Rotation-minimizing frames on space curves — theory, algorithms, applications

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— synopsis —

- rotation-minimizing frames (RMFs) on space curves
- applications of RMFs — “defects” of the Frenet frame
- RMFs for spatial Pythagorean–hodograph (PH) curves
- characterization of PH curves with exact rational RMFs
- a “philosophical” interlude — theory versus practice
- directed frames — camera orientation control
  computation of rotation-minimizing directed frames
- polar differential geometry — anti-hodograph, Frenet directed frame, polar curvature and torsion
- conclusions and practical advice on standard answers to questions asked at conferences
rotation-minimizing frames on space curves

- an adapted frame \((e_1, e_2, e_3)\) on a space curve \(r(\xi)\) is a system of three orthonormal vectors, such that \(e_1 = \frac{r'}{|r'|}\) is the curve tangent and \((e_2, e_3)\) span the curve normal plane at each point.

- on any given space curve, there are infinitely many adapted frames — the Frenet frame is perhaps the most familiar.
  
  R. L. Bishop (1975), There is more than one way to frame a curve, *Amer. Math. Monthly* **82**, 246–251

- for a rotation–minimizing frame (RMF), the normal–plane vectors \((e_2, e_3)\) exhibit no instantaneous rotation about \(e_1\).
  

- angular orientation of RMF relative to Frenet frame = integral of curve torsion w.r.t. arc length (⇒ one–parameter family of RMFs)
  
• **spatial PH curves** admit exact evaluation of torsion integral, but expression for RMF contains transcendental terms


• **piecewise–rational RMF approximation** on polynomial & rational curves


• **Euler–Rodrigues frame (ERF)** is better reference than Frenet frame for identifying curves with **rational RMFs** (RRMF curves)


  **ERF = rational** adapted frame defined on spatial PH curves that is non–singular at inflection points
• “implicit” algebraic condition for rational RMFs on spatial PH curves — no rational RMFs for non–degenerate cubics
  

• sufficient–and–necessary conditions on Hopf map coefficients of spatial PH quintics for rational RMF
  

• simplified (quadratic) RRMF conditions for quaternion and Hopf map representations of spatial PH quintics
  

• general RRMF conditions for spatial PH curves of any degree
  

• spatial motion design by RRMF quintic Hermite interpolation
  
elementary differential geometry of space curves

Frenet frame \((t(\xi), n(\xi), b(\xi))\) on space curve \(r(\xi)\) defined by

\[
\begin{align*}
  t &= \frac{r'}{|r'|}, \quad n = \frac{r' \times r''}{|r' \times r''|} \times t, \quad b = \frac{r' \times r''}{|r' \times r''|}
\end{align*}
\]

\(t\) defines instantaneous direction of motion along curve; 
\(n\) points toward center of curvature; \(b = t \times n\) completes frame
variation of frame \((t(\xi), n(\xi), b(\xi))\) along curve \(r(\xi)\) specified in terms of parametric speed, curvature, torsion functions

\[
\sigma = |r'|, \quad \kappa = \frac{|r' \times r''|}{|r'|^3}, \quad \tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}
\]

by Frenet–Serret equations

\[
\begin{bmatrix}
  t' \\
  n' \\
  b'
\end{bmatrix} = \sigma
\begin{bmatrix}
  0 & \kappa & 0 \\
  -\kappa & 0 & \tau \\
  0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
  t \\
  n \\
  b
\end{bmatrix}
\]

- \((t, n)\) span osculating plane (second–order contact at each point)
- \((n, b)\) span normal plane (cuts curve orthogonally at each point)
- \((b, t)\) span rectifying plane (envelope of these planes defines \textit{rectifying developable}, allows curve to be flattened onto a plane)
“defects” of Frenet frame on space curves

• \((t, n, b)\) do not depend \textit{rationally} on curve parameter \(\xi\)

• normal–plane vectors \((n, b)\) become \textit{indeterminate} and can suddenly “flip” at \textit{inflection points} of curve, where \(\kappa = 0\)

• exhibits “\textit{unnecessary rotation}” in the curve normal plane

\[
\frac{dt}{ds} = d \times t, \quad \frac{dn}{ds} = d \times n, \quad \frac{db}{ds} = d \times b
\]

\textit{Darboux vector} \(d = \kappa b + \tau t = \text{Frenet frame rotation rate}\)

component \(\tau t\) describes instantaneous rotation in normal plane

(unnecessary for “smoothly varying” adapted orthonormal frame)
total curvature $|\mathbf{d}| = \sqrt{\kappa^2 + \tau^2}$ = angular velocity of Frenet frame

rotation–minimizing adapted frame $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ satisfying

$$\frac{d\mathbf{t}}{ds} = \omega \times \mathbf{t}, \quad \frac{d\mathbf{u}}{ds} = \omega \times \mathbf{u}, \quad \frac{d\mathbf{v}}{ds} = \omega \times \mathbf{v}$$

RMF characteristic property — angular velocity $\omega$ satisfies $\omega \cdot \mathbf{t} \equiv 0$

no instantaneous rotation of normal–plane vectors $(\mathbf{u}, \mathbf{v})$ about tangent $\mathbf{t}$

→ rotation–minimizing frame much better than Frenet frame for applications in animation, path planning, swept surface constructions, etc.

among all adapted frames on a space curve, the RMF identifies least elastic energy associated with twisting (as distinct from bending)
Frenet frame (center) & rotation-minimizing frame (right) on space curve

motion of an ellipsoid oriented by Frenet & rotation-minimizing frames
sudden reversal of Frenet frame through an inflection point

surface constructed by sweeping an ellipse along a space curve using Frenet frame (center) & rotation-minimizing frame (right)
Pythagorean-hodograph (PH) curves

\[ r(\xi) = \text{PH curve in } \mathbb{R}^n \iff \text{coordinate components of } r'(\xi) \]

elements of “Pythagorean \((n + 1)\)-tuple of polynomials”

PH curves incorporate **special algebraic structures** in their hodographs
*(complex number \& quaternion models for planar \& spatial PH curves)*

- rational offset curves \[ r_d(\xi) = r(\xi) + d \mathbf{n}(\xi) \]
- polynomial arc-length function \[ s(\xi) = \int_0^\xi |r'(\xi)| \, d\xi \]
- closed-form evaluation of energy integral \[ E = \int_0^1 \kappa^2 \, ds \]
- real-time CNC interpolators, rotation-minimizing frames, etc.
Pythagorean quartuples of polynomials

\[x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \iff \begin{cases} 
  x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\
  y'(t) = 2 \left[ u(t)q(t) + v(t)p(t) \right] \\
  z'(t) = 2 \left[ v(t)q(t) - u(t)p(t) \right] \\
  \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) 
\end{cases} \]


**Quaternion representation** \( \mathcal{A}(t) = u(t) + v(t) i + p(t) j + q(t) k \)

→ **Spatial Pythagorean hodograph** \( r'(t) = (x'(t), y'(t), z'(t)) = \mathcal{A}(t) i \mathcal{A}^*(t) \)

**Hopf map representation** \( \alpha(t) = u(t) + i v(t), \beta(t) = q(t) + i p(t) \)

→ \( (x'(t), y'(t), z'(t)) = (|\alpha(t)|^2 - |\beta(t)|^2, 2 \text{Re}(\alpha(t)\overline{\beta}(t)), 2 \text{Im}(\alpha(t)\overline{\beta}(t))) \)

**Equivalence** — identify “\( i \)” with “\( i \)” and set \( \mathcal{A}(t) = \alpha(t) + k \beta(t) \)
rotation-minimizing frames on spatial PH curves

new basis in normal plane

\[
\begin{bmatrix}
  u \\
v
\end{bmatrix}
= 
\begin{bmatrix}
  \cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  n \\
b
\end{bmatrix}
\]

where \( \theta = -\int \tau \, ds \) : cancels “unnecessary rotation” in normal plane

free integration constant \( \Rightarrow \exists \) one–parameter family of RMFs

options for construction of RMF \((t, u, v)\) on spatial PH quintics:

- **analytic reduction** — involves rational function integration, logarithmic dependence on curve parameter

- **rational approximation** — use Padé (rational Hermite) approach: simple algorithm & rapid convergence

- **exact rational RMFs** — identify sufficient and necessary conditions for rational RMFs on spatial PH curves
comparison of Frenet & rotation-minimizing frames

spatial PH quintic  Frenet frame  rotation–minimizing frame
compared with the rotation-minimizing frame \((t, u, v)\), the Frenet frame \((t, n, b)\) exhibits a lot of “unnecessary” rotation (in the curve normal plane)
any space curve with a rational RMF must be a PH curve (since only PH curves have rational unit tangents)

Choi & Han (2002): for PH curve with hodograph \( r'(\xi) = A(\xi) i \overline{A}(\xi) \)

\[
\mathbf{t}(\xi) = \frac{A(\xi) i \overline{A}(\xi)}{|A(\xi)|^2}, \quad \mathbf{p}(\xi) = \frac{A(\xi) j \overline{A}(\xi)}{|A(\xi)|^2}, \quad \mathbf{q}(\xi) = \frac{A(\xi) k \overline{A}(\xi)}{|A(\xi)|^2}
\]

defines the Euler–Rodrigues frame (ERF) — \((\mathbf{t}, \mathbf{p}, \mathbf{q})\) is better “reference” than Frenet frame \((\mathbf{t}, \mathbf{n}, \mathbf{b})\) for seeking rational RMFs on spatial PH curves

ERF is not intrinsic (depends on chosen basis \((i, j, k)\) for \(\mathbb{R}^3\)) — but is inherently rational and non–singular at inflection points

RMF vectors \((\mathbf{u}, \mathbf{v})\) must be obtainable from ERF vectors \((\mathbf{p}, \mathbf{q})\) through rational rotation in curve normal plane at each point of \(r(\xi)\)
Han (2008): is there a rational rotation in the normal plane defined by two real polynomials \( a(\xi), b(\xi) \) that maps the ERF vectors \( p(\xi), q(\xi) \) onto the RMF vectors \( u(\xi), v(\xi) \)?

\[
\begin{align*}
    u(\xi) &= \frac{a^2(\xi) - b^2(\xi)}{a^2(\xi) + b^2(\xi)} p(\xi) - \frac{2a(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)} q(\xi), \\
    v(\xi) &= \frac{2a(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)} p(\xi) + \frac{a^2(\xi) - b^2(\xi)}{a^2(\xi) + b^2(\xi)} q(\xi).
\end{align*}
\]

If such polynomials \( a(\xi), b(\xi) \) exist, we have an RRMF curve — i.e., a PH curve with a rational rotation–minimizing frame.
“implicit” algebraic condition for RRMF curves

Han (2008): PH curve defined by \( A(\xi) = u(\xi) + v(\xi) \mathbf{i} + p(\xi) \mathbf{j} + q(\xi) \mathbf{k} \) is an RRMF curve if and only if polynomials \( a(\xi), b(\xi) \) exist such that

\[
\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2}
\]

Hopf map representation with \( \alpha(\xi) = u(\xi) + iv(\xi), \beta(\xi) = q(\xi) + ip(\xi) \) requires existence of complex polynomial \( w(\xi) = a(\xi) + ib(\xi) \) such that

\[
\frac{\overline{\alpha}\alpha' - \overline{\alpha}'\alpha + \overline{\beta}\beta' - \overline{\beta}'\beta}{|\alpha|^2 + |\beta|^2} = \frac{ww' - \overline{w}'w}{|w|^2}
\]

Han (2008): no RRMF cubics exist, except degenerate (planar) curves
characterization of RRMF quintics

Farouki, Giannelli, Manni, Sestini (2009): use Hopf map form with

\[
\begin{align*}
\alpha(t) &= \alpha_0 (1-t)^2 + \alpha_1 2(1-t)t + \alpha_2 t^2, \\
\beta(t) &= \beta_0 (1-t)^2 + \beta_1 2(1-t)t + \beta_2 t^2.
\end{align*}
\]

defines RRMF quintic \(\iff\) \(w_0, w_1, w_2 \in \mathbb{C}, \gamma \in \mathbb{R}\) exist such that

\[
\begin{align*}
|\alpha_0|^2 + |\beta_0|^2 &= \gamma |w_0|^2, \\
\overline{\alpha}_0 \alpha_1 + \overline{\beta}_0 \beta_1 &= \gamma \overline{w}_0 w_1, \\
\overline{\alpha}_0 \alpha_2 + \overline{\beta}_0 \beta_2 + 2 (|\alpha_1|^2 + |\beta_1|^2) &= \gamma (\overline{w}_0 w_2 + 2 |w_1|^2), \\
\overline{\alpha}_1 \alpha_2 + \overline{\beta}_1 \beta_2 &= \gamma \overline{w}_1 w_2, \\
|\alpha_2|^2 + |\beta_2|^2 &= \gamma |w_2|^2.
\end{align*}
\]

NOTE: can take \(w_0 = 1\) without loss of generality
Proposition 1. A PH quintic has a rational rotation-minimizing frame if and only if the coefficients $\alpha_0, \alpha_1, \alpha_2$ and $\beta_0, \beta_1, \beta_2$ of the two quadratic complex polynomials $\alpha(t)$ and $\beta(t)$ satisfy the constraints

$$(|\alpha_0|^2 + |\beta_0|^2) |\bar{\alpha}_1 \alpha_2 + \bar{\beta}_1 \beta_2|^2 = (|\alpha_2|^2 + |\beta_2|^2) |\alpha_0 \bar{\alpha}_1 + \beta_0 \bar{\beta}_1|^2,$$

$$(|\alpha_0|^2 + |\beta_0|^2) (\alpha_0 \beta_2 - \alpha_2 \beta_0) = 2(\alpha_0 \bar{\alpha}_1 + \beta_0 \bar{\beta}_1)(\alpha_0 \beta_1 - \alpha_1 \beta_0).$$

one real + one complex constraint on $\alpha_0, \alpha_1, \alpha_2$ and $\beta_0, \beta_1, \beta_2$

$\Rightarrow$ RRMF quintics have three less freedoms than general PH quintics

Algorithm to construct RRMF quintics: freely choose $\alpha_0, \alpha_2$ and $\beta_0, \beta_2$ & obtain $\alpha_1, \beta_1$ in terms of one free parameter, from RRMF constraints
example RRMF quintic construction

choose \( \alpha_0 = 1 + 2i, \ \beta_0 = -2 + i, \ \alpha_2 = 2 - i, \ \beta_2 = -1 + 2i \)

\[ \implies \alpha_1 = \frac{1 + i}{\sqrt{2}}, \ \beta_1 = \frac{-3 + i}{\sqrt{2}} \] and \( (w_0, w_1, w_2) = \left( 1, \frac{1}{\sqrt{2}}, \frac{3 - 4i}{5} \right) \)
polynomials defining RMF vectors \((u, v)\) in terms of ERF vectors \((p, q)\)

\[
a(t) = (1 - t)^2 + \frac{1}{\sqrt{2}} 2(1 - t)t + \frac{3}{5} t^2, \quad b(t) = -\frac{4}{5} t^2.
\]

comparison of angular speeds for ERF and RMF
“lingering doubts” about RRMF quintic conditions

- constraints are of rather high degree — 4 and 6
- not invariant when “0” and “2” subscripts swapped (corresponds to the re-parameterization $t \rightarrow 1 - t$)
- do not easily translate to quaternion representation

Problem revisited in Farouki (2010), to appear

- to avoid asymmetry, do not assume $w_0 = 1$
- consider PH quintics in canonical form with $r'(0) = (1, 0, 0)$
- strategic switching between quaternion & Hopf map forms
improved sufficient–and–necessary conditions

Proposition 2. A spatial PH quintic defined by the quaternion polynomial
\[ A_0(1 - \xi)^2 + A_1 2(1 - \xi)\xi + A_2 \xi^2 \]
has a rational RMF if and only if
\[ A_0 i A_2^* + A_2 i A_0^* = 2 A_1 i A_1^*. \]

Proposition 3. A spatial PH quintic defined by the complex polynomials
\[ \alpha_0(1 - \xi)^2 + \alpha_1 2(1 - \xi)\xi + \alpha_2 \xi^2 \quad \text{and} \quad \beta_0(1 - \xi)^2 + \beta_1 2(1 - \xi)\xi + \beta_2 \xi^2 \]
has a rational RMF if and only if
\[ \text{Re}(\alpha_0 \alpha_2 - \beta_0 \beta_2) = |\alpha_1|^2 - |\beta_1|^2, \quad \alpha_0 \beta_2 + \alpha_2 \beta_0 = 2 \alpha_1 \beta_1. \]

- new conditions are only quadratic in coefficients
- easy transformation quaternion ⇐ Hopf map forms
- obvious invariance on swapping “0” and “2” subscripts
RRMF quintics constructed from new conditions
rational RMFs on space curves of any degree


**Proposition 4.** For \( \mathbf{A}(t) = u(t) + v(t) \mathbf{i} + p(t) \mathbf{j} + q(t) \mathbf{k} \), the condition

\[
\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2}
\]

*can be satisfied if and only if a polynomial \( h(t) \) exists, such that*

\[
(\sqrt{u^2 + v^2 + p^2 + q^2})^2 = h(t)
\]

*Polynomial \( \rho = (uv' - u'v - pq' + p'q)^2 + (uq' - u'q - vp' + v'p)^2 \) plays a key role in the theory of double PH curves, with \( |\mathbf{r}'(t)| \) and \( |\mathbf{r}'(t) \times \mathbf{r}''(t)| \) both polynomials in \( t \) — rational Frenet frames and rational curvatures.*
theory versus practice — a “philosophical” interlude

theoretical astronomer —
don’t believe an observation until there’s a theory to explain it

observational astronomer —
don’t believe a theory until there’s an observation to confirm it

“In theory, there is no difference between theory and practice. In practice, there is.”

Yogi Berra
Yankees baseball player, aspiring philosopher
famous sayings of Yogi Berra, sportsman-philosopher

- Baseball is ninety percent mental, and the other half is physical.
- Always go to other people’s funerals — otherwise they won’t come to yours.
- It was impossible to get a conversation going, everyone was talking too much.
- You better cut the pizza into four pieces, because I’m not hungry enough to eat six.
- You got to be very careful if you don’t know where you are going, because you might not get there.
- Nobody goes there anymore. It’s too crowded.
adapted & directed frames on space curve $r(\xi)$

- **adapted frame** $(e_1, e_2, e_3) \Rightarrow e_1$ is the unit curve tangent, $t = r'/|r'|$

- infinitely many choices of normal plane vectors $e_2, e_3$ orthogonal to $t$

- angular velocity $\omega$ of rotation-minimizing adapted frame $(e_1, e_2, e_3)$ is characterized by $\omega \cdot t \equiv 0$

---

- **directed frame** $(e_1, e_2, e_3) \Rightarrow e_1$ is the unit polar vector, $o = r/|r|$

- infinitely many choices of image plane vectors $e_2, e_3$ orthogonal to $o$

- angular velocity $\omega$ of rotation-minimizing directed frame $(e_1, e_2, e_3)$ is characterized by $\omega \cdot o \equiv 0$
rotation-minimizing directed frames — applications


- camera orientation planning for cinematography, video inspection, computer games, virtual reality, etc.
- minimize surgeon disorientation in endoscopic surgery
- related problem: field de-rotator for altazimuth telescope
- maintenance for aircraft engines, gas turbines, pipes, etc.
- for many applications, RMDF image orientation can be achieved through software transformations
camera orientation frame along space curve $r(\xi)$

- assume target object fixed at origin (for moving target, consider only relative motion between camera & target)

- unit polar vector $\mathbf{o}(\xi) = \frac{r(\xi)}{|r(\xi)|}$ defines camera optical axis

- let camera image plane, orthogonal to $\mathbf{o}(\xi)$, be spanned by two unit vectors $\mathbf{u}(\xi)$ and $\mathbf{v}(\xi)$

- if $r(\xi), r'(\xi)$ linearly independent, set $\mathbf{v}(\xi) = \frac{r(\xi) \times r'(\xi)}{|r(\xi) \times r'(\xi)|}$

- set $\mathbf{u}(\xi) = \mathbf{v}(\xi) \times \mathbf{o}(\xi)$ — $(\mathbf{o}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ defines a right-handed orthonormal directed frame along $r(\xi)$
compare **directed frame** defined above

\[
o = \frac{r}{|r|}, \quad u = \frac{r \times r'}{|r \times r'|} \times o, \quad v = \frac{r \times r'}{|r \times r'|}
\]  

(1)

with **Frenet frame** from differential geometry

\[
t = \frac{r'}{|r'|}, \quad n = \frac{r' \times r''}{|r' \times r''|} \times t, \quad b = \frac{r' \times r''}{|r' \times r''|}
\]  

(2)

note that \((t, n, b) \rightarrow (o, u, v)\) under map \((r', r'') \rightarrow (r, r')\)

call (1) the **Frenet directed frame**, (2) the **Frenet adapted frame**

define **anti-hodograph** (indefinite integral) \(s(\xi) = \int r(\xi) \, d\xi\)

\[
\Rightarrow \text{ Frenet directed frame of a curve } r(\xi)
\]

\[= \text{ Frenet adapted frame of its anti-hodograph, } s(\xi)\]
properties of “anti-hodograph” —  \( s(\xi) = \int r(\xi) \, d\xi \)

- curve **hodographs** (derivatives) \( r'(\xi) \) are widely used in CAGD
- **anti-derivative** of function \( f(\xi) \) is indefinite integral, \( s(\xi) = \int f(\xi) \, d\xi \)
- infinitely many anti-hodographs — just translates of each other
- \( s(\xi_*) \) is a **cusp** of anti-hodograph \( \Rightarrow r(\xi) \) traverses origin at \( \xi = \xi_* \)
- \( s(\xi_*) \) is an **inflection** of anti-hodograph \( \Rightarrow \) tangent line to \( r(\xi) \) goes through origin for \( \xi = \xi_* \)
- polynomial curve \( \iff \) polynomial anti-hodograph, but this correspondence does not extend to **rational** anti-hodographs (integral of rational function may incur transcendental terms)
polar differential geometry of space curve $r(\xi)$

$$\rho = |r|, \quad \lambda = \frac{|r \times r'|}{|r|^3}, \quad \nu = \frac{(r \times r') \cdot r''}{|r \times r'|^2}$$

polar distance, polar curvature, polar torsion of $r(\xi)

= parametric speed, curvature, torsion of anti-hodograph, $s(\xi) = \int r(\xi)$

• polar curvature $\lambda(\xi) \equiv 0 \iff r(\xi) = \text{line through origin}$

• polar torsion $\nu(\xi) \equiv 0 \iff r(\xi) = \text{in plane through origin}$

• hence, $\lambda(\xi) \equiv 0 \Rightarrow \kappa(\xi) \equiv 0$ and $\nu(\xi) \equiv 0 \Rightarrow \tau(\xi) \equiv 0$

• $\lambda = 0$ identifies polar inflection — $r$ and $r'$ linearly dependent

• polar helix $\frac{\lambda(\xi)}{\nu(\xi)} = \text{constant} \iff r(\xi) = \text{on cone with apex at origin}$
Frenet-Serret equations for directed frame \((o, u, v)\)

\[
\begin{bmatrix}
o' \\
u' \\
v'
\end{bmatrix} = \rho \begin{bmatrix}
0 & \lambda & 0 \\
-\lambda & 0 & \nu \\
0 & -\nu & 0
\end{bmatrix} \begin{bmatrix}
o \\
u \\
v
\end{bmatrix}
\]

polar distance, polar curvature, polar torsion of \(r(\xi)\)

\[\rho = |r|, \quad \lambda = \frac{|r \times r'|}{|r|^3}, \quad \nu = \frac{(r \times r') \cdot r''}{|r \times r'|^2}\]

arc-length derivatives of \((o, u, v)\)

\[
\frac{do}{ds} = e \times o, \quad \frac{du}{ds} = e \times u, \quad \frac{dv}{ds} = e \times v.
\]

polar Darboux vector \(e = \frac{\rho}{\sigma} (\lambda v + \nu o)\)

angular velocity of directed frame \(\omega = |e| = \frac{\rho}{\sigma} \sqrt{\lambda^2 + \nu^2}\)
corresponding properties of the Frenet adapted and directed frames on space curves

<table>
<thead>
<tr>
<th>Frenet adapted frame</th>
<th>Frenet directed frame</th>
</tr>
</thead>
<tbody>
<tr>
<td>tangent vector $t$</td>
<td>polar vector $o$</td>
</tr>
<tr>
<td>principal normal $n$</td>
<td>principal axis $u$</td>
</tr>
<tr>
<td>binormal vector $b$</td>
<td>bi-axis vector $v$</td>
</tr>
<tr>
<td>normal plane = span($n$, $b$)</td>
<td>image plane = span($u$, $v$)</td>
</tr>
<tr>
<td>osculating plane = span($t$, $n$)</td>
<td>motion plane = span($o$, $u$)</td>
</tr>
<tr>
<td>rectifying plane = span($b$, $t$)</td>
<td>orthogonal plane = span($v$, $o$)</td>
</tr>
<tr>
<td>parametric speed $\sigma$</td>
<td>polar distance $\rho$</td>
</tr>
<tr>
<td>curvature $\kappa$</td>
<td>polar curvature $\lambda$</td>
</tr>
<tr>
<td>torsion $\tau$</td>
<td>polar torsion $\upsilon$</td>
</tr>
</tbody>
</table>

Each property of the Frenet directed frame of $\mathbf{r}(\xi)$ coincides with the corresponding property of the Frenet adapted frame of its anti-hodograph, $s(\xi) = \int \mathbf{r}(\xi) \, d\xi$
connection between Frenet adapted & directed frames

(t, n, b) and (o, u, v) are both orthonormal frames for $\mathbb{R}^3$

$$\begin{bmatrix} o \\ u \\ v \end{bmatrix} = \begin{bmatrix} o \cdot t & o \cdot n & o \cdot b \\ u \cdot t & u \cdot n & u \cdot b \\ v \cdot t & v \cdot n & v \cdot b \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

elements of matrix $M \in \text{SO}(3)$ in terms of $r, r', r'', \rho = |r|, \sigma = |r'|$:

\[ o \cdot t = \frac{r \cdot r'}{\rho \sigma}, \quad o \cdot n = -\frac{(r \times r') \cdot (r' \times r'')}{\rho \sigma |r' \times r''|}, \quad o \cdot b = \frac{(r \times r') \cdot r''}{\rho |r' \times r''|}, \]

\[ u \cdot t = \frac{|r \times r'|}{\rho \sigma}, \quad u \cdot n = \frac{r \cdot r' (r \times r') \cdot (r' \times r'')}{\rho \sigma |r \times r'| |r' \times r''|}, \quad u \cdot b = -\frac{(r \cdot r') (r \times r') \cdot r''}{\rho |r \times r'| |r' \times r''|}, \]

\[ v \cdot t = 0, \quad v \cdot n = \frac{\sigma (r \times r') \cdot r''}{|r \times r'| |r' \times r''|}, \quad v \cdot b = \frac{(r \times r') \cdot (r' \times r'')}{|r \times r'| |r' \times r''|}. \]
computation of rotation-minimizing directed frames

let \((o, p, q)\) be rotation-minimizing directed frame on \(r(\xi)\)

obtain \((p, q)\) from \((u, v)\) by rotation in image plane

\[
\begin{bmatrix}
  p \\
  q \\
\end{bmatrix}
= \begin{bmatrix}
  \cos \psi & \sin \psi \\
  -\sin \psi & \cos \psi \\
\end{bmatrix}
\begin{bmatrix}
  u \\
  v \\
\end{bmatrix}
\]

using anti-hodograph transformation, \(\psi = -\int v \rho \, d\xi\)
(i.e., integral of polar torsion w.r.t. anti-hodograph arc length)

- **RMDF angular velocity** \(\omega\) omits \(v \circ\) term from polar Darboux vector
- **infinitely many directed RMFs**, corresponding to different integration constants (maintain fixed angles relative to each other)
- **angle function** \(\psi(\xi)\) can be determined exactly for spatial P curves
  by rational function integration
example: circular camera path $r(\theta) = (r \cos \theta, r \sin \theta, h)$

$$ o = \left( \frac{r \cos \theta, r \sin \theta, h}{\sqrt{r^2 + h^2}} \right), \quad u = (-\sin \theta, \cos \theta, 0), \quad v = \left( \frac{-h \cos \theta, -h \sin \theta, r}{\sqrt{r^2 + h^2}} \right). $$

note — principal axis vector $u$ coincides with curve tangent $t$

$$ \rho = r \sqrt{r^2 + h^2}, \quad \lambda = \frac{r}{r^2 + h^2}, \quad \nu = \frac{h}{r^2 + h^2}. $$

polar distance, polar curvature, polar torsion — all constant

$$ \psi = -\frac{\theta}{\sqrt{1 + (r/h)^2}}. $$

RMDF orientation relative to Frenet directed frame — linear in $\theta$
directed frames on circular path, $r(\theta) = (r \cos \theta, r \sin \theta, h)$

Left: polar vectors. Center: image-plane vectors for directed Frenet frame. Right: image-plane vectors for the rotation-minimizing directed frame.
views of ellipsoid with camera image plane oriented using Frenet directed frame (upper) and rotation-minimizing directed frame (lower)
example: helical path \( r(\theta) = (r \cos \theta, r \sin \theta, k \theta) \) with \( c = k/r \)

\[
\mathbf{o} = \frac{(\cos \theta, \sin \theta, c \theta)}{\sqrt{1 + c^2 \theta^2}},
\]

\[
\mathbf{u} = \frac{(-c^2 \theta (\cos \theta + \theta \sin \theta) - \sin \theta, c^2 \theta (\theta \cos \theta - \sin \theta) + \cos \theta, c)}{\sqrt{1 + c^2 \theta^2} \sqrt{1 + c^2 + c^2 \theta^2}},
\]

\[
\mathbf{v} = \frac{(c (\sin \theta - \theta \cos \theta), -c (\cos \theta + \theta \sin \theta), 1)}{\sqrt{1 + c^2 + c^2 \theta^2}}.
\]

\[
\rho = r \sqrt{1 + c^2 \theta^2}, \quad \lambda = \frac{\sqrt{1 + c^2 + c^2 \theta^2}}{r (1 + c^2 \theta^2)^{3/2}}, \quad \nu = \frac{c \theta}{r (1 + c^2 + c^2 \theta^2)}.
\]

polar distance, polar curvature, polar torsion

\[
\psi = \tan^{-1} \frac{\sqrt{1 + c^2 \theta^2}}{c} - \tan^{-1} \frac{1}{c} - \frac{\sqrt{1 + c^2 \theta^2} - 1}{c}.
\]

RMDF orientation relative to Frenet directed frame
directed frames on helical path, \( r(\theta) = (r \cos \theta, r \sin \theta, k\theta) \)

Left: polar vectors. Center: image-plane vectors for Frenet directed frame. Right: image-plane vectors for the rotation-minimizing directed frame.
views of ellipsoid with camera image plane oriented using Frenet directed frame (upper) and rotation-minimizing directed frame (lower)
• theory, algorithms, applications for rotation-minimizing frames

• **RRMF curves** = PH curves with rational rotation–minimizing frames

• quaternion and Hopf map characterizations of RRMF quintics

• divisibility characterization for RRMF curves of any degree

• Yogi Berra’s insights on relationship between theory and practice

• rotation-minimizing directed frames in camera orientation control

• anti-hodograph and polar differential geometry of space curves
ANY QUESTIONS ??

It is better to ask a simple question, and perhaps seem like a fool for a moment, than to be a fool for the rest of your life.

old Chinese proverb

Please note —

Answers to all questions will be given exclusively in the form of Yogi Berra quotations.
some famous Yogi Berra responses

• If you ask me anything I don’t know, *I’m not gonna answer.*

• I wish I knew the answer to that, because *I’m tired of answering that question.*

Concerning *future research directions* . . .

• *The future ain’t what it used to be.*