \( \mathcal{B}_T \)

\[
r\varepsilon_o(r) u_{B,t} - \text{div} \{ r\varepsilon_o(r) \nabla u_\varepsilon \} = \frac{1}{2}(1 - \mu_0)D_u \frac{u_{x_2}^+ - u_{x_2}^-}{\varepsilon_o} + \frac{\delta_{x_2}}{\varepsilon_o} \frac{D_u u_{x_2}^+ - u_{x_2}^-}{2}
\]

\[
r\varepsilon_o(r) v_{B,t} - \text{div} \{ r\varepsilon_o(r) \nabla v_\varepsilon \} = \frac{1}{2}(1 - \mu_0)D_v \frac{v_{x_2}^+ - v_{x_2}^-}{\varepsilon_o} + \frac{\delta_{x_2}}{\varepsilon_o} \frac{D_v v_{x_2}^+ - v_{x_2}^-}{2}
\]

(10.7)

where \( \nabla \) and \( \text{div} \) are relative to \( \mathcal{B} \). Moreover

\[
u_\varepsilon, z(r, 0, t) = u_\varepsilon, z(r, H, t) = 0 \quad \text{and} \quad v_\varepsilon, z(r, 0, t) = v_\varepsilon, z(r, H, t) = 0 \quad r \in (r_o, R), \quad t \in (0, T).
\]

The regularity requirement (10.6) is not sufficient to insure that \( u_\varepsilon, v_\varepsilon \) have a trace in \( L^1(\mathcal{B}_T) \), so that (10.7) must be understood in a suitable weak sense: see Subsection 10.2.5 below.
10.2.4 The Limit in the Outer Shell

The restrictions of \( \{u_z\} \) and \( \{v_z\} \) to the outer shell \( S_{\varepsilon,T} \) converge to functions \( u_s, v_s \) defined in \( S_T \) and satisfying

\[
u_s, v_s \in C(0, T; L^2(S)) : \quad (u_{S,z}, u_{S,\theta}), \quad (v_{S,z}, v_{S,\theta}) \in L^2(S_T).
\]

These are related to the interior limits \( u \) and \( v \) and to \( u^* \) and \( v^* \) by sharing their traces, i.e.,

\[
u_S = u|_S \quad v_S = v|_S \quad \text{in} \quad L^2(S_T)
\]

as well as at the activated level \( z^* \)

\[
u_S(z^*) = u^*|_{\|\cdot\|=R} \quad v_S(z^*) = v^*|_{\|\cdot\|=R} \quad \text{in} \quad L^2(\{\|\cdot\|=R\} \times (0, T))
\]

Moreover

\[
u_S(0, z^*, t) = u^*_B(R, z^*, t) \quad v_S(0, z^*, t) = v^*_B(R, z^*, t) \quad \text{in} \quad L^2(S_T \cap \{|\theta|=0\}).
\]

Since \( z \to u_S(\theta, z, t) \) is continuous in \( L^2((0, 2\pi] \times (0, T]) \), (10.8) and (10.9) imply

\[
u(\|\cdot\|=R) = u^*_t(\|\cdot\|=R) \quad \text{in} \quad L^2((0, 0, 2\pi] \times (0, T]).
\]

Therefore while the interior limit \( u(\|\cdot\|=R) \) and the limit \( u^*_t(\|\cdot\|=R) \) in the activated interdiscal space might differ for \( \{|\|\cdot\||<R\} \), they coincide for \( \|\cdot\|=R \). The limit \( u_S \) satisfies, weakly in \( S_T \)

\[
u_{S,t} - D_u \Delta_S u_S = -\frac{(1 - \mu_o) D_u}{\sigma_{\varepsilon_o}} u_p|_{\|\cdot\|=R} - \delta_{z^*} \frac{\nu D_u}{\sigma_{\varepsilon_o}} u_{s,\rho}|_{\|\cdot\|=R} - \delta_{\{\theta=0\}} \frac{2\theta u_{s,\rho}(R)}{\sigma_{\varepsilon_o}} u_{B,\rho}|_{r=R}
\]

where \( \delta_{\{\theta=0\}} \) is the Dirac mass concentrated on the line \( \{\theta=0\} \) traced on the outer shell by the limiting incisure \( B \). In the same way \( v_S \) satisfies

\[
u_{S,t} - D_v \Delta_S v_S = -\frac{(1 - \mu_o) D_v}{\sigma_{\varepsilon_o}} v_p|_{\|\cdot\|=R} - \delta_{z^*} \frac{\mu D_v}{\sigma_{\varepsilon_o}} v_{s,\rho}|_{\|\cdot\|=R} - \delta_{\{\theta=0\}} \frac{2\theta v_{s,\rho}(R)}{\sigma_{\varepsilon_o}} v_{B,\rho}|_{r=R} - \frac{1}{\sigma_{\varepsilon_o}} \{g_1(\hat{v}) - g_2(\hat{u})\}
\]
Moreover \( u_S \) and \( v_S \) take the variational boundary data

\[
\hat{u}_z, \hat{v}_z(\theta, 0, t) = \tilde{u}_z, \tilde{v}_z(\theta, H, t) = 0, \quad \theta \in (0, 2\pi), \ t \in (0, T).
\]

The regularity requirements on the limit in the interior and on the special level

are not sufficient to insure that \( u_{\rho}, v_{\rho} \) and \( u_{\ast \rho}, v_{\ast \rho} \) have traces in \( L^1(S_T) \) and 
\( L^1(\{|\tau| = R\}T) \) respectively; thus (10.10) and (10.11) must be understood in
the standard weak sense (see § 10.2.5).

### 10.2.5 Weak Form of the Homogenized Limit

The functions \( (u, u_\ast, u_S, u_B) \) and \( (v, v_\ast, v_S, v_B) \) are in the indicated regularity
classes and satisfy

\[
(1 - \mu_o) \left\{ \int_{\Omega_T - B_T} \left\{ - u_\varphi_t + D_u \nabla u \cdot \nabla \varphi + (\gamma o u - f(v)) \varphi \right\} dx dt 
\right. 
\]

\[
\left. - \int_{\Omega - B} u_\varphi(x, 0) \ dx \right\}_{\text{interior}} 
\]

\[
+ \sigma \varepsilon_o \left\{ \int_{S_T} \left\{ - u_S \varphi_t + D_u \nabla u \cdot \nabla \varphi \right\} d\eta dt - \int_S u_\varphi(x, 0) d\eta \right\}_{\text{outer shell}} 
\]

\[
+ \left\{ \int_{B_T} 2r \theta \varepsilon_o(r) \left\{ - u_B \varphi_t + D_u \nabla u \cdot \nabla \varphi \right\} dr dz dt 
\right. 
\]

\[
\left. - \int_B 2r \theta \varepsilon_o(r) u_\varphi(x, 0) dr dz \right\}_{\text{incisure}} 
\]

\[
+ \nu \varepsilon_o \left\{ \int_{D_{\ast T}} \left\{ - u_\ast \varphi_t + D_u \nabla u_\ast \cdot \nabla \varphi + (\gamma o u_\ast - f(v_\ast)) \varphi 
\right. 
\]

\[
\left. + \frac{1}{\nu \varepsilon_o} u_\ast f_1(v_\ast, \tau, z_\ast, t) \varphi \right\} d\tau dt 
\right. 
\]

\[
\left. - \int_{D_{\ast}} u_\varphi(\tau, z_\ast, 0) d\tau \right\}_{\text{special level}} = 0
\]
and

\[
(1 - \mu_o) \left\{ \int_{\Omega_T - B_T} \left\{ - v_0 \psi_t + D_v \nabla \psi \cdot \nabla \psi \right\} dx \, dt - \int_{\Omega - B} v_0 \psi(x, 0) \, dx \right\} \text{ interior}
\]

\[
+ \sigma \varepsilon \left\{ \int_{S_T} \left\{ - v_S \psi_t + D_v \nabla S \psi \cdot \nabla S \psi + \frac{1}{\sigma \varepsilon} (g_1(v_S) - g_2(u_S)) \psi \right\} d\eta \, dt \right\} \text{ outer shell}
\]

\[
+ \nu \varepsilon \left\{ \int_{D_\delta - T} \left\{ - v_\delta \psi_t + D_v \nabla \psi \cdot \nabla \psi \right\} d\varpi \, dt \right\} \text{ incisure}
\]

\[
+ \nu \varepsilon \left\{ \int_{D_\delta, T} \left\{ - v_\delta \psi_t + D_v \nabla \psi \cdot \nabla \psi \right\} d\varpi \, dt \right\} \text{ special level}
\]

= 0.

for all test functions \( \varphi, \psi \in W^{1,2}(\Omega_T) \), such that

\[
\varphi, \psi \big|_{S_T} \in W^{1,2}(S_T); \quad \varphi, \psi \big|_{D_\delta - T} \in W^{1,2}(D_\delta - T); \quad \varphi, \psi \big|_{B_T} \in W^{1,2}(B_T(\delta))
\]

for all \( 0 < \delta \ll 1 \), where the restrictions are intended in the sense of the traces. Moreover

\[
\sqrt{r \theta \varepsilon}(\varphi) \big|_{B_T}, \psi \big|_{B_T}, \nabla \varphi \big|_{B_T}, \nabla \psi \big|_{B_T} \in L^2(B_T).
\]

References


