

Nonstationary Subdivision Schemes and Totally Positive Refinable Functions ^{*}

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Abstract

In this paper we construct a class of totally positive refinable functions, obtained by a suitable use of certain nonstationary subdivision schemes. These functions are characterized by having small support and their smoothness can be established a priori. Further properties, such as partition of unity and reproduction of linear functions, are analyzed. Finally, Bernstein-like bases are constructed.

1 Introduction

Total positivity plays a main role in several problems of approximation theory, as well as in CAGD [8]. Examples of totally positive functions are provided by the well-known B-splines or, more in general, by a class of parametric refinable functions, that present a great flexibility in several applications [11],[12].

The intimate connection between refinability and subdivision schemes is well known and recently suggested the possibility of exploiting nonstationary subdivision schemes for the construction of some totally positive functions [4].

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In this paper we present the construction of a new class of totally positive functions, obtained by a suitable use of certain nonstationary subdivision schemes. These functions are characterized by having small support and a smoothness that can be established a priori.

The outline of the paper is as follows. In Sect. 2 we collect some definitions concerning totally positive refinable functions, both in the stationary and nonstationary framework. In Sect. 3 some basic facts about subdivision schemes are recalled. Sect. 4 is devoted to the introduction of a new class of nonstationary subdivision schemes that generates totally positive refinable functions; some interesting properties of these function are also given. Finally, in Sect. 5 we construct the corresponding Bernstein-like bases.

2 Totally Positive Refinable Functions

Let ϕ_m , $m \geq 0$, be a collection of nonstationary refinable functions, i.e. functions satisfying a set of *nonstationary refinement equations* of the type

$$\phi_m = \sum_{\alpha \in \mathbb{Z}} a_\alpha^m \phi_{m+1}(2 \cdot -\alpha), \quad m \geq 0, \quad (2.1)$$

where the sequences $\mathbf{a}^m = \{a_\alpha^m\}_{\alpha \in \mathbb{Z}}$, $m \geq 0$, are the *refinement masks* [3],[9]. In the following we assume \mathbf{a}^m having compact support.

When $\mathbf{a}^m = \mathbf{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}}$ for all $m \geq 0$, (2.1) reduces to the *stationary refinement equation*

$$\phi = \sum_{\alpha \in \mathbb{Z}} a_\alpha \phi(2 \cdot -\alpha). \quad (2.2)$$

The existence and the properties of both (2.1) and (2.2) are related to the properties of the *symbols* of the masks \mathbf{a}^m and \mathbf{a} , respectively, i.e. the Laurent polynomials

$$a^m(z) := \sum_{\alpha \in \mathbb{Z}} a_\alpha^m z^\alpha, \quad m \geq 0, \quad a(z) := \sum_{\alpha \in \mathbb{Z}} a_\alpha z^\alpha. \quad (2.3)$$

In particular, necessary conditions for the existence and uniqueness of solutions to (2.1) and (2.2) are $a^m(1) = 2$ and $a(1) = 2$, respectively.

In this context, we are interested in functions that are both refinable and totally positive. We recall that a function f is said to be totally positive if

for any sequences $x_1 < \dots < x_p$, $x_i \in \mathbb{R}$, and $\alpha_1 < \dots < \alpha_p$, $\alpha_j \in \mathbb{Z}$, $p \geq 1$, the collocation matrix

$$F \begin{pmatrix} x_1, \dots, x_p \\ \alpha_1, \dots, \alpha_p \end{pmatrix} := \det_{i,j=1,\dots,p} f(x_i - \alpha_j) \quad (2.4)$$

is totally positive.

While there are several examples of totally positive refinable functions in the stationary case, as the well-known B-splines on integer knots and the GP class introduced in [11], not many examples are known in the nonstationary setting. Typical examples are the *up-function*, introduced in [16] in the context of the solution of functional differential equations, and the exponential B-splines [15]. In particular, the up-function can be viewed as the solution ϕ_0 of the nonstationary refinement equations

$$\phi_m = \sum_{\alpha \in \mathbb{Z}} c_\alpha^m \phi_{m+1}(2 \cdot -\alpha), \quad m \geq 0, \quad (2.5)$$

where

$$c_\alpha^m = \frac{1}{2^m} \binom{m+1}{\alpha}, \quad \alpha = 0, \dots, m+1, \quad (2.6)$$

are the entries of the refinement mask of the B-spline of degree m [3],[5]. The power of the nonstationary framework is highlighted by the remarkable property of the up-function that it belongs to $C^\infty(\mathbb{R})$ while still having a small support, namely $[0, 2]$ (see Fig. 1, left). This means that nonstationary schemes allow us to obtain basic limit functions with small support, since it is related to the starting mask, and high smoothness, that is given by the limit mask.

By using the masks (2.6), it is not possible to construct other nonstationary refinable functions since m is the unique freedom degree. Instead, new classes of nonstationary refinable functions can be obtained by combining the B-spline masks and the GP masks $\mathbf{q}^{(n,h)} = \{q_\alpha^{(n,h)}\}_{0 \leq \alpha \leq n+1}$, whose explicit expression is [11]

$$q_\alpha^{(n,h)} = \frac{1}{2^h} \left[\binom{n+1}{\alpha} + 4(2^{h-n} - 1) \binom{n-1}{\alpha-1} \right], \quad \alpha = 0, \dots, n+1, \quad (2.7)$$

where $n \geq 2$ is a fixed integer and $h \geq n$ is a real parameter. (We set $\binom{n}{l} = 0$ when $0 < l$ or $l > n$.) The corresponding symbols are

$$q^{(n,h)}(z) = \frac{1}{2^h} (z+1)^{n-1} (z^2 + 2(2^{h-n+1} - 1)z + 1). \quad (2.8)$$

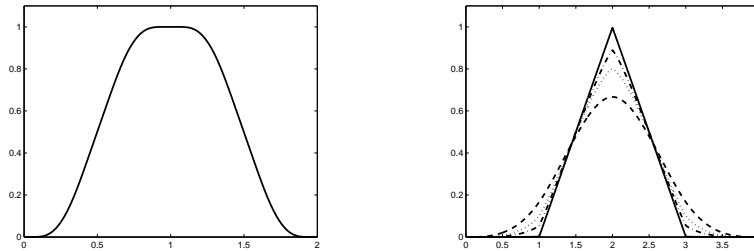


Figure 1: Up-function (left) and GP refinable function (right) with $n = 3$ and $h = 4$ (dotted line), $h = 5$ (dash-dotted line), $h = 10$ (solid line). The dashed line represents the cubic B-spline.

When $\mathbf{q}^{(n,h)}$, with n and h held fix, is used in a stationary refinement equation, the associated refinable function, say $\sigma^{(n,h)}$, has support $[0, n + 1]$ and belongs to $C^{n-2}(\mathbb{R})$. The parameter h acts as a tension parameter: while h ranges from n to ∞ , the shape of the refinable function changes continuously from the shape of the B-spline of degree n to the one of the B-spline of degree $n - 2$ without reaching it (see Fig. 1, right). In Sect. 4 we will show how to combine the masks of the B-splines and the masks $\mathbf{q}^{(n,h)}$ in nonstationary subdivision schemes in order to construct nonstationary refinable functions with small support and high smoothness.

3 Nonstationary Subdivision Schemes

Refinement masks can be associated to *subdivision schemes*, i.e. iterative schemes based on simple *refinement rules* generating denser and denser sequences of points convergent to a continuous curve or surface. By defining the k -level subdivision operator $S_{\mathbf{a}^k} : \ell(\mathbb{Z}) \rightarrow \ell(\mathbb{Z})$ as

$$(S_{\mathbf{a}^k} \boldsymbol{\lambda})_\alpha := \sum_{\beta \in \mathbb{Z}} a_{\alpha-2\beta}^k \lambda_\beta, \quad \alpha \in \mathbb{Z}, \quad (3.1)$$

the *nonstationary subdivision scheme* is given by [3], [7]

$$\boldsymbol{\lambda}^0 := \boldsymbol{\lambda}, \quad \boldsymbol{\lambda}^{k+1} := S_{\mathbf{a}^k} \boldsymbol{\lambda}^k, \quad k \geq 0. \quad (3.2)$$

When the refinement rule is the same at each iteration, i.e. $\mathbf{a}^k = \mathbf{a}$ for all k , the scheme is said to be *stationary* [2], [7].

A subdivision scheme is said to be C^ν if, for any initial sequence $\boldsymbol{\lambda} \in \ell^\infty(\mathbb{Z})$ (the linear space of bounded scalar sequences), there exists a *limit function* $f_{\boldsymbol{\lambda}} \in C^\nu(\mathbb{R})$ satisfying

$$\limsup_{k \rightarrow \infty} \sup_{\alpha \in \mathbb{Z}} \left| f_{\boldsymbol{\lambda}}(2^{-k}\alpha) - \lambda_\alpha^k \right| = 0 \quad (3.3)$$

with $f_{\boldsymbol{\lambda}} \neq 0$ for at least some initial data $\boldsymbol{\lambda}$. As usual, we shall denote the limit function as

$$f_{\boldsymbol{\lambda}} := S_{\{\mathbf{a}^m\}}^\infty \boldsymbol{\lambda} = \lim_{k \rightarrow \infty} S_{\mathbf{a}^{m+k}} \cdots S_{\mathbf{a}^{m+1}} S_{\mathbf{a}^m} \boldsymbol{\lambda}, \quad k > 0. \quad (3.4)$$

The key ingredient to study the convergence of a nonstationary subdivision scheme and the smoothness of its limit function is the *asymptotic equivalence* of subdivision schemes. In particular, we restrict here to the asymptotic equivalence between a nonstationary subdivision scheme and a stationary one [6].

Definition 1. A nonstationary subdivision scheme $S_{\{\mathbf{a}^m\}}$ is said to be asymptotically equivalent to a stationary one $S_{\{\mathbf{a}\}}$, in symbols $S_{\{\mathbf{a}^m\}} \approx S_{\{\mathbf{a}\}}$, if

$$\sum_{m=0}^{\infty} \|S_{\{\mathbf{a}^m\}} - S_{\{\mathbf{a}\}}\|_\infty < \infty, \quad (3.5)$$

where $\|S_{\{\mathbf{a}\}}\|_\infty := \max_{\alpha \in \{0,1\}} \left\{ \sum_{\beta \in \mathbb{Z}} |a_{\alpha-2\beta}| \right\}$.

The convergence and smoothness of a nonstationary subdivision scheme asymptotically equivalent to a stationary scheme can be deduced by the following proposition, which is a consequence of Theorems 7 and 10 in [6].

Proposition 3.1. Let $S_{\{\mathbf{a}^m\}}$ be a nonstationary scheme with $\text{supp}\{\mathbf{a}^m\} \subset [0, n]$, $n < \infty$. If $S_{\{\mathbf{a}^m\}} \approx S_{\{\mathbf{a}\}}$, where $S_{\{\mathbf{a}\}}$ is a C^0 stationary scheme with compact support, then $S_{\{\mathbf{a}^m\}}$ is C^0 . Moreover, if

$$a^k(z) = \frac{(1+z)^\gamma}{2^\gamma} b^k(z), \quad k \geq K \geq 0, \quad (3.6)$$

where $S_{\{\mathbf{b}^m\}}$ is C^0 , then $S_{\{\mathbf{a}^m\}}$ is C^γ .

It is well known that subdivision schemes and refinable functions are strictly related each other. Actually, if the starting sequence of a convergent subdivision scheme is the delta-sequence $\boldsymbol{\delta}_0 = \{\delta_{\alpha,0}\}_{\alpha \in \mathbb{Z}}$, the *basic limit functions*

$$\phi_m := S_{\{\mathbf{a}^m\}}^\infty \boldsymbol{\delta}_0, \quad m \geq 0, \quad \phi := S_{\{\mathbf{a}\}}^\infty \boldsymbol{\delta}_0, \quad (3.7)$$

are just the solutions to the refinement equations (2.1) and (2.2), respectively.

Moreover, the nonstationary refinement equations (2.1) can be associated to a *nonstationary cascade algorithm* as follows [9].

Definition 2. *Given a sequence of masks $\{\mathbf{a}^m\}_{m \geq 0}$, the associated nonstationary cascade algorithm generates the sequences of functions $\{h_{m,k}\}_{m \geq 0, k > 0}$, by the algorithm*

$$h_{m,k} = \sum_{\alpha \in \mathbb{Z}} a_\alpha^k h_{m+1,k-1}(2 \cdot -\alpha), \quad m \geq 0, \quad k > 0. \quad (3.8)$$

For the convergence, the starting functions $h_{m,0}$ have to satisfy the conditions

$$\begin{cases} h_{m,0} \rightarrow \tilde{h}_0 \in L^2(\mathbb{R}), & \text{as } m \rightarrow \infty, \\ \widehat{h}_{m+k,0}(2^{-k}\omega) \rightarrow 1, \quad \omega \in \mathbb{R} & \text{as } k \rightarrow \infty, \end{cases} \quad (3.9)$$

uniformly in m and locally uniformly in ω . Conditions (3.9) are easily satisfied if $h_{m,0} = \tilde{h}_0$ for all $m \geq 0$, where \tilde{h}_0 is a given function with $\widehat{\tilde{h}_0}(0) = 1$. It is also convenient to choose a stable starting sequence, i.e. a sequence $\{h_{m,0}\}_{m \geq 0}$ such that

$$A_m \sum_{\alpha \in \mathbb{Z}} |f_\alpha|^2 \leq \left\| \sum_{\alpha \in \mathbb{Z}} f_\alpha h_{m,0}(\cdot - \alpha) \right\|^2 \leq B_m \sum_{\alpha \in \mathbb{Z}} |f_\alpha|^2, \quad m \geq 0, \quad (3.10)$$

where $0 < A_m \leq B_m$.

The nonstationary cascade algorithm (3.8) is related to the nonstationary subdivision scheme (3.2) by the equations

$$h_{m,k} = \sum_{\alpha \in \mathbb{Z}} s_\alpha^{m,k} h_{m+k,0}(2^k \cdot -\alpha), \quad k > 0, \quad m \geq 0, \quad (3.11)$$

where

$$s^{m,k} := S_{\mathbf{a}^{m+k-1}} \cdots S_{\mathbf{a}^{m+1}} S_{\mathbf{a}^m} \boldsymbol{\delta}_0. \quad (3.12)$$

The convergence of the nonstationary cascade algorithm can be inferred from the convergence of the corresponding subdivision scheme, as the following theorem from [9] shows.

Theorem 3.2. (Goodmann, Lee) *Given a mask sequence $\{\mathbf{a}^m\}_{m \geq 0}$ such that $S_{\{\mathbf{a}^m\}} \approx S_{\{\mathbf{a}\}}$, and a stable starting sequence $\{h_{m,0}\}_{m \geq 0}$, one has*

$$\phi_m = S_{\{\mathbf{a}^m\}}^\infty \delta_0 \quad \Leftrightarrow \quad \lim_{k \rightarrow \infty} \|h_{m,k} - \phi_m\|_\infty = 0, \quad m \geq 0. \quad (3.13)$$

4 New Classes of Nonstationary Subdivision Schemes

New classes of nonstationary subdivision schemes, whose basic limit functions are totally positive refinable functions, can be generated by combining B-splines masks and GP refinement masks. A first example was given in [4] where $C^{n-1}(\mathbb{R})$ totally positive refinable functions having support $[0, n]$ have been introduced. Here, we will construct totally positive refinable functions having the *same smoothness* but *smaller support* than the functions in [4].

Consider the class of nonstationary subdivision schemes

$$\mathcal{A} := \{S_{\{\mathbf{a}^{(n,m)}\}}, n \geq 2, m \geq 0, \mu > 1\}, \quad (4.1)$$

with k -level refinement masks given by

$$\begin{cases} a_\alpha^{(n,k)} := \frac{1}{2^k} \binom{k+1}{\alpha}, & 0 \leq k \leq \nu - 1, \\ a_\alpha^{(n,k)} := \frac{1}{2^{n+k-\mu}} \left[\binom{n+1}{\alpha} + 4(2^{k-\mu} - 1) \binom{n-1}{\alpha-1} \right], & \nu \leq k, \end{cases} \quad (4.2)$$

where ν is a given integer with $1 \leq \nu \leq n$ and $\mu > 1$ is a real parameter. The corresponding symbols are

$$\begin{cases} a^{(n,k)}(z) = \frac{1}{2^k} (1+z)^{k+1}, & 0 \leq k \leq \nu - 1, \\ a^{(n,k)}(z) = \frac{1}{2^{n+k-\mu}} (1+z)^{n-1} (z^2 + 2(2^{1+k-\mu} - 1)z + 1), & \nu \leq k. \end{cases} \quad (4.3)$$

Since in the first $k \leq \nu - 1$ iterations of the scheme we use the masks of the B-splines of degree k , while we use the GP mask $\mathbf{q}^{(n, n+k-\mu)}$ for $k \geq \nu$, we expect to generate refinable functions with support not greater than $[0, n+1]$ and smoothness $n - 1$.

Theorem 4.1. Any $S_{\{\mathbf{a}^{(n,m)}\}} \in \mathcal{A}$ is C^{n-1} . Moreover, the associated basic limit function $\phi^{(n,m)}$ belongs to $C^{n-1}(\mathbb{R})$ and has $\text{supp } \phi^{(n,m)} = [0, R_{(n,m)}(\nu)]$ where

$$R_{(n,m)}(\nu) = \begin{cases} (n-1-\nu)2^{m-\nu} + (m+2), & 0 \leq m \leq \nu-1, \\ n+1, & m \geq \nu. \end{cases} \quad (4.4)$$

Proof. First of all we prove the convergence of the nonstationary scheme with symbols

$$b^{(k)}(z) = \begin{cases} (1+z), & 0 \leq k \leq \nu-1, \\ \frac{1}{2^{1+k-\mu}}(z^2 + 2(2^{1+k-\mu} - 1)z + 1), & \nu \leq k. \end{cases} \quad (4.5)$$

Observe that $b^{(\infty)}(z) := \lim_{k \rightarrow \infty} b^{(k)}(z) = \frac{1}{2}(1+z)^2$ is the symbol of the hat function. Since

$$\|S_{\{\mathbf{b}^{(k)}\}} - S_{\{\mathbf{b}^{(\infty)}\}}\|_{\infty} = \begin{cases} \frac{1}{2}, & 0 \leq k \leq \nu-1, \\ 1 - 2^{-k-\mu}, & \nu \leq k, \end{cases}$$

and

$$\sum_{k=\nu}^{\infty} (1 - 2^{-k-\mu}) < \infty,$$

(cf. [4]), it follows $S_{\{\mathbf{b}^{(k)}\}} \approx S_{\{\mathbf{b}^{(\infty)}\}}$, thus $S_{\{\mathbf{b}^{(k)}\}}$ is C^0 (cf. Proposition 1). Now, observe that

$$a^{(n,k)}(z) = \frac{1}{2^{n-1}}(1+z)^{n-1}b^{(k)}(z), \quad k \geq \nu; \quad (4.6)$$

thus, the smoothness result follows from Proposition 1.

As for the support, one has [7]

$$\text{supp } \phi^{(n,m)} = \left[\sum_{k=m}^{\infty} 2^{m-k-1} L_k, \sum_{k=m}^{\infty} 2^{m-k-1} R_k \right],$$

where $[L_k, R_k] = \text{supp } \mathbf{a}^{(n,k)}$.

Since $L_k = 0$, $k \geq 0$, and $R_k = \begin{cases} k+1 & \text{for } 0 \leq k \leq \nu-1 \\ n+1 & \text{for } \nu \leq k \end{cases}$, the claim follows from straightforward computations. \square

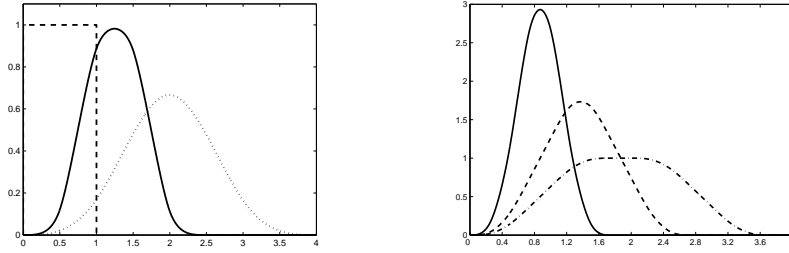


Figure 2: Left: Nonstationary refinable functions $\phi^{(3,0)}$ for $\nu = 1$ and $\mu = 1.5$ (solid line) in comparison with the characteristic function (dashed line) and the cubic spline (dotted line). Right: $\phi^{(3,0)}$ (solid line), $\phi^{(3,1)}$ (dashed line), $\phi^{(3,2)}$ (dash-dotted line) for $\nu = 3$ and $\mu = 1.5$.

We remark that every $\phi^{(n,m)}$ belongs to $C^{n-1}(\mathbb{R})$, but just the refinable functions with $0 \leq m \leq \nu - 1$ have support smaller than $[0, n + 1]$. In particular, one has

$$\begin{aligned} \text{supp } \phi^{(n,0)} &= \left[0, \frac{n}{2} + 1\right], & \nu &= 1, \\ \text{supp } \phi^{(n,m)} &= [0, (m + 2) - 2^{m-n}], & \nu &= n, \quad 0 \leq m \leq n - 1. \end{aligned}$$

As an example, in the table below we give the values of $R_{(3,m)}(\nu)$. Some refinable functions $\phi^{(3,m)}$, belonging to $C^2(\mathbb{R})$, are displayed in Fig. 2.

	$R_{(3,0)}(\nu)$	$R_{(3,1)}(\nu)$	$R_{(3,2)}(\nu)$	$R_{(3,m)}(\nu), m \geq 3$
$\nu = 1$	$5/2$	4	4	4
$\nu = 2$	2	3	4	4
$\nu = 3$	$15/8$	$11/4$	$7/2$	4

Total positivity and central symmetry of the nonstationary refinable functions here constructed can be deduced by the associated nonstationary cascade algorithm, as the following theorem shows.

Theorem 4.2. *Any $\phi^{(n,m)}$, generated by a nonstationary subdivision scheme in \mathcal{A} , is centrally symmetric, i.e. $\phi^{(n,m)}(x) = \phi^{(n,m)}(R_{(n,m)}(\nu) - x)$, and totally positive.*

Proof. Since any nonstationary subdivision scheme in \mathcal{A} is convergent, the corresponding nonstationary cascade algorithm is convergent as well (cf. Th. 1). By choosing the B-splines B_n of degree n as starting functions, we get the following approximation of the basic limit function:

$$h_{m,k} = \sum_{\alpha \in \mathbb{Z}} s_{\alpha}^{(n,m,k)} B_n(2^k \cdot -\alpha), \quad \lim_{k \rightarrow \infty} h_{m,k} = \phi^{(n,m)},$$

where $s^{(n,m,k)} := S_{\mathbf{a}(n,m+k-1)} \cdots S_{\mathbf{a}(n,m+1)} S_{\mathbf{a}(n,m)} \delta_0$.

Recalling that any $h_{m,0}$ is centrally symmetric and totally positive and any mask $a^{(n,k)}$ is a centrally symmetric and totally positive sequence, by induction any $h_{m,k}$ is centrally symmetric and totally positive as well, and so is any $\phi^{(n,m)}$. \square

In order to use the functions $\phi^{(n,m)}$ in applications, we need to study the properties of the system of integer translates

$$\Phi^{(n,m)} := \{\phi^{(n,m)}(\cdot - \alpha), \alpha \in \mathbb{Z}\}, \quad m \geq 0, \quad n \geq 2. \quad (4.7)$$

Proposition 4.3. *For any $n \geq 2$ and $m \geq 0$, the system $\Phi^{(n,m)}$ forms a linearly independent system that is a partition of unity.*

Proof. For any $n \geq 2$ and $m \geq 0$ the Fourier transform $\widehat{\phi}^{(n,m)}(\omega) = \prod_{k \geq m} a^{(n,k)}(e^{i\frac{\omega}{2^k}})$ has no complex periodic zeros. Thus, the system $\Phi^{(n,m)}$ is linearly independent [13]. Partition of unity follows from the properties $\sum_{\alpha \in \mathbb{Z}} a_{\alpha-2\beta}^{(n,k)} = 1$, $\alpha = 0, 1$. \square

Remark. The same reasonings as in Proposition 2 and Theorems 3-4 can be applied to the subdivision scheme (2.5) generating the up-function. Thus, the up-function is totally positive and the system of its integer translates is linearly independent, and forms a partition of unity.

The following theorem concerns the reproduction of linear functions that is of interest in the construction of an approximation operator of Bernstein-Schoenberg type.

Theorem 4.4. *For $n \geq 3$ and $m > 0$, there exist real numbers $\xi_{\alpha}^{(n,m)}$, $\alpha \in \mathbb{Z}$, such that*

$$x = \sum_{\alpha \in \mathbb{Z}} \xi_{\alpha}^{(n,m)} \phi^{(n,m)}(x - \alpha), \quad x \in \mathbb{R}. \quad (4.8)$$

Proof. From the factorization (4.5-4.6) it follows that for $n \geq 3$ and $m > 0$ any symbols $a^{(n,k)}$ has a factor $(1+z)^2$. This means that at any level k , the k -level subdivision operator $S_{\mathbf{a}^{(n,k)}}$ preserves linear functions and so does the basic limit function $\phi^{(n,m)}$ (cf. [14]). \square

Observe that the r.h.s. of (4.8) is nothing other than the Bernstein-Schoenberg type operator

$$\mathcal{S}^{(n,m)} f = \sum_{\alpha \in \mathbb{Z}} f(\xi_{\alpha}^{(n,m)}) \phi^{(n,m)}(\cdot - \alpha) \quad (4.9)$$

applied to $f(x) = x$. From partition of unity and (4.8) it turns out that for $n \geq 3$ and $m > 0$ $\mathcal{S}^{(n,m)}$ reproduces linear functions.

5 Bernstein-like Bases

We are interested in constructing totally positive bases on the interval $[0,1]$ for refinable spaces. We observe that bases obtained by a simple truncation may exhibit instabilities near the endpoints. To avoid this problem, we will construct particular bases on $[0,1]$ exhibiting the main properties of the Bernstein polynomial bases, i.e. central symmetry, total positivity, and the presence of zeros at the endpoints. To this end, we introduce the following definition.

Definition 3. *A Bernstein-like basis $W = \{w_{\alpha}, 0 \leq \alpha \leq N\}$ is a totally positive basis of $[0,1]$ that is centrally symmetric, is a partition of unity and satisfies a zero property of order N at the endpoints, i.e.*

$$\begin{cases} w_0(0) = 1, & w_{\alpha}^{(\gamma)}(0) = 0, & 1 \leq \alpha \leq N, & 0 \leq \gamma \leq \alpha - 1, \\ w_N(1) = 1, & w_{N-\alpha}^{(\gamma)}(1) = 0, & 1 \leq \alpha \leq N, & 0 \leq \gamma \leq \alpha - 1. \end{cases} \quad (5.1)$$

In order to generate a basis satisfying conditions (5.1) starting from a given totally positive basis, we can follow a reasoning line analogous to that ones given in [1, 12].

Let us consider the totally positive basis $\Sigma^{(n,h)}$, formed by the integer translates of a given stationary GP refinable function $\sigma^{(n,h)}$. Let $\Sigma_{[0,1]}^{(n,h)} :=$

$\Sigma^{(n,h)}|_{[0,1]}$ and let $H_{(n,h)}$ be the totally positive matrix constructed by the algorithm described in [12]. This procedure yields the basis

$$U^{(n,h)} = H_{(n,h)}^{-1} \Sigma_{[0,1]}^{(n,h)} \quad (5.2)$$

whose properties are given in the following theorem.

Theorem 5.1. *The basis $U^{(n,h)} = \{u_0^{(n,h)}, \dots, u_n^{(n,h)}\}$ is a Bernstein-like basis. Moreover, for $n \geq 3$, $U^{(n,h)}$ reproduces linear functions.*

We observe that when $n = h$ (B-spline case), $U^{(n,h)}$ reduces to the polynomial Bernstein basis of degree n . In Fig. 3 (left) the Bernstein-like basis $U^{(3,4)}$ is displayed.

Other Bernstein-like bases can be obtained by applying the same procedure to $\Phi_{[0,1]}^{(n,m)} := \Phi^{(n,m)}|_{[0,1]}$. In more detail, let

$$\psi^{(n,m)}(x) := \phi^{(n,m)} \left(x + \frac{1}{2} (\lceil R_{(n,m)}(\nu) \rceil - R_{(n,m)}(\nu)) \right)$$

and denote by $\Psi^{(n,m)}$ the basis of the integer translates of $\psi^{(n,m)}$. The basis

$$W^{(n,m)} = K_{(n,m)}^{-1} \Phi_{[0,1]}^{(n,m)},$$

where $\Psi_{[0,1]}^{(n,m)} := \Psi^{(n,m)}|_{[0,1]}$ and $K_{(n,m)}$ is the matrix constructed by applying the quoted algorithm to $\Psi_{[0,1]}^{(n,m)}$, is a Bernstein-like basis when $|\text{supp } \phi^{(n,m)}| \in \mathbb{N}$. In the case when $|\text{supp } \phi^{(n,m)}|$ is fractional, $W^{(n,m)}$ still has a behavior analogous to that one of a Bernstein-like basis, as the example in Fig. 3 (right) shows.

6 Conclusion

We have constructed a class of centrally symmetric, totally positive refinable functions that are obtained by a suitable use of certain nonstationary subdivision schemes. The integer translates of these functions form a linear independent system that is a partition of unity and reproduces linear functions. The main feature of these functions is in that they have a high spatial localization; in particular, these functions achieve a prescribed smoothness having a smaller support than other totally positive refinable functions, such

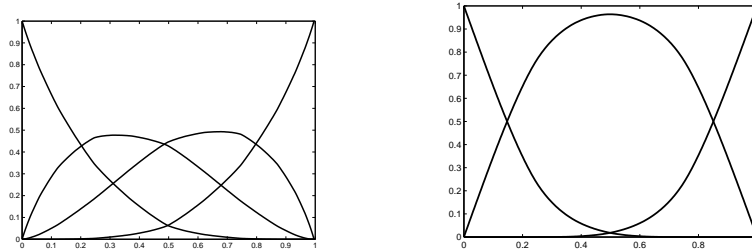


Figure 3: Bernstein-like basis $U^{(3,4)}$ (left) and $W^{(3,0)}$ for $\nu = 1$ and $\mu = 1.5$ (right).

as the B-splines or the functions introduced in [4]. All these properties make them suitable for the reconstruction of curves. Moreover, the corresponding Bernstein-like bases can be used for the construction of quasi-interpolant operators.

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