CHAPTER 1

Semigroups and Evolution Equations: Functional Calculus, Regularity and Kernel Estimates

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Abstract

This is a survey on recent developments of the theory of one-parameter semigroups and evolution equations with special emphasis on functional calculus and kernel estimates. Also other topics as asymptotic behavior for large time and holomorphic semigroups are discussed. As main application we consider elliptic operators with various boundary conditions.

Introduction

The theory of one-parameter semigroups provides a framework and tools to solve evolutionary problems. It is impossible to give an account of this rich and most active field. In this chapter we rather try to present a survey on a particular subject, namely functional calculus, maximal regularity and kernel estimates which, in our eyes, has seen a most spectacular development, and which, so far, is not presented in book form. We comment on these three subjects:

1. *Functional calculus* (Section 4). If A is a self-adjoint operator, one can define $f(A)$ for all bounded complex-valued measurable functions defined on the spectrum of A. It was McIntosh who initiated and developed a theory of functional calculus for a less restricted large class of operators, namely *sectorial operators*; i.e., operators whose spectrum is included in a sector and whose resolvent satisfies a certain estimate. Negative generators of bounded holomorphic semigroups are sectorial operators and are our main subject of investigation. And indeed, for these operators $f(A)$ can be defined for a large class of holomorphic functions defined on a sector containing the spectrum. Taking $f(z) = \mathrm{e}^{-tz}$ leads to the semigroup e^{-tA}, the function $f(z) = z^\alpha$ to the fractional power of such an operator A. One important reason to study functional calculus is the Dore–Venni theorem. In its hypotheses functional calculus plays a role; the conclusion is the invertibility of the sum of two operators A and B. Thus, the Dore–Venni theorem asserts that the equation

$$Ax + Bx = y$$

has a unique solution $x \in D(A) \cap D(B)$. To say that the solution is at the same time in both domains can be rephrased by saying that the solution has "*maximal regularity*", a crucial property in many circumstances.

2. *Form methods* (Section 5). On Hilbert space the functional calculus behaves particularly well as we show in Sections 4 and 5. Most interesting is the close connection with form methods. Basically, the following is true: an operator is associated with a form if and only if it has a bounded H^∞-calculus. Form methods, based on the fundamental Lax–Milgram lemma, allow a most efficient treatment of elliptic and parabolic problems as we show later.

3. *Maximal regularity and Fourier transform* (Section 6). The following particular problem of maximal regularity is important for solving nonlinear equations: The generator A of a semigroup T is said to have property (*MR*) if $T * f \in W^{1,2}((0,1); X)$ for all $f \in L^2((0,1); X)$. On Hilbert spaces every generator of a holomorphic semigroup has (*MR*); but a striking result of Kalton–Lancien asserts that this fact characterizes Hilbert spaces (among a large class of Banach spaces). On the other hand, in recent years it has been understood which role "*unconditional properties*" play for operator-valued Fourier transform and Cauchy problems. So one may characterize property (*MR*) by R-boundedness, a property defining "unconditional boundedness" of sets of operators.

4. *Kernel estimates* (Section 7). Gaussian estimates for the kernels of parabolic equations have been investigated for many years. It is most interesting in its own right that the solutions of a parabolic equation with measurable coefficients are very close to the Gaussian semigroup. But Gaussian estimates have also striking consequences for the underlying semigroup. For example, we show that they do imply boundedness of the H^∞-calculus.

5. *Elliptic operators* (Section 8). The theory presented here can be applied to elliptic operators with measurable coefficients to which Section 8 is devoted. We will explain *Kato's square root problem*, the most difficult question of coincidence of form domain and the domain of the square root, which has been solved recently by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian.

We start the chapter by putting together some basic properties of semigroups which are particularly useful in the sequel. Special attention is given to holomorphic semigroups (Section 2) and to the theory of asymptotic behavior (Section 3). As prototype example in this account serve the Laplacian with Dirichlet and Neumann boundary conditions: On spaces of continuous functions this operator will be considered in Section 2, later its L^p-properties are established. Concerning the results on asymptotic behavior we concentrate on those which can be applied to parabolic equations in Section 8.

Most of the results are presented without proof, referring to the literature. Frequently, only particular cases which are easy to formulate are presented; and in a few cases we give proofs. Some of them are new, not very well known or particularly elegant. The article presents a special choice, guided by personal taste, even in this narrow subject. We hope that the numerous references allow reader to go beyond that choice and that the list of monographs at the end helps them to view the subject in a broader context.

1. Semigroups

In this introductory section we present semigroups from three different points of view. We mention few properties but refer to the various text books concerning the theory.

1.1. *The algebraic approach*

Let X be a complex Banach space, let $C(\mathbb{R}_+, X)$ be the space of all continuous functions defined on $\mathbb{R}_+ := [0, \infty)$ with values in X. *A C_0-semigroup* is a mapping $T : \mathbb{R}_+ \to \mathcal{L}(X)$ such that

(a) $T(\cdot)x \in C(\mathbb{R}_+, X)$ for all $x \in X$;
(b) $T(0) = I$;
(c) $T(s+t) = T(s)T(t)$, $s, t \in \mathbb{R}_+$.

Given a C_0-semigroup T on X, one defines the *generator* A of T as an unbounded operator on X by

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}$$

with domain $D(A) := \{x \in X\colon \lim_{t\downarrow 0} \frac{T(t)x-x}{t} \text{ exists}\}$. Then $D(A)$ is dense in X and A is closed and linear. In other words, A is the derivative of T in 0 (in the strong sense) and for this reason one also calls A the *infinitesimal generator* of T.

The second approach involves the Cauchy problem.

1.2. *The Cauchy problem*

Let A be a closed linear operator on X. Let $J \subset \mathbb{R}$ be an interval. A *mild solution* of the differential equation

$$\dot{u}(t) = Au(t), \quad t \in J, \tag{1.1}$$

is a function $u(t) \in C(J, X)$ such that $\int_s^t u(r)\,\mathrm{d}r \in D(A)$ for all $s, t \in J$ and $A\int_s^t u(r)\,\mathrm{d}r = u(t) - u(s)$. A *classical solution* is a function $u \in C^1(J, X)$ such that $u(t) \in D(A)$ and $\dot{u}(t) = Au(t)$ for all $t \in J$. Since A is closed, a mild solution u is a classical solution if and only if $u \in C^1(J, X)$.

THEOREM [[ABHN01], 3.1.12]. *Let $J = [0, \infty)$. The following assertions are equivalent*:

(i) *A generates a C_0-semigroup T*;

(ii) *for all $x \in X$, there exists a unique mild solution u of* (1.1) *satisfying $u(0) = x$.*

In that case $u(t) = T(t)x$, $t \geqslant 0$.

The theorem implies in particular that the generator A determines uniquely the semigroup. Here is the third approach.

1.3. *Semigroups and Laplace transforms*

Let T be a C_0-semigroup with generator A. Then the *growth bound*

$$\omega(T) := \inf\bigl\{\omega \in \mathbb{R}\colon \exists M \text{ such that } \bigl\|T(t)\bigr\| \leqslant M\mathrm{e}^{\omega t} \text{ for all } t \geqslant 0\bigr\}$$

satisfies $-\infty \leqslant \omega(T) < \infty$, and if $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > \omega(T)$, then λ is in the *resolvent set* $\rho(A)$ of A and

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)x\,\mathrm{d}t := \lim_{\tau\to\infty}\int_0^\tau \mathrm{e}^{-\lambda t}T(t)x\,\mathrm{d}t$$

for all $x \in X$, where $R(\lambda, A) = (\lambda - A)^{-1}$.

THEOREM [[ABHN01], Theorem 3.1.7, p. 113]. *Let $T\colon \mathbb{R}_+ \to \mathcal{L}(X)$ be strongly continuous. Let A be an operator on X, $\lambda_0 \in \mathbb{R}$ such that $(\lambda_0, \infty) \subset \rho(A)$ and*

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t}T(t)x\,\mathrm{d}t, \quad \lambda > \lambda_0,$$

for all $x \in X$. Then T is a C_0-semigroup and A its generator.

Thus generators of C_0-semigroups are precisely those operators whose resolvent is a Laplace transform. Laplace transform techniques play an important role in semigroup theory (see [[ABHN01]] for a systematic theory).

1.4. *More general C_0-semigroups*

In order to talk about Dirichlet boundary conditions we need more general semigroups (cf. Section 2.5). A strongly continuous function $T : (0, \infty) \to \mathcal{L}(X)$ is called a (*nondegenerate locally bounded*) *semigroup* if

(a) $T(t)T(s) = T(t+s)$, $t, s > 0$;
(b) $\sup_{0<t\leqslant 1} \|T(t)\| < \infty$;
(c) $T(t)x = 0$ for all $t > 0$ implies $x = 0$.

As a consequence $\omega(T) < \infty$ and there exists a unique operator A such that $(\omega(T), \infty) \subset \rho(A)$ and $R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t} T(t)x \, \mathrm{d}t$ for all $x \in X$ and $\lambda > \omega(T)$. We call A the *generator* of T (see [[ABHN01], 3.2]). Then T is a C_0-semigroup if and only if $\overline{D(A)} = X$. If X is reflexive, then this is automatically true.

1.5. *The inhomogeneous Cauchy problem*

If A generates a C_0-semigroup T, then also the inhomogeneous Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t) + f(t), & t \in [0, \tau], \\ u(0) = x \end{cases} \tag{1.2}$$

is well posed. More precisely, let $x \in X$, $f \in L^1((0, \tau); X)$. A *mild solution* of (1.2) is a continuous function $u : [0, \tau] \to X$ such that $\int_0^t u(s) \, \mathrm{d}s \in D(A)$ and

$$u(t) - x = A \int_0^t u(s) \, \mathrm{d}s + \int_0^t f(s) \, \mathrm{d}s$$

for all $t \in [0, \tau]$. Define $T * f$ by $T * f(t) = \int_0^t T(t-s) f(s) \, \mathrm{d}s$.

PROPOSITION. *The function u given by $u(t) = T(t)x + T * f(t)$ is the unique mild solution of* (1.2).

1.5.1. *Classical solutions.* Thus, in the particular case where $x = 0$, the mild solution of (1.2) is $u = T * f$. It is very rare that this u is a classical solution for all f. Let T be a C_0-semigroup on X with generator A.

THEOREM (Baillon; see [EG92]). *Assume that X is reflexive or $X = L^1$ (or more generally that $c_0 \not\subset X$). If for all $f \in C([0, \tau]; X)$ one has $T * f \in C^1([0, \tau], X)$, then A is bounded.*

In Section 5 we will devote much attention to the question when

$$T * f \in W^{1,p}\big((0,\tau); X\big) \quad \text{for all } f \in L^p\big((0,\tau); X\big).$$

2. Holomorphic semigroups

A semigroup $T:(0,\infty) \to X$ (in the sense of Section 1.4) is called *holomorphic* if there exists $\theta \in (0, \pi/2]$ such that T has a holomorphic extension $T: \Sigma_\theta \to \mathcal{L}(X)$ which is bounded on $\{z \in \Sigma_\theta\colon\ |z| \leqslant 1\}$. In that case, this holomorphic extension is unique and satisfies $T(z_1 + z_2) = T(z_1)T(z_2)$ for all $z_1, z_2 \in \Sigma_\theta$. If T is a C_0-semigroup, then

$$\lim_{\substack{z\to 0\\ z\in\Sigma_\theta}} T(z)x = x$$

for all $x \in X$, and we call T a *holomorphic C_0-semigroup*. If the extension T is bounded on Σ_θ, we call T a *bounded holomorphic semigroup*. Thus, for this property, it does not suffice that T is bounded on $[0, \infty)$. Take for example, $T(t) = \mathrm{e}^{\mathrm{i}t}$ on $X = \mathbb{C}$.

2.1. *Characterization of bounded holomorphic semigroups*

Let A be an operator on X. The following assertions are equivalent:

(i) *A generates a bounded holomorphic semigroup T*;

(ii) *one has $\lambda \in \rho(A)$ whenever* $\operatorname{Re}\lambda > 0$ *and* $\sup_{\operatorname{Re}\lambda>0} \|\lambda R(\lambda, A)\| < \infty$;

(iii) *there exists $\alpha \in (0, \pi/2)$ such that* $\mathrm{e}^{\pm\mathrm{i}\alpha}A$ *generates a bounded semigroup.*

In that case $(T(\mathrm{e}^{\pm\mathrm{i}\alpha}t))_{t\geqslant 0}$ is the semigroup generated by $\mathrm{e}^{\pm\mathrm{i}\alpha}A$. Moreover, T is a C_0-semigroup if and only if $D(A)$ is dense. This is automatic whenever X is reflexive.

2.2. *Characterization of holomorphic semigroups*

An operator A generates a holomorphic semigroup T if and only if there exists ω such that $A - \omega$ generates a bounded holomorphic semigroup S. In that case $T(t) = \mathrm{e}^{\omega t}S(t)$, $t \geqslant 0$.

2.3. *Boundary groups*

Let A be the generator of a C_0-semigroup T having a holomorphic extension to the half-plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C}\colon\ \operatorname{Re}\lambda > 0\}$. Then $\mathrm{i}A$ generates a C_0-group U if and only if $\sup_{\operatorname{Re}z>0, |z|\leqslant 1} \|T(z)\| < \infty$.

In that case we call U the *boundary group* of T and write $U(s) =: T(\mathrm{i}s)$, $s \in \mathbb{R}$.

2.4. *The Gaussian semigroup*

Consider the *Gaussian semigroup* G defined on $L^1 + L^\infty := L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ by

$$\big(G(z)f\big)(x) = (4\pi z)^{-n/2} \int_{\mathbb{R}^n} f(y) \mathrm{e}^{-(x-y)^2/4z}\, \mathrm{d}y \tag{2.1}$$

for all $x \in \mathbb{R}^n$, $f \in L^1 + L^\infty$, $\operatorname{Re} z > 0$. Then G is a holomorphic semigroup which is bounded on Σ_θ for each $0 < \theta < \pi/2$. The generator $\Delta_{1+\infty}$ of G is given by

$$D(\Delta_{1+\infty}) = \big\{ f \in L^1 + L^\infty \colon\ \Delta f \in L^1 + L^\infty \big\},$$
$$\Delta_{1+\infty} f = \Delta f \quad \text{in } \mathcal{D}\big(\mathbb{R}^n\big)'.$$

Let E be one of the spaces $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, $C^{\mathrm{b}}(\mathbb{R}^n) = \{f \in L^\infty(\mathbb{R}^n)\colon\ f$ is continuous$\}$;

$$BUC\big(\mathbb{R}^n\big) := \big\{ f \in C^{\mathrm{b}}\big(\mathbb{R}^n\big)\colon\ f \text{ is uniformly continuous} \big\},$$
$$C_0\big(\mathbb{R}^n\big) := \Big\{ f \in C^{\mathrm{b}}\big(\mathbb{R}^n\big)\colon\ \lim_{|x|\to\infty} \big|f(x)\big| = 0 \Big\},$$

which are all subspaces of $L^1 + L^\infty$. Then the restriction $G(t)|_E$ defines a holomorphic semigroup G_E on E which is bounded on Σ_θ for each $\theta \in (0, \pi/2)$. Its generator is Δ_E given by

$$D(\Delta_E) = \{ f \in E\colon\ \Delta f \in E \},$$
$$\Delta_E f = \Delta f \quad \text{in } \mathcal{D}\big(\mathbb{R}^n\big)'.$$

On $E = L^p(\mathbb{R}^n)$, $1 \leqslant p < \infty$, $BUC(\mathbb{R}^n)$ and $C_0(\mathbb{R}^n)$ the semigroup G_E is a C_0-semigroup.

2.5. *The Dirichlet Laplacian*

In this section we introduce the Laplacian with Dirichlet boundary conditions on the space $C(\overline{\Omega})$. It is a most basic example and we show in an elementary way that it generates a holomorphic semigroup.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We assume that Ω is *Dirichlet regular*; i.e., for all $\varphi \in C(\partial\Omega)$ there exists a solution of

$$D(\varphi) \begin{cases} h \in C\big(\overline{\Omega}\big), & h|_{\partial\Omega} = \varphi, \\ \Delta h = 0 & \text{in } \mathcal{D}(\Omega)'. \end{cases}$$

Such a solution is unique and automatically in $C^\infty(\Omega)$. If Ω has Lipschitz boundary, then Ω is Dirichlet regular, but much milder geometric assumptions suffice (see, e.g., [[DL88], Chapter II]). We consider the operator A defined on $C(\overline{\Omega})$ by

$$D(A) = \{u \in C_0(\Omega) \colon \ \Delta u \in C(\overline{\Omega})\},$$
$$Au = \Delta u \quad \text{in } \mathcal{D}(\Omega)',$$

where $C_0(\Omega) := \{u \in C(\overline{\Omega}) \colon \ u_{|\partial\Omega} = 0\}$ and $\mathcal{D}(\Omega)'$ denotes the space of all distributions. We call A the *Dirichlet Laplacian* on $C(\overline{\Omega})$.

THEOREM. *The operator A generates a bounded holomorphic semigroup T on $C(\overline{\Omega})$.*

For the proof we need the following form of the maximum principle.

MAXIMUM PRINCIPLE. *Let $v \in C(\overline{\Omega})$ such that $\lambda v - \Delta v = 0$ in $\mathcal{D}(\Omega)'$, where* $\operatorname{Re}\lambda > 0$. *Then*

$$\sup_{x \in \partial\Omega} |v(x)| = \sup_{x \in \overline{\Omega}} |v(x)|.$$

PROOF. Suppose that $\|v\|_{L^\infty(\Omega)} > \sup_{x\in\partial\Omega} |v(x)|$. Let $K := \{x \in \Omega \colon \ |v(x)| = \|v\|_\infty\}$ and $v_\varepsilon = \rho_\varepsilon * v$, where ρ_ε is a mollifier. Then $v_\varepsilon \to v$, $\Delta v_\varepsilon = \rho_\varepsilon * \Delta v \to \Delta v$ uniformly on compact subsets of Ω as $\varepsilon \downarrow 0$. Let $\Omega_1 \subset \Omega$ be relatively compact such that $K \subset \Omega_1$ and $\overline{\Omega}_1 \subset \Omega$. Then, for small $\varepsilon > 0$, there exists $x_\varepsilon \in \Omega_1$ such that $|v_\varepsilon(x_\varepsilon)| = \sup_{x\in\Omega_1} |v_\varepsilon(x)|$. Then

$$\operatorname{Re}\big[\Delta v_\varepsilon(x_\varepsilon)\overline{v_\varepsilon(x_\varepsilon)}\big] \leqslant 0. \tag{2.2}$$

In fact, consider the function $f(y) = \operatorname{Re} v_\varepsilon(y)\overline{v_\varepsilon(x_\varepsilon)}$. Then f has a local maximum in x_ε. Hence $\Delta f(x_\varepsilon) \leqslant 0$. Let $x_{\varepsilon_n} \to x_0$. Since $v_{\varepsilon_n} \to v$ uniformly on Ω_1, it follows that $x_0 \in K$. From (2.2) we deduce that $\operatorname{Re}[\overline{v(x_0)}\Delta v(x_0)] \leqslant 0$. Hence,

$$\begin{aligned}\operatorname{Re}\lambda |v(x_0)|^2 &\leqslant \operatorname{Re}\lambda |v(x_0)|^2 - \operatorname{Re}\big[\overline{v(x_0)}\Delta v(x_0)\big] \\ &= \operatorname{Re}\big[\overline{v(x_0)}\big(\lambda v(x_0) - \Delta v(x_0)\big)\big] = 0.\end{aligned}$$

Since $x_0 \in K$, it follows that $v = 0$, contradicting the assumption. □

PROOF OF THE THEOREM. (a) Similarly as the Maximum principle above, one shows that A is dissipative.

(b) We show that $0 \in \rho(A)$. Let $f \in C(\overline{\Omega})$. Denote by $\tilde{f}$ the extension of f to $\mathbb{R}^n$ by 0 and let $v = E * \tilde{f}$, where E is the Newtonian potential. Then $v \in C(\mathbb{R}^n)$ and $\Delta v = f$ in $\mathcal{D}(\Omega)'$. Let $\varphi = v|_{\partial\Omega}$ and consider the solution h of the Dirichlet problem $D(\varphi)$. Then $u = v - h \in D(A)$ and $Au = f$. We have shown that A is surjective. Since the solution of $D(0)$ is unique, the operator A is injective. Since A is closed, it follows that $0 \in \rho(A)$.

(c) It follows from (a) and (b) that A is m-dissipative. In particular, $\lambda \in \rho(A)$ whenever $\operatorname{Re}\lambda > 0$.

(d) Denote by Δ_0 the Laplacian on $C_0(\mathbb{R}^n)$ which generates a bounded holomorphic semigroup by 2.4. Thus there exists $M \geqslant 0$ such that

$$\left\| \lambda R(\lambda, \Delta_0) \right\| \leqslant M \qquad \text{if } \operatorname{Re} \lambda > 0.$$

We show that $\|\lambda R(\lambda, A)\| \leqslant 2M$ if $\operatorname{Re}\lambda > 0$ which proves the Theorem. In fact, let $\operatorname{Re}\lambda > 0$, $f \in C(\overline{\Omega})$, $g = R(\lambda, A)f$. Let $\tilde{g} = R(\lambda, \Delta_\infty)\tilde{f}$. Then $v = g - \tilde{g} \in C(\overline{\Omega})$. Moreover, $v|_{\partial\Omega} = -\tilde{g}|_{\partial\Omega}$ and $\lambda v - \Delta v = 0$ in $\mathcal{D}(\Omega)'$. By the Maximum principle, one has

$$\begin{aligned} \sup_{x \in \overline{\Omega}} \left| v(x) \right| &= \sup_{x \in \partial\Omega} \left| v(x) \right| = \sup_{x \in \partial\Omega} \left| \tilde{g}(x) \right| \\ &\leqslant \frac{M}{|\lambda|} \left\| \tilde{f} \right\|_{L^\infty(\mathbb{R}^n)} = \frac{M}{|\lambda|} \|f\|_{L^\infty(\Omega)}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|g\|_{L^\infty(\Omega)} &\leqslant \|v\|_{L^\infty(\Omega)} + \left\| \tilde{g} \right\|_{L^\infty(\Omega)} \\ &\leqslant 2\frac{M}{|\lambda|} \|f\|_{L^\infty(\Omega)}. \end{aligned}$$

□

FURTHER PROPERTIES. *One has $\omega(T) < \infty$ and $T(t)$ is positive and compact for all $t > 0$. The restriction T_0 of T to $C_0(\Omega)$ is a C_0-semigroup.*

REFERENCE. The elegant elementary argument above is due to Lumer and Paquet [LP79], where it is given for more general elliptic operators. See also [[ABHN01], Chapter 6] for a different presentation, and [LS99] for more general results.

2.6. *The Neumann Laplacian on $C(\overline{\Omega})$*

Let $\Omega \subset \mathbb{R}^n$ be open. There is a natural realization Δ_Ω^N of the Laplacian on $L^2(\Omega)$ with Neumann boundary conditions. The operator Δ_Ω^N is self-adjoint and generates a bounded, holomorphic, positive C_0-semigroup $(\mathrm{e}^{t\Delta_\Omega^N})_{t \geqslant 0}$ on $L^2(\Omega)$. We refer to Section 5.3.3 for the precise definition.

THEOREM [FT95]. *Let Ω be bounded with Lipschitz-boundary. Then $C(\overline{\Omega})$ is invariant under the semigroup $(\mathrm{e}^{t\Delta_\Omega^N})_{t \geqslant 0}$ and the restriction is a bounded holomorphic C_0-semigroup T on $C(\overline{\Omega})$.*

The main point in the proof is to show invariance of $C(\overline{\Omega})$ by the semigroup or the resolvent. Holomorphy can be shown with the help of Gaussian estimates (Section 7.4.3). We also mention that the semigroup T induced on $C(\overline{\Omega})$ is *compact*, i.e., each $T(t)$ is compact for $t > 0$. This follows from ultracontractivity (Sections 7.3.3 and 7.3.7).

Not for each open, bounded set Ω the space $C(\overline{\Omega})$ is invariant: the Theorem is false on the domain

$$\Omega = \left\{(x, y) \in \mathbb{R}^2\colon\ |x| < 1, |y| < 1\right\} \setminus [0, 1) \times \{0\};$$

see [Bie03].

The Theorem also holds for Robin boundary conditions, we refer to [War03a].

2.7. *Wentzell boundary conditions*

As a further example we mention a very different kind of boundary condition. Let $m \in C[0, 1]$ be strictly positive. Let

$$(Lu)(x) = m(x) \cdot x(1 - x)u''(x), \quad x \in (0, 1),$$

for $u \in C^2(0, 1)$. Define the operator A on $C[0, 1]$ by

$$D(A) = \left\{u \in C[0, 1] \cap C^2(0, 1)\colon\ \lim_{\substack{x\to 0\\ x\to 1}} (Lu)(x) = 0\right\},$$
$$Au = Lu.$$

Then A generates a holomorphic C_0-semigroup T of angle $\pi/2$. Moreover, T is positive and contractive.

We refer to Campiti and Metafune [CM98] for this and more general degenerate elliptic operators with *Wentzell boundary conditions*.

2.8. *Dynamic boundary conditions*

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set of class C^2. We will introduce a realization of the Laplacian in $C(\overline{\Omega})$ with Wentzell–Robin boundary conditions. By $C^1_\nu(\overline{\Omega})$ we denote the space of all functions $f \in C(\overline{\Omega})$ for which the outer normal derivative

$$\frac{\partial f}{\partial \nu}(z) = -\lim_{t\downarrow 0} \frac{f(z - t \cdot \nu(z)) - f(z)}{t}$$

exists uniformly for $z \in \partial\Omega$; see [[DL88], Vol. 1, Sect. II.1.3b]. Let $\beta, \gamma \in C(\partial\Omega)$ and suppose that $\beta(z) > 0$ for all $z \in \partial\Omega$. Define the operator A on $C(\overline{\Omega})$ by

$$D(A) := \left\{f \in C^1_\nu\big(\overline{\Omega}\big)\colon\ \Delta f \in C\big(\overline{\Omega}\big), \Delta f + \beta\frac{\partial f}{\partial \nu} + \gamma f = 0 \text{ on } \partial\Omega\right\},$$
$$Af := \Delta f.$$

THEOREM. *The operator A generates a positive, compact and holomorphic C_0-semigroup on $C(\overline{\Omega})$.*

Favini, Goldstein, Goldstein and Romanelli [FGGR02] were the first to prove that A is a generator with the help of dissipativity. An approach by form methods was then given in [AMPR03]. Warma [War03b] proved analyticity in the case where Ω is an interval. It was Engel [Eng04] who succeeded to prove that the semigroup is holomorphic in the general case.

The boundary conditions incorporated into the domain of A express in fact *dynamic boundary conditions* for the evolution equation. To see this, denote by T the semigroup generated by A. Let $f \in C(\overline{\Omega})$ and let $u(t) = T(t)f$. Then $u \in C^1((0,\infty), C(\overline{\Omega}))$ and $\dot{u}(t) = \Delta u(t)$, $t > 0$, on Ω. Moreover, $\Delta u(t) \in C(\overline{\Omega})$ and $\Delta u(t) = -\beta \frac{\partial u}{\partial \nu}(t) - \gamma u(t)$ on $\partial\Omega$. Hence,

$$\dot{u}(t) = -\beta \frac{\partial u}{\partial \nu}(t) - \gamma u(t) \quad \text{on } \partial\Omega, t > 0.$$

3. Asymptotics

Most important and interesting is the study of the asymptotic behavior of a semigroup $T(t)$ for $t \to \infty$. The philosophy is, as for other questions, that one knows better the generator and its resolvent than the semigroup. Thus the challenge is to deduce the asymptotic behavior from spectral properties of the generator. Here we will describe some principal results with emphasis on those which can be applied to parabolic equations in Section 8. We refer to [[ABHN01]] and [[Nee96]] for a systematic theory of the asymptotic behavior of semigroups and to [[EN00]] for other kinds of examples.

3.1. *Exponential stability*

Let T be a C_0-semigroup with generator A. By

$$s(A) = \sup\{\operatorname{Re}\lambda \colon \lambda \in \sigma(A)\}$$

we denote the *spectral bound* of A. We say that T is *exponentially stable* if $\omega(T) < 0$; i.e., if there exist $\varepsilon > 0$, $M \geqslant 0$ such that

$$\|T(t)\| \leqslant M e^{-\varepsilon t}, \quad t \geqslant 0.$$

It is easy to see that T is exponentially stable if and only if

$$\lim_{t\to\infty} \|T(t)\| = 0.$$

FUNDAMENTAL QUESTION. *Does $s(A) < 0$ imply that T is exponentially stable*? In general this is not true. The realization of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t}(t,s) = s\frac{\partial u}{\partial s}(t,s), & t > 0, s > 1, \\ u(0,s) = u_0(s), & s > 1, \end{cases}$$

in the Sobolev space $W^{1,2}(1,\infty)$ leads to a C_0-semigroup T whose generator A has spectral bound $s(A) < -\frac{1}{2}$ but T is unbounded [[ABHN01], p. 350]. An example of a hyperbolic equation is given by Renardy [Ren94].

However, additional hypotheses are known, which lead to a positive answer. Here we consider three important cases corresponding to a regularity assumption, semigroups on Hilbert space and a positivity assumption.

3.1.1. *Eventually norm continuous semigroups.* A C_0-semigroup T is called *eventually norm-continuous* if $\lim_{t\downarrow 0} \|T(t_0 + t) - T(t_0)\| = 0$ for some $t_0 > 0$, and T is called *norm-continuous* if this holds for all $t_0 > 0$. Of course, each holomorphic semigroup has this property.

THEOREM. *Let A be the generator of an eventually norm-continuous semigroup T. If $s(A) < 0$, then T is exponentially stable.*

For further extensions we refer to [[ABHN01], Chapter 5], [Bla01], [BBN01].

We mention some perturbation results: If A generates an eventually norm continuous C_0-semigroup T on X and $B \in \mathcal{L}(X)$ is compact, then $A + B$ also generates an eventually norm-continuous C_0-semigroup. The compactness assumption cannot be omitted, in general. It can be omitted if T is norm-continuous on $(0,\infty)$. Eventually norm-continuous semigroups appear in models for cell growth. We refer to [[Nag86], A-II.1.30 and C-IV.2.15].

3.1.2. *The Gearhart–Prüss theorem.* *Let H be a Hilbert space. Assume that $s(A) < 0$ and* $\sup_{\operatorname{Re}\lambda>0} \|R(\lambda, A)\| < \infty$. *Then T is exponentially stable.*

There are several proofs of this result which all depend on the fact that the vector-valued Fourier transform is an isomorphism for Hilbert spaces. Prüss' proof [Prü84] uses Fourier-series. The above result is not true on L^p-spaces for $1 \leqslant p \leqslant \infty$, $p \neq 2$; see [ArBu02], Example 3.7.

3.1.3. *Positive semigroups on L^p-spaces.* In the next result we consider a C_0-semigroup T on $L^p(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a measure space and $1 \leqslant p < \infty$. The semigroup is called *positive* if $T(t)f \geqslant 0$ for each $0 \leqslant f \in L^p(\Omega, \Sigma, \mu)$. Of course, here $f \geqslant 0$ means that $f(x) \geqslant 0$ μ-a.e. For positive semigroups, there is an easy criterion for negative spectral bound: One has

$$s(A) < 0 \text{ if and only if } A \text{ is invertible and } A^{-1} \leqslant 0.$$

THEOREM (Weis [Wei95]). *Let A be the generator of a positive C_0-semigroup T on $L^p(\Omega)$, $1 \leqslant p < \infty$. If $s(A) < 0$, then T is exponentially stable.*

For a proof and further references we refer to [[ABHN01], 5.3.6] and [[Nee96]]. A similar result is true on $C_0(\Omega)$, where Ω is a locally compact space [[ABHN01], 5.3.8], but false on a space $L^p \cap L^q$ [[ABHN01], 5.1.11].

3.2. *Ergodic semigroups*

Let T be a bounded C_0-semigroup on a Banach space X. Denote by A the generator of T and by A^* the adjoint of A. We say that T is *ergodic* if

$$Px = \lim_{t \to \infty} \frac{1}{t} \int_0^t T(s)x \, \mathrm{d}s$$

exists for all $x \in X$.

ERGODIC THEOREM. *The following assertions are equivalent*:
(i) T *is ergodic*;
(ii) $X = \ker A \oplus \overline{R(A)}$;
(iii) $\ker A$ *separates* $\ker A^*$.
In that case P *is the projection onto* $\ker A$ *along* $\overline{R(A)}$.

Here we denote by $R(A) = \{Ax\colon\ x \in D(A)\}$ the range of A. One has always $\ker A \cap \overline{R(A)} = \{0\}$. To say that $\ker A$ separates $\ker A^*$ means that for all $x^* \in \ker A^*$, $x^* \neq 0$, there exists $x \in \ker A$ such that $\langle x^*, x\rangle \neq 0$. Note that $\ker A^*$ always separates $\ker A$ (by the Hahn–Banach theorem). Thus on reflexive spaces, ergodicity is automatic.

THEOREM. *Every bounded C_0-semigroup on a reflexive space is ergodic.*

It is interesting to know whether reflexivity is the best possible hypothesis on the Banach space in order to guarantee automatic ergodicity. And indeed it is, under some additional hypothesis. In fact, on a general Banach space no method is known to construct nontrivial C_0-semigroups. For this reason one has to suppose some geometric property. We will assume that X has a Schauder basis. See Section 4.5.2 for the precise definition and also for a method to construct diagonal semigroups under this hypothesis. The following theorem is a semigroup version of a result on power bounded operators by Fonf–Lin–Wojtaszczyk [FLW01] which was recently given by Mugnolo [Mug02].

THEOREM. *Let X be a Banach space with a Schauder basis. Assume that each bounded C_0-semigroup on X is ergodic. Then X is reflexive.*

We conclude by an example. Denote by G_p the Gaussian semigroup, on $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, and by Δ_p its generator (see Section 2.4). Then G_1 is not ergodic since

$\ker \Delta_1 = 0$ and $\ker \Delta_\infty = \mathbb{R} \cdot 1$. The semigroup G_p is ergodic for $1 < p < \infty$. But one can even show that $\lim_{t\to\infty} G_p(t) = 0$ strongly in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Strong convergence, and not merely convergence in mean, is the subject of the next section and the result for the Gaussian semigroup follows from Section 3.3.2.

3.3. *Convergence and asymptotically almost periodicity*

In the preceding section we described when a semigroup converges in mean. Now we investigate a stronger property, namely strong convergence (see Theorem 2 of Section 3.3.2). More generally, we consider semigroups which can be decomposed into a semigroup converging strongly to zero and an almost periodic group.

Let T be a bounded C_0-semigroup on X with generator A. We say that T is *asymptotically almost periodic* if $X = X_0 \oplus X_{\mathrm{ap}}$, where

$$X_0 := \Big\{ x \in X\colon \lim_{t\downarrow 0} T(t)x = 0 \Big\}$$

$$X_{\mathrm{ap}} := \overline{\operatorname{span}}\big\{ x \in X\colon \exists \eta \in \mathbb{R}, T(t)x = \mathrm{e}^{\mathrm{i}\eta t} x \big\}.$$

Note that X_0 and X_{ap} are invariant under the semigroup. Moreover, there exists a bounded C_0-semigroup U on X_{ap} such that $T(t)|_{X_{\mathrm{ap}}} = U(t)$, $t \geqslant 0$. Let $\sigma_p(A) := \{\lambda \in \mathbb{C}\colon \exists x \in D(A), x \neq 0, Ax = \lambda x\}$ be the *point spectrum* of A. If $\sigma_p(A) \cap \mathrm{i}\mathbb{R} \subset \{0\}$, then $X_{\mathrm{ap}} = \ker A$. In that case T is asymptotically almost periodic if and only if $Px := \lim_{t\to\infty} T(t)x$ converges for all $x \in X$ (and not just only in mean as considered in the previous section).

3.3.1. *Compact resolvent.* Let A be an operator with nonempty resolvent set $\rho(A)$. We say that A has *compact resolvent* if $R(\lambda, A)$ is compact for all (equivalently one) $\lambda \in \rho(A)$. This is equivalent to saying that the injection $D(A) \hookrightarrow X$ is compact where $D(A)$ carries the graph norm. It implies that $\sigma(A) = \sigma_p(A)$ is a sequence converging to ∞ (unless $\dim X < \infty$).

THEOREM [[ABHN01], p. 361]. *Let T be a bounded C_0-semigroup whose generator has compact resolvent. Then T is asymptotically almost periodic.*

This result can be generalized to the case where $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is countable with more involved proofs, though.

3.3.2. *Countable spectrum.* Let T be a bounded C_0-semigroup with generator A.

THEOREM 1. *Assume that X is reflexive and $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is countable. Then T is asymptotically almost periodic.*

If X is not reflexive, one needs an ergodicity hypothesis.

THEOREM 2. *Assume that*

(a) $\sigma(A) \cap \mathrm{i}\mathbb{R}$ *is countable*;

(b) $\sigma_p(A^*) \cap \mathrm{i}\mathbb{R} \subset \{0\}$;
(c) T *is ergodic.*
Then $Px = \lim_{t\to\infty} T(t)x$ *converges for all* $x \in X$.

REFERENCES: [[ABHN01], Chapter 5], [Vu97]. So far, there does not exist a (spectral) characterization of strong convergence. But we refer to Chill and Tomilov [CT03,CT04] for very interesting (not purely spectral) conditions.

3.4. *Positive semigroups*

The asymptotic behavior of positive semigroups is most interesting and has applications to many areas, for example population dynamics and transport theory (see [[EN00]], [[Mok97]], [[Nag86]]). Here we think more of applications to parabolic equations and establish a result on convergence to a rank-1-projection which will be applied to a second-order parabolic equation (see Section 3.5.1 and (8.5)). We also present some other results which are obtained by putting together the results on countable spectrum of Section 3.3 and Perron–Frobenius theory for positive semigroups. Throughout this section we assume that X is a space of the following two kinds:
(a) $X = L^p(\Omega)$, $1 \leqslant p < \infty$, where (Ω, Σ, μ) is a σ-finite measure space, or
(b) $X = C_0(K)$, where K is locally compact.
Here we let $C_0(K) := \{f : K \to \mathbb{C}$ continuous: for each $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset K$ such that $|f(x)| \leqslant \varepsilon$ for $x \in K \setminus K_\varepsilon\}$. Of course, if K is compact, then $C_0(K) = C(K)$.

By X_+ we denote the cone of all functions f in X which are positive almost everywhere if $X = L^p$ and everywhere if $X = C_0(K)$. Let T be a positive C_0-semigroup on X; i.e., T satisfies $T(t)X_+ \subset X_+$ for all $t > 0$. Denote by A the generator of T. At first we recall that

$$s(A) = \omega(T) \tag{3.1}$$

and $s(A) \in \sigma(A)$ if $\sigma(A) \neq \emptyset$, two important special properties of positive C_0-semigroups on X (see Section 1.3 for the definition of $\omega(T)$, and [[ABHN01], Theorems 5.3.6 and 5.3.1] for the proofs).

THEOREM. *Let T be a bounded, ergodic, eventually norm-continuous, positive C_0-semigroup on X. Then*

$$Pf = \lim_{t\to\infty} T(t)f$$

exists for all $f \in X$.

As a consequence P is a positive projection, the projection onto $\ker A$ along $\overline{R(A)}$. Recall that each holomorphic semigroup is eventually norm-continuous and ergodicity is automatic if $X = L^p$, $1 < p < \infty$.

It is a remarkable result of Perron–Frobenius theory that $\sigma(A) \cap \mathrm{i}\mathbb{R} \subset \{0\}$ whenever T is positive, bounded and eventually norm-continuous, [[Nag86], C-III.2.10, p. 202 and A-II.1.20, p. 38]. Thus the theorem follows from Theorem 2 in Section 3.3.

3.5. *Positive irreducible semigroup*

We assume again that $X = L^p(\Omega)$, $1 \leqslant p < \infty$, or $X = C_0(K)$ as in Section 3.4. An element $f \in L^p(\Omega)$ is called *strictly positive* if $f(x) > 0$ a.e. An element $f \in C_0(K)$ is called *strictly positive* if $f(x) > 0$ for all $x \in K$. Finally, a functional $\varphi \in C_0(K)^*$ is called *strictly positive* if

$$\langle \varphi, f \rangle > 0 \quad \text{for all } f \in C_0(K)_+ \setminus \{0\}.$$

Let $u \in X$ and $\varphi \in X^*$ be strictly positive such that $\langle \varphi, u \rangle = 1$. Then $Pf = \langle \varphi, f \rangle u$ defines a projection on X. We call P a *strictly positive rank-*1*-projection*.

DEFINITION [[Nag86], p. 306]. A positive C_0-semigroup is *irreducible* if for some (equivalently all) $\lambda > s(A)$ the function $R(\lambda, A)f$ is strictly positive for all $f \in X_+ \setminus \{0\}$.

Irreducibility has many remarkable consequences (see [[Nag86], p. 306]). For example, if $X = C_0(K)$, it implies that $\sigma(A) \neq \emptyset$. This also remains true on $X = L^p(\Omega)$, $1 \leqslant p < \infty$, if in addition we assume that $T(t_0)$ is compact for some $t_0 > 0$. However, this case is more difficult and depends on a deep result of de Pagter [[Nag86], C-III-3.7].

3.5.1. *Convergence to a rank-*1*-projection.* In the following theorem we assume that $T(t_0)$ is compact for some $t_0 > 0$. This implies that $T(t)$ is compact for all $t > t_0$ and that T is norm-continuous on $[t_0, \infty)$. For example, if T is holomorphic and A has compact resolvent, then $T(t)$ is compact for all $t > 0$.

THEOREM. *Let T be a positive, irreducible C_0-semigroup on X. Assume that $T(t_0)$ is compact for some $t_0 > 0$. Then $s(A) > -\infty$ and there exists a strictly positive rank-*1*-projection P such that*

$$\left\| \mathrm{e}^{-s(A)t} T(t) - P \right\| \leqslant M \mathrm{e}^{-\varepsilon t}, \quad t \geqslant 0,$$

for some $M \geqslant 0$, $\varepsilon > 0$.

Thus the rescaled semigroup converges exponentially to a rank-1-projection. Such rescaled convergence is sometimes called *balanced exponential growth* and plays an important role for models describing cell growth (see [Web87]). Here the theorem will be applied to parabolic equations (see Section 8.5).

PROOF OF THEOREM. Since $\sigma(A) \neq \emptyset$ we can assume that $s(A) > -\infty$. Since $T(t_0)$ is compact, T is quasicompact [[Nag86], B-IV.2.8, p. 214]. Now the result follows from [[Nag86], C-IV.2.1, p. 343, and C-III.3.5(d), p. 310]. $\square$

3.5.2. *Convergence to a periodic group.* The following result is a combination of Perron–Frobenius theory for positive semigroups and the results on countable spectrum of Section 3.3. Let T be a bounded C_0-semigroup with generator A. Assume that A has compact resolvent. Then we know from Section 3.3 that $X = X_0 \oplus X_{\mathrm{ap}}$, where X_0 and X_{ap} are invariant under T. Moreover, there exists a C_0-group U on X_{ap} such that $U(t) = T(t)|_{X_{\mathrm{ap}}}$ for $t \geqslant 0$.

THEOREM. *In addition to the assumptions made above, if T is positive and irreducible, then U is periodic.*

PROOF. If $s(A) < 0$, then $X_{\mathrm{ap}} = \{0\}$. If $\sigma(A) \cap \mathrm{i}\mathbb{R} = \{0\}$, then $X_{\mathrm{ap}} = \ker A$; i.e., $U(t) \equiv I$. If $\sigma(A) \cap \mathrm{i}\mathbb{R} \neq \{0\}$, it follows from [[Nag86], C-III.3.8, p. 313] that $\sigma(A) \cap \mathrm{i}\mathbb{R} = \mathrm{i}\frac{2\pi}{\tau}\mathbb{Z}$ for some $\tau > 0$. It follows from the definition of X_{ap} that $T(t+\tau)x = T(t)x$, $t \geqslant 0$, for all $x \in X_{\mathrm{ap}}$. □

4. Functional calculus

In this section we consider sectorial operators. These are unbounded operators whose spectra lie in a sector. Generators of bounded C_0-semigroups are of this type. If A is such an operator, we will define closed operators $f(A)$ for a large class of holomorphic functions defined on a sector. Of particular importance are fractional powers A^α. We will introduce the class *BIP* which is important to establish theorems of maximal regularity but also for results on interpolation. Throughout this chapter X is a complex Banach space.

4.1. *Sectorial operators*

Given an angle $0 < \varphi < \pi$, we consider the open sector

$$\Sigma_\varphi := \big\{r\mathrm{e}^{\mathrm{i}\alpha} \colon\ r > 0, |\alpha| < \varphi\big\}.$$

Let X be a Banach space. An operator A is called *φ-sectorial* if

$$\sigma(A) \subset \overline{\Sigma_\varphi} \tag{4.1}$$

and

$$\sup_{\lambda \in \mathbb{C} \setminus \Sigma_\varphi} \big\|\lambda R(\lambda, A)\big\| < \infty. \tag{4.2}$$

An operator A on X is called *sectorial* if there exists $0 < \varphi < \pi$ such that A is φ-sectorial. In that case we define the *sectoriality angle* $\varphi_{\mathrm{sec}}(A)$ of A by

$$\varphi_{\mathrm{sec}}(A) := \inf\big\{\varphi \in (0, \pi)\colon\ \text{(4.1) and (4.2) are valid}\big\}. \tag{4.3}$$

Note that $\varphi_{\mathrm{sec}}(A) \in [0, \pi)$.

4.1.1. *Simple criterion.* If $(-\infty, 0) \subset \rho(A)$ and

$$\left\|\lambda R(\lambda, A)\right\| \leqslant c \text{ for all } \lambda < 0 \text{ and some } c \geqslant 0, \text{ then } A \text{ is sectorial.}$$

This follows from the power series expansion of the resolvent.

4.1.2. *Bounded holomorphic semigroups.* An operator A is sectorial with $\varphi_{\mathrm{sec}}(A) < \frac{\pi}{2}$ if and only if $-A$ generates a bounded holomorphic semigroup T. It is a C_0-semigroup if and only if $D(A)$ is dense. This is automatically the case if X is reflexive.

4.1.3. *Bounded semigroups.* If $-A$ generates a bounded C_0-semigroup, then A is sectorial and $\varphi_{\mathrm{sec}}(A) \leqslant \frac{\pi}{2}$.

4.1.4. *Injective operators.* Let A be a sectorial operator with range $R(A) := \{Ax\colon x \in D(A)\}$. If A is injective, then A^{-1} with domain $D(A^{-1}) = R(A)$ is sectorial and $\varphi_{\mathrm{sec}}(A^{-1}) = \varphi_{\mathrm{sec}}(A)$.

4.1.5. *Reflexive spaces.* If A is an injective sectorial operator on a reflexive space, then $D(A) \cap R(A)$ is dense.

4.1.6. *Warning.* The notion of *sectorial* operators is not universal in the literature. In particular, it does not coincide with Kato's definition in [[Kat66]]. In [PS90], [DHP01] and [[Prü93]] the additional assumption that A is injective, densely defined with dense image is incorporated into the definition of sectorial. This implies in particular that $D(A) \cap R(A)$ is dense.

4.2. *The sum of commuting operators*

Many interesting problems can be formulated in the following form. Let A and B be operators on a Banach space: Under which conditions is the problem

$$Ax + Bx = y \tag{P}$$

well posed? This means that for all $y \in X$ there exists a unique $x \in D(A) \cap D(B)$ solving (P). In this section we will establish a spectral theoretical approach which leads to a "weak solution". We define the operator $A + B$ on the domain $D(A + B) := D(A) \cap D(B)$ by $(A + B)x = Ax + Bx$.

Assume that $\rho(A) \cap \rho(B) \neq \emptyset$. We say that A and B *commute* if $R(\lambda, A)R(\lambda, B) = R(\lambda, B)R(\lambda, A)$ for all (equivalently one) $\lambda \in \rho(A) \cap \rho(B)$. This is equivalent to saying that for some (equivalently all) $\lambda \in \rho(A)$ one has $R(\lambda, A)D(B) \subset D(B)$ and $BR(\lambda, A)x = R(\lambda, A)Bx$ for all $x \in D(B)$. For example, if A and B generate C_0-semigroups S and T, then A and B commute if and only if $S(t)T(t) = T(t)S(t)$ for all $t \geqslant 0$. In that case $U(t) = T(t)S(t)$ is a C_0-semigroup, the operator $A + B$ is closable and $\overline{A + B}$ is the generator of U.

THEOREM. *Let A, B be two commuting sectorial operators such that*

$$\varphi_{\mathrm{sec}}(A) + \varphi_{\mathrm{sec}}(B) < \pi.$$

Then $A + B$ has a unique extension $(A + B)^{\sim}$ such that $(\omega, \infty) \subset \rho((A + B)^{\sim})$ for some $\omega \in \mathbb{R}$ and such that $(A + B)^{\sim}$ commutes with A. Moreover,

$$\sigma\big((A + B)^{\sim}\big) \subset \sigma(A) + \sigma(B). \tag{4.4}$$

If $D(A)$ is dense in X, then $A + B$ is closable and $(A + B)^{\sim}$ is the closure of $A + B$.

Note that by (4.4), $(A + B)^{\sim}$ is invertible if $0 \in \rho(A)$ or $0 \in \rho(B)$. This result is due to Da Prato and Grisvard [DPG75], Théorème 3.7, under the assumption that $D(A)$ is dense. The spectral inclusion (4.2.1) was proved in [[Prü93], Theorem 8.5] and in [ARS94], where also nondensely defined operators are considered.

If A or B is invertible, then by the Theorem, for each $y \in X$, there exists a unique $x \in D((A + B)^{\sim})$ such that

$$(A + B)^{\sim} x = y.$$

One aim of the functional calculus to be developed here is to establish conditions under which $x \in D(A) \cap D(B)$, which is expressed by saying that the solution has *maximal regularity*.

4.3. *The elementary functional calculus*

Let A be a sectorial operator and let $\varphi_{\mathrm{sec}}(A) < \varphi < \pi$.

4.3.1. *Invertible operators.* By $H_1^\infty(\Sigma_\varphi)$ we denote the space of all bounded holomorphic functions $f : \Sigma_\varphi \to \mathbb{C}$ such that

$$\big|f(z)\big| \leqslant c|z|^{-\gamma}, \quad |z| \geqslant 1, \tag{4.5}$$

where $c \geqslant 0, \gamma > 0$ depend on f. Then $H_1^\infty(\Sigma_\varphi)$ is an algebra for pointwise operations. Let $\varphi_{\mathrm{sec}}(A) < \varphi_1 < \varphi$. Consider the path

$$\Gamma(r) = \begin{cases} -r\mathrm{e}^{-\mathrm{i}\varphi_1} & \text{if } r < 0, \\ r\mathrm{e}^{\mathrm{i}\varphi_1} & \text{if } r \geqslant 0. \end{cases}$$

For $f \in H_1^\infty(\Sigma_\varphi)$ we define $f(A) \in \mathcal{L}(X)$ by

$$f(A) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma f(\lambda) R(\lambda, A)\, \mathrm{d}\lambda. \tag{4.6}$$

This definition does not depend on the choice of φ_1. The mapping $f \mapsto f(A)$ is an algebra homomorphism from $H_1^\infty(\Sigma_\varphi)$ into $\mathcal{L}(X)$ such that

$$f_\lambda(A) = R(\lambda, A) \quad \text{for } \lambda \in \mathbb{C} \setminus \Sigma_\varphi,$$

where $f_\lambda(z) = \frac{1}{\lambda - z}$. Now let $x \in D(A)$. Then

$$f(A)x = (I + A)g(A)x, \tag{4.7}$$

where $g(\lambda) = \frac{f(\lambda)}{1+\lambda}$. This follows from Cauchy's theorem since $R(\lambda, A) - \frac{1}{1+\lambda} R(\lambda, A)(I + A) = \frac{1}{1+\lambda} I$. For $f \in H^\infty(\Sigma_\varphi)$ we define $f(A)$ by

$$f(A) = (I + A)g(A) \tag{4.8}$$

with $g(\lambda) = \frac{f(\lambda)}{1+\lambda}$. Since $g \in H_1^\infty(\Sigma_\varphi)$, one has $g(A) \in \mathcal{L}(X)$ and so $f(A)$ is a closed operator with domain

$$D\big(f(A)\big) := \big\{x \in X\colon\ g(A)x \in D(A)\big\}.$$

This definition is consistent with (4.6) if $f \in H_1^\infty(\Sigma_\varphi)$. Next we relax the hypothesis of invertibility.

4.3.2. *Injective operators.* Let A be a sectorial operator and $\varphi > \varphi_{\mathrm{sec}}(A)$. Denote by $H_0^\infty(\Sigma_\varphi)$ the algebra of all bounded holomorphic functions $f : \Sigma_\varphi \to \mathbb{C}$ satisfying an estimate

$$\begin{aligned} \big|f(z)\big| &\leqslant c|z|^{-\gamma}, \quad |z| > 1, \\ \big|f(z)\big| &\leqslant c|z|^{\gamma}, \quad |z| \leqslant 1, \end{aligned}$$

where $c \geqslant 0$ and $\gamma > 0$ depend on f. For $f \in H_0^\infty(\Sigma_\varphi)$ we define $f(A) \in \mathcal{L}(X)$ by (4.6). Now assume that A is injective. The operator $B = A(I + A)^{-2}$ is bounded and injective with range $D(A) \cap R(A)$. Hence, $A + 2I + A^{-1} = B^{-1}$ with domain $D(A) \cap R(A)$ is a closed, injective operator. Now let $f \in H^\infty(\Sigma_\varphi)$. Let $g(\lambda) = \frac{\lambda}{(1+\lambda)^2} f(\lambda)$. Then $g \in H_0^\infty(\Sigma_\varphi)$. Hence, $g(A) \in \mathcal{L}(X)$. We define $f(A)$ by

$$\begin{aligned} f(A) &= \big(A + 2I + A^{-1}\big)g(A) \\ &= (I + A)^2 A^{-1} g(A). \end{aligned} \tag{4.9}$$

Then $f(A)$ is a closed operator. A similar argument as in Section 4.3.1 shows that this new definition is consistent with the previous one (4.6) if $f \in H_0^\infty(\Sigma_\varphi)$. Moreover, if A is invertible, then the definition is consistent with Section 4.3.1.

In this way we defined a closed operator $f(A)$ for each $f \in H^\infty(\Sigma_\varphi)$. We may extend the definition to an even larger class of holomorphic functions. Let $\Psi(\lambda) = \frac{\lambda}{(1+\lambda)^2}$ and let

$L(A) = (I + A)^2 A^{-1}$. Denote by $B(\Sigma_\varphi)$ the space of all holomorphic functions f on Σ_φ such that $\Psi^n f \in H_0^\infty(\Sigma_\varphi)$ for some $n \in \mathbb{N}$. Then we define the closed operator $f(A)$ by

$$f(A) = L(A)^n \bigl(\Psi^n f\bigr)(A). \tag{4.10}$$

This definition is consistent with the previous ones.

We call the functional calculus defined in Sections 4.3.1 or 4.3.2 the *elementary functional calculus*. The following properties justify this name.

4.3.3. *Properties of the functional calculus.* Let A be an injective sectorial operator. For $\lambda \in \mathbb{C} \setminus \Sigma_{\varphi_{\mathrm{sec}}(A)}$ one has

$$R(\lambda, A) = f_\lambda(A),$$

where $f_\lambda(z) = \frac{1}{z-\lambda}$. The set $H(A) := \{f \in B(\Sigma_\varphi)\colon\ f(A) \in \mathcal{L}(X)\}$ is a subalgebra of $B(\Sigma_\varphi)$ and

$$f \mapsto f(A)$$

is an algebra homomorphism [McI86,LeM98b].

4.3.4. *Strip-type operators.* Instead of sectors we consider here a horizontal strip

$$St_\omega := \bigl\{z \in \mathbb{C}\colon\ |\operatorname{Im} z| < \omega\bigr\}$$

where $\omega > 0$. We say that an operator B on X is of *strip type*, if there exists a strip St_ω such that

(a) $\sigma(B) \subset St_\omega$ and

(b) $\sup_{|\operatorname{Im}\lambda| \geqslant \omega} \|R(\lambda, B)\| < \infty$.

We denote by

$$\omega_{\mathrm{st}}(B) = \inf\bigl\{\omega > 0\colon\ \text{(a) and (b) hold}\bigr\}$$

the strip-type of B. Two kinds of examples are important: the case where $\mathrm{i}B$ generates a C_0-group and $B = \log A$ to which the two following subsections are devoted.

4.3.5. *Groups.* Let B be an operator such that $\mathrm{i}B$ generates a C_0-group U. Denote by

$$\omega_U := \inf\bigl\{\omega \in \mathbb{R}\colon\ \exists M \geqslant 0 \text{ such that } \bigl\|U(t)\bigr\| \leqslant M\mathrm{e}^{\omega|t|} \text{ for all } t \in \mathbb{R}\bigr\}$$

the *group type* of U. Since the resolvent of $\pm\mathrm{i}B$ is the Laplace transform of the semigroup $(U(\pm t))_{t\geqslant 0}$, it follows that B is a strip type operator and

$$\omega_{\mathrm{st}}(B) \leqslant \omega_U. \tag{4.11}$$

In general, it may happen that

$$\omega_{\mathrm{st}}(B) < \omega_U$$

(see [Wol81]). But on Hilbert space, strip type and group type coincide by the Gearhart–Prüss theorem; i.e.,

$$\omega_{\mathrm{st}}(B) = \omega_U \tag{4.12}$$

whenever $\mathrm{i}B$ generates a C_0-group U on a Hilbert space. Next we give another important example of a strip-type operator.

4.3.6. *The logarithm.* Let A be an injective sectorial operator. Let $\varphi > \varphi_{\mathrm{sec}}(A)$. Let $g(\lambda) = \frac{\lambda}{(1+\lambda)^2} \log \lambda$. Then $g \in H_0^\infty(\Sigma_\varphi)$. We define

$$\log A = (I + A)^2 A^{-1} g(A).$$

Thus $\log A$ is a closed operator by the same argument as in Section 4.3.2. We note some of the properties of the logarithm. Recall that A^{-1} is sectorial. One has

$$\log A^{-1} = -\log A. \tag{4.13}$$

THEOREM (Haase [Haa03a], Nollau [Nol69]). *Let A be a sectorial operator. Then* $\log A$ *is a strip-type operator and*

$$\omega_{\mathrm{st}}(\log A) = \varphi_{\mathrm{sec}}(A).$$

Now it may happen that $\mathrm{i} \log A$ generates a C_0-group. Then this group is given by the imaginary powers of A, which we consider in Section 4.4.

Further references: [Oka99,Oka00a,Oka00b].

4.4. *Fractional powers and BIP*

In the section we introduce fractional powers A^α of a sectorial operator A. We first consider the case where A is invertible which is simpler to present.

4.4.1. *Fractional powers of invertible operators.* Let A be an invertible sectorial operator. Let $\alpha \in \mathbb{C}$ such that $\mathrm{Re}\,\alpha > 0$. Then the function $f_\alpha(z) = z^{-\alpha}$ is in $H_1^\infty(\Sigma_\varphi)$ for all $0 < \varphi < \pi$. Thus we may define

$$A^{-\alpha} = f_\alpha(A) \in \mathcal{L}(X), \quad \mathrm{Re}\,\alpha > 0, \tag{4.14}$$

by (4.6). It follows that

$$A^{-\alpha} A^{-\beta} = A^{-(\alpha+\beta)}, \quad \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0. \tag{4.15}$$

In fact, $(A^{-\alpha})_{\mathrm{Re}\,\alpha>0}$ is a bounded holomorphic semigroup. Its generator is $-\log A$ (which was defined in Section 4.3.6). The operator A is densely defined if and only if $\log A$ is so (i.e., if and only if $(A^{-\alpha})_{\mathrm{Re}\,\alpha>0}$ is a C_0-semigroup).

A particular case of Section 4.4.1 occurs when $-A$ generates an exponentially stable C_0-semigroup T (i.e., $\|T(t)\| \leqslant M\mathrm{e}^{-\varepsilon t}$ for all $t \geqslant 0$ and some $M \geqslant 0$, $\varepsilon > 0$). Then A is invertible and $A^{-\alpha}$ can be expressed in terms of the semigroup instead of the resolvent as above, namely,

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\,\mathrm{d}t \tag{4.16}$$

for $\mathrm{Re}\,\alpha > 0$.

Next we consider a more general class of sectorial operators.

4.4.2. *Fractional powers of injective sectorial operators.* Let A be an injective sectorial operator and let $\pi > \varphi > \varphi_{\mathrm{sec}}(A)$. Let $\alpha \in \mathbb{C}$, $f_\alpha(z) = z^\alpha$, $z \in \Sigma_\varphi$. Then $f_\alpha \in B(\Sigma_\varphi)$. Thus

$$A^\alpha = f_\alpha(A), \quad \alpha \in \mathbb{C},$$

is defined according to Section 4.3.2. It is a closed injective operator and the following properties are valid:

$$A^{-\alpha} = \left(A^\alpha\right)^{-1} = \left(A^\alpha\right)^{-1}, \quad \alpha \in \mathbb{C}; \tag{4.17}$$

$$A^\alpha A^\beta \subset A^{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{C}; \tag{4.18}$$

$$A^\alpha A^\beta = A^{\alpha+\beta}, \quad \mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > 0. \tag{4.19}$$

If $\mathrm{Re}\,\alpha > 0$, then

$$D\big((A+\omega)^\alpha\big) = D\big(A^\alpha\big) \quad \text{for all } \omega \geqslant 0. \tag{4.20}$$

Moreover, for $0 < \alpha < \frac{\pi}{\varphi_{\mathrm{sec}}(A)}$, the operator A^α is sectorial and

$$\varphi_{\mathrm{sec}}\big(A^\alpha\big) = \alpha\varphi_{\mathrm{sec}}(A) \quad \text{and} \quad \log\big(A^\alpha\big) = \alpha \log A; \tag{4.21}$$

moreover,

$$\big(A^\alpha\big)^\beta = A^{\alpha\beta}, \quad \mathrm{Re}\,\beta > 0. \tag{4.22}$$

Formula (4.21) is interesting: it shows in particular that $-A^\alpha$ generates a bounded holomorphic semigroup if $0 < \alpha$ is small enough.

There is an enormous amount of literature on fractional powers and we refer in particular to the recent monograph [[MS01]].

Next we consider imaginary powers $A^{\mathrm{i}s}$ of an injective sectorial operator. They play an important role for regularity theory and also for interpolation theory.

4.4.3. *Characterization of BIP.* Let A be a sectorial injective operator on X. Then by Section 4.4.2 we can define the closed operators $A^{\mathrm{i}s}$, $s \in \mathbb{R}$, and also the closed operator $\log A$ (by Section 4.3.6).

THEOREM. *The following assertions are equivalent*:

(i) $D(A) \cap R(A)$ *is dense and* $A^{\mathrm{i}s} \in \mathcal{L}(X)$ *for all* $s \in \mathbb{R}$.

(ii) *The operator* $\mathrm{i} \log A$ *generates a* C_0*-group* U.

In that case $U(s) = A^{\mathrm{i}s}$, $s \in \mathbb{R}$.

DEFINITION. We say that A has *bounded imaginary powers* and write $A \in BIP$, if A is injective, sectorial and if the two equivalent conditions of Theorem are satisfied.

Note that this includes the hypothesis that $D(A) \cap R(A)$ be dense in X. Recall however, that $D(A) \cap R(A)$ is automatically dense in X if X is reflexive and A is a sectorial, injective operator. It is obvious that

$$A \in BIP \quad \text{if and only if} \quad A^{-1} \in BIP, \tag{4.23}$$

References: [Haa03a], [[Prü93]].

4.4.4. *BIP for invertible operators.* Property *BIP* can be formulated in a different way if the operator is invertible. Let A be sectorial of dense domain. Assume that $0 \in \rho(A)$. Then, for $\operatorname{Re} \alpha > 0$, the operator $A^{-\alpha} \in \mathcal{L}(X)$ was defined in Section 4.4.1, and we had seen that $(A^{-\alpha})_{\operatorname{Re} \alpha > 0}$ is a holomorphic C_0-semigroup which is bounded on Σ_φ for each $0 < \varphi < \frac{\pi}{2}$. The generator of this semigroup is $-\log A$. Now it follows from Section 2.3 that $A \in BIP$ if and only if the holomorphic C_0-semigroup $(A^{-\alpha})_{\operatorname{Re} \alpha > 0}$ has a trace. We formulate this as a theorem.

THEOREM. *Let A be a sectorial densely defined operator with* $0 \in \rho(A)$. *The following assertion are equivalent*:

(i) $A \in BIP$;

(ii) *there exists* $c > 0$ *such that* $\|A^{-\alpha}\| \leqslant c$ *whenever* $\operatorname{Re} \alpha > 0$, $|\alpha| \leqslant 1$.

4.4.5. *The BIP-type.* Let $A \in BIP$. Then we denote by

$$\varphi_{bip}(A) = \inf\bigl\{\omega \in \mathbb{R} \colon \exists M \geqslant 0 \text{ such that } \bigl\|A^{\mathrm{i}s}\bigr\| \leqslant M \mathrm{e}^{\omega |s|} \text{ for all } s \in \mathbb{R}\bigr\}$$

the group-type of the C_0-group $(A^{\mathrm{i}s})_{s \in \mathbb{R}}$. We call $\varphi_{bip}(A)$ the *BIP-type* of A. One always has

$$\varphi_{\mathrm{sec}}(A) \leqslant \varphi_{bip}(A). \tag{4.24}$$

Recall from (4.10) that $\varphi_{\mathrm{sec}}(A) = \omega_{\mathrm{st}}(\log A)$. If X is a Hilbert space, then $\omega_{\mathrm{st}}(\log A) = \varphi_{bip}(A)$ (by (4.12)). Thus we obtain

$$\varphi_{bip}(A) = \varphi_{\mathrm{sec}}(A) \quad \text{on Hilbert space.} \tag{4.25}$$

This is McIntosh's result with a new proof due to Haase. The identity (4.25) is no longer true on Banach spaces. Recently, an example is given by Haase [Haa03a,Haa03b] where

$$\varphi_{\mathrm{sec}}(A) < \pi < \varphi_{bip}(A) \quad \text{on a UMD-space } X. \tag{4.26}$$

In fact, $X = L^p(\mathbb{R}, \omega_1\,\mathrm{d}x) \cap L^q(\mathbb{R}, \omega_2\,\mathrm{d}x)$ for $1 < p < 2 < q < \infty$ and ω_1, ω_2 strictly positive measurable functions. Another example of different sectorial and *BIP*-angle is given independently by Kalton [Kal03], where the operator has a bounded H^∞-calculus, in addition.

4.4.6. *An inverse theorem for BIP operators.* If $A \in BIP$, then $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ is a C_0-group. *Which groups occur in this manner*? There is a very satisfying answer if the underlying Banach space has some geometric properties. We refer to Section 6.1.3 for the definition of UMD-spaces. Here we just recall that each space L^p, $1 < p < \infty$, is a UMD-space. Let $(U(s))_{s\in\mathbb{R}}$ be a C_0-group with generator $\mathrm{i}B$. Then B is a strip-type operator. Recall the definition of $\omega_{\mathrm{st}}(B)$ from Section 4.3.4.

THEOREM (Monniaux). *Let* $\mathrm{i}B$ *be the generator of a* C_0*-group* U *on a UMD-space* X. *Assume that*

$$\omega_{\mathrm{st}}(B) < \pi.$$

Then there exists a unique sectorial operator A *such that* $A^{\mathrm{i}s} = U(s)$ *for all* $s \in \mathbb{R}$. *We call* A *the analytic generator of* U.

This theorem is due to Monniaux [Mon99] for the case where $\omega_U < \pi$ (the group-type of U). It was Haase [Haa03b] who observed that the weaker assumption $\omega_{\mathrm{st}}(B) < \pi$ suffices. Of course, in the situation of the theorem one has $B = \log A$. One may ask more generally which strip type operators B are of the form $\log A$ for some sectorial operator A. This seems to be unknown. It is known though, that the hypothesis on the Banach space cannot be omitted in the above theorem (see [Mon99] for an example). More precisely, on each Banach space X, to each C_0-group U one can associate the analytic generator A. If A is sectorial, then $A^{\mathrm{i}s} = U(s)$, $s \in \mathbb{R}$. But if the space X is not a UMD-space, then it may happen that $\rho(A) = \emptyset$. In particular, A is not sectorial. The notion of analytic generators is due to Cioranescu and Zsidó [CZ76] before the notion and properties of UMD-spaces were known. Their aim was to treat Tomita–Takesaki theory for von Neumann algebras.

4.4.7. *Bounded analytic generators.* We now describe when the C_0-group $(A^{\mathrm{i}s})_{s\in\mathbb{R}}$ is the boundary group of a holomorphic semigroup (see Section 2.3 for this notion).

PROPOSITION [Mon99]. *Let* U *be a* C_0*-group of type* $< \pi$ *on a UMD-space* X. *The following assertions are equivalent*:

(i) *The analytic generator* A *of* U *is bounded*;

(ii) *there exists a holomorphic* C_0*-semigroup* $(T(z))_{\mathrm{Re}\, z>0}$ *such that* $U(t)x = \lim_{r\downarrow 0} T(\mathrm{i}t + r)x$ *for all* $x \in X$.

In that case one has $A = T(1)$.

4.4.8. *The Dore–Venni theorem.* Next we establish the famous Dore–Venni theorem [DoVe87] on maximal regularity. It gives an answer to the question of closedness of $A+B$ which arose in Section 4.2. We refer to Section 6.1.3 for the definition and properties of UMD-spaces.

The following version of the Dore–Venni theorem is due to Prüss and Sohr [PS90].

THEOREM. *Let X be a UMD-space. Let $A, B \in BIP$ such that $\varphi_{bip}(A) + \varphi_{bip}(B) < \pi$. Assume that A and B commute. Then $A+B$ is closed. Moreover, $A+B \in BIP$ and*

$$\varphi_{bip}(A+B) \leqslant \max\bigl\{\varphi_{bip}(A), \varphi_{bip}(B)\bigr\}.$$

If one of the operators is invertible, it is possible to deduce this result from Monniaux' inverse theorem (Section 4.4.6). We sketch the proof.

PROOF [MON99]. Assume that $0 \in \rho(B)$. Then by Section 4.4.7 the group $(B^{-\mathrm{i}s})_{s\in\mathbb{R}}$ is the boundary of a holomorphic C_0-semigroup T and $B^{-1} = T(1)$. Let $W(t) = A^{\mathrm{i}t}B^{-\mathrm{i}t}$. By Monniaux' inverse theorem there exists a sectorial operator C such that $C^{\mathrm{i}t} = A^{\mathrm{i}t}B^{-\mathrm{i}t}$. It is not difficult to show that $C = AB^{-1}$ with domain $D(C) = \{x \in X\colon\ B^{-1}x \in D(A)\}$. In particular, $(AB^{-1} + I)$ is invertible. Let $y \in X$. We have to find $x \in D(A) + D(B)$ such that $Ax + Bx = y$; i.e., $(AB^{-1} + I)Bx = y$. Thus $x = B^{-1}(AB^{-1} + I)^{-1}y$ is a solution. □

4.4.9. *A noncommutative Dore–Venni theorem.* The assumption that A and B commute can be relaxed by imposing a growth condition on the commutator of the resolvent.

THEOREM (Monniaux and Prüss [MP97]). *Assume that X is a UMD-space. Let $A,\ B \in BIP$ with $\varphi_{bip}(A) + \varphi_{bip}(B) < \pi$. Assume that $0 \in \rho(A)$. Let $\psi_A > \varphi_{bip}(A)$, $\psi_B > \varphi_{bip}(B)$ be angles such that $\psi_A + \psi_B < \pi$. Assume that there exist constants $0 \leqslant \alpha < \beta < 1$ and $c \geqslant 0$ such that*

$$\bigl\|AR(\lambda, A)\bigl[A^{-1}R(\mu, B) - R(\mu, B)A^{-1}\bigr]\bigr\| \leqslant \frac{c}{(1 + |\lambda|^{1-\alpha})|\mu|^{1+\beta}}$$

for all $\lambda \in \Sigma_{\pi-\psi_A}$ and all $\mu \in \Sigma_{\pi-\psi_B}$. Then there exists $\omega \in \mathbb{R}$ such that $A + B + \omega$ is closed and sectorial.

This result can be applied to Volterra equations [MP97], to nonautonomous evolution equations [HM00a,HM00b] and also to prove maximal regularity of the Ornstein–Uhlenbeck operator [MPRS00].

4.4.10. *Interpolation spaces and BIP.* Let A be an injective sectorial operator. Then $D(A^\alpha)$ is a Banach space for the norm

$$\|x\|_{D(A^\alpha)} := \|x\| + \bigl\|A^\alpha x\bigr\|,$$

$0 < \alpha < 1$.

PROPOSITION ([[Tri95], p. 103], [Yag84], [[MS01], Theorem 11.6.1]). *Let $A \in BIP$. Then the complex interpolation space $[X, D(A)]_\alpha$ is isomorphic to $D(A^\alpha)$ for $0 < \alpha < 1$.*

Now assume in addition that A is invertible. Then Dore [Dor99a] (see also [Dor99b] for the noninvertible case) showed that the part of A in the real interpolation space $(X, D(A))_{\alpha,p}$ has a bounded H^∞-calculus whenever $0 < \alpha < 1$, $1 < p < \infty$. (See Section 4.5 for the definition of H^∞-calculus.) If X is a Hilbert space, then $(X, D(A))_{\alpha,2} = [X, D(A)]_\alpha$. Thus, if X is a Hilbert space, the part of A in $[X, D(A)]_\alpha$ has a bounded H^∞-calculus. Observe that the part of A in $D(A^\alpha)$ is similar to A. Thus, if $D(A^\alpha) = [X, D(A)]_\alpha$, then A has a bounded H^∞-calculus.

We have proved the converse of the above proposition and may formulate the following characterization on Hilbert spaces.

THEOREM. *Let A be a sectorial, invertible operator on a Hilbert space and let $0 < \alpha < 1$. The following assertions are equivalent*:

(i) $A \in BIP$;

(ii) $D(A^\alpha) = [X, D(A)]_\alpha$;

(iii) *the H^∞-calculus is bounded.*

For further results on Hilbert spaces we refer to Section 5 and to the paper by Auscher, McIntosh and Nahmrod [AMN97].

This interesting relation between *BIP* and interpolation spaces seemed to be the motivation of much research on operators with bounded imaginary powers in the seventies (see, e.g., Seeley's paper [See67]). Long time after this, the Dore–Venni theorem lead to a different direction, namely regularity theory.

4.5. *Bounded H^∞-calculus for sectorial operators*

Let A be an injective sectorial operator on a Banach space X and let $\pi \geqslant \varphi > \varphi_{\mathrm{sec}}(A)$. Then for $f \in H^\infty(\Sigma_\varphi)$ the closed operator $f(A)$ was defined in Section 4.3.

DEFINITION. We say that A *has a bounded $H^\infty(\Sigma_\varphi)$-calculus* if

$$f(A) \in \mathcal{L}(X) \quad \text{for all } f \in H^\infty(\Sigma_\varphi). \tag{4.27}$$

In that case, the mapping

$$f \mapsto f(A)$$

is a continuous algebra homomorphism from $H^\infty(\Sigma_\varphi)$ into $\mathcal{L}(X)$.

We note the following: Let $\varphi_{\mathrm{sec}}(A) < \varphi_1 < \varphi_2 \leqslant \pi$. If A has a bounded $H^\infty(\Sigma_{\varphi_1})$-calculus then also the $H^\infty(\Sigma_{\varphi_2})$-calculus is bounded. Thus, the weakest property in this context is to have a bounded $H^\infty(\Sigma_\pi)$-calculus for the cut plane $\Sigma_\pi = \mathbb{C} \setminus \{x \in \mathbb{R}\colon\ x < 0\}$.

DEFINITION. We write $A \in H^\infty$ to say that A is injective, sectorial and has a bounded $H^\infty(\Sigma_\pi)$-calculus. We let

$$\varphi_{H^\infty}(A) := \inf\bigl\{\sigma \in \bigl(\varphi_{\mathrm{sec}}(A), \pi\bigr]\colon\ A \text{ has a bounded } H^\infty(\Sigma_\varphi)\text{-calculus}\bigr\}.$$

4.5.1. *Characterization via H_0^∞.* Let A be an injective sectorial operator with dense domain and dense range, and let $\varphi > \varphi_{\mathrm{sec}}(A)$. Then A has a bounded $H^\infty(\Sigma_\varphi)$-calculus if and only if there exists $c > 0$ such that

$$\bigl\| f(A)\bigr\|_{\mathcal{L}(X)} \leqslant c\|f\|_{H^\infty(\Sigma_\varphi)} \tag{4.28}$$

for all $f \in H_0^\infty(\Sigma_\varphi)$. Note that $H_0^\infty(\Sigma_\varphi)$ is not norm-dense in $H^\infty(\Sigma_\varphi)$. Still, there is a canonical way to extend the calculus.

4.5.2. *Unbounded H^∞-calculus.* It is not an easy matter to decide whether a given operator has a bounded H^∞-calculus. Later we will give several criteria and examples. Here we show a way to construct counterexamples. Let X be a Banach space. A sequence $(e_n)_{n\in\mathbb{N}}$ is called a *Schauder basis* of X if, for each $x \in X$, there exists a unique sequence $(x_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} x_n e_n = x. \tag{4.29}$$

The Schauder basis is called *unconditional* if for each $x = \sum_{n=1}^\infty x_n e_n \in X$ the sequence $\sum_{n=1}^\infty c_n x_n e_n$ converges in X for each $c = (c_n)_{n\in\mathbb{N}} \in \ell^\infty$. Otherwise we call $(e_n)_{n\in\mathbb{N}}$ a *conditional* Schauder basis. If X has an unconditional Schauder basis it also has a conditional Schauder basis [[Sin70], Chapter II, Theorem. 23.2]. In particular, each separable Hilbert space has a conditional Schauder basis. Thus the following example is in particular valid in a separable Hilbert space.

EXAMPLE (Unbounded $H^\infty(\Sigma_\pi)$-calculus). Let X be a Banach space with a conditional Schauder basis $(e_n)_{n\in\mathbb{N}}$. Define the operator A on X by

$$Ax = \sum_{n=1}^{\infty} 2^n x_n e_n \tag{4.30}$$

with domain the set of all $x = \sum_{n=1}^\infty x_n e_n \in X$ such that (4.30) converges. Then A is sectorial with $\varphi_{\mathrm{sec}}(A) = 0$. However, A does not have a bounded $H^\infty(\Sigma_\pi)$-calculus.

For the proof it is helpful to consider more general diagonal operators. Denote by BV the space of all scalar sequences $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ such that

$$\|\alpha\|_{BV} = |\alpha_1| + \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$$

Then BV is a Banach space (isomorphic to ℓ^1). If $\sum_{n=1}^{\infty} y_n$ converges in X, then $\sum_{n=1}^{\infty} \alpha_n y_n$ converges for each $\alpha \in BV$. This follows from Abel's partial summation rule

$$\sum_{n=1}^{m} \alpha_n y_n = \sum_{n=1}^{m} \alpha_n (s_{n+1} - s_n) = \sum_{n=2}^{m} s_n (\alpha_{n-1} - \alpha_n) + s_{m+1}\alpha_m - s_1\alpha_1,$$

where $s_1 = 0$, $s_n = \sum_{k=1}^{n-1} y_k$ for $n \geqslant 2$. Thus, if $\alpha \in BV$, then

$$D_\alpha x = \sum_{n=1}^{\infty} \alpha_n x_n e_n \tag{4.31}$$

for $x = \sum_{n=1}^{\infty} x_n e_n$ defines an operator $D_\alpha \in \mathcal{L}(X)$. Moreover,

$$\|D_\alpha\| \leqslant c\|\alpha\|_{BV}, \quad \alpha \in BV, \tag{4.32}$$

by the closed graph theorem (or a direct estimate). Next we consider unbounded diagonal operators. Let $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{C}$. Then

$$D_\alpha x = \sum_{n=1}^{\infty} \alpha_n x_n e_n \tag{4.33}$$

with $D(D_\alpha) := \{x = \sum_{n=1}^{\infty} x_n e_n\colon$ (4.33) converges$\}$ defines a closed operator on X. This is obvious since the coordinate functionals

$$\sum_{n=1}^{\infty} x_n e_n \mapsto x_m$$

are continuous.

PROPOSITION. *Let* $0 < \alpha_1 < \alpha_n < \alpha_{n+1}$ *such that* $\lim_{n\to\infty} \alpha_n = \infty$. *Then* D_α *is a sectorial operator of type* $\varphi_{\mathrm{sec}}(D_\alpha) = 0$. *Moreover,* $0 \in \rho(D_\alpha)$ *and* D_α *has compact resolvent.*

PROOF. Let $0 < \varphi < \pi$. Let $\pi \geqslant |\theta| > \varphi$, $\lambda = r\mathrm{e}^{\mathrm{i}\theta}$, $r > 0$. Consider the sequence $\beta_n = \frac{1}{r\mathrm{e}^{\mathrm{i}\theta} - \alpha_n}$. Then

$$\beta_n - \beta_{n+1} = \frac{1}{r}\left\{\frac{1}{\mathrm{e}^{\mathrm{i}\theta} - \alpha_n/r} - \frac{1}{\mathrm{e}^{\mathrm{i}\theta} - \alpha_{n+1}/r}\right\},$$

$$\sum_{n=1}^{\infty} |\beta_n - \beta_{n+1}| \leqslant \frac{1}{r}\sum_{n=1}^{\infty} \int_{\alpha_{n/r}}^{\alpha_{n+1}/r} \frac{1}{|\mathrm{e}^{\mathrm{i}\theta} - x|^2}\,\mathrm{d}x$$

$$\leqslant \frac{1}{r}\int_0^{\infty} \frac{1}{|\mathrm{e}^{\mathrm{i}\theta} - x|^2}\,\mathrm{d}x = \frac{c_\varphi}{r},$$

where c_φ does not depend on θ or r. Thus $\|D_\beta\| \leqslant \frac{c}{r}$. It is easy to see that $D_\beta = (\lambda - D_\alpha)^{-1}$. This shows that D_α is sectorial of type 0. We omit the proof of compactness of D_β (see [LeM00]). □

Now let A be the operator of the example; i.e., $A = D_\alpha$ with $\alpha_n = 2^n$. In order to show that A has no bounded $H^\infty(\Sigma_\pi)$-calculus we use the following (see [[Gar81], Theorem 1.1. on pp. 287 and 284 ff]).

INTERPOLATION LEMMA. *Let $b = (b_n)_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. Then there exists $f \in H^\infty(\Sigma_\pi)$ such that*

$$f\big(2^n\big) = b_n \quad \textit{for all } n \in \mathbb{Z}.$$

Let $f \in H^\infty(\Sigma_\pi)$. It is easy to see from the definition that $e_n \in D(f(A))$ and

$$f(A)e_n = f\big(2^n\big)e_n.$$

Now assume that $f(A) \in \mathcal{L}(X)$. Then it follows that

$$\begin{aligned}\lim_{m\to\infty}\sum_{n=1}^{m} f\big(2^n\big)x_n e_n &= \lim_{m\to\infty} f(A)\sum_{n=1}^{m} x_n e_n \\ &= f(A)x\end{aligned}$$

converges for all $x \in X$. Thus, by the Interpolation Lemma, $\sum_{n=1}^{\infty} b_n x_n e_n$ converges for all $b \in \ell^\infty$ and all $x \in X$. This contradicts the fact that the basis is conditional.

A first example of this kind had been given by Baillon and Clément [BC91]. Further developments are contained in [Ven93], [LeM00] and [Lan98].

4.5.3. *BIP versus bounded H^∞-calculus.* It is clear from the definition that $A \in H^\infty$ implies $A \in BIP$ whenever A is a sectorial, injective operator. It is remarkable that in that case

$$\varphi_{bip}(A) = \varphi_{H^\infty}(A) \tag{4.34}$$

(see [CDMY96]). Conversely, if X is a Hilbert space, then *BIP* implies boundedness of the H^∞-calculus by Section 4.4.10. This is not true on L^p-spaces for $p \neq 2$ as the following example shows.

EXAMPLE [Lan98]. Let $1 < p < \infty$, $p \neq 2$. There exists a sectorial injective operator $A \in BIP$ on L^p which does not have a bounded $H^\infty(\Sigma_\pi)$-calculus. We consider the space $L^p_{2\pi}$ of all scalar-valued 2π-periodic measurable functions on $\mathbb{R}$ such that

$\|f\|_p := \frac{1}{2\pi}(\int_0^{2\pi} |f(t)|^p \,\mathrm{d}t)^{1/p} < \infty$. By e_n we denote the nth trigonometric polynomial $e_n(t) = \mathrm{e}^{\mathrm{i}nt}$, $n \in \mathbb{Z}$. For $f \in L^p_{2\pi}$, we denote by

$$\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(t)\mathrm{e}^{\mathrm{i}nt} \,\mathrm{d}t$$

the nth Fourier coefficient. Then $\{e_n\colon n \in \mathbb{Z}\}$ is a Schauder basis of $L^p_{2\pi}$; i.e.,

$$f = \lim_{n\to\infty} \sum_{n=-m}^{m} \hat{f}(n)e_n =: \sum_{n=-\infty}^{+\infty} \hat{f}(n)e_n$$

for all $f \in L^p$. Moreover, the *Riesz-projection*

$$R : f \mapsto \sum_{n=0}^{\infty} \hat{f}(n)e_n$$

is bounded on $L^p_{2\pi}$. Consider the operator A on $L^p_{2\pi}$ given by

$$Af = \sum_{n=-\infty}^{+\infty} 2^n \hat{f}(n)e_n \tag{4.35}$$

with domain $D(A) = \{f \in L^p_{2\pi}\colon \text{(4.35) converges}\}$. Since the Riesz-projection is bounded we can write A as the direct sum of $A_1 \oplus A_2$, where

$$A_1 f = \sum_{n=0}^{\infty} 2^n \hat{f}(n)e_n,$$

$$A_2 f = \sum_{n=1}^{\infty} 2^{-n} \hat{f}(-n)e_{-n}$$

with appropriate domains. The first is injective and sectorial by Section 4.5.2 the second is the inverse of an operator of the Proposition in Section 4.5.2. Thus A is sectorial and injective. By the same argument as in Section 4.5.2 one sees that A has no bounded $H^\infty(\Sigma_\pi)$-calculus if $p \neq 2$. Now for $a \in \mathbb{R}$ consider the shift operator L_a given by $(L_a u)(t) = u(t+a)$ for $u \in L^p_{2\pi}$. Then $L_a \in \mathcal{L}(L^p_{2\pi})$ and $\|L_a\| = 1$. Moreover, $(L_a u)^\wedge(n) = \mathrm{e}^{\mathrm{i}na}\hat{u}(n)$. Let $a = s\log 2$. Then it follows that

$$A^{\mathrm{i}s} = L_a.$$

Thus $A \in BIP$ and $\|A^{\mathrm{i}s}\|_{\mathcal{L}(L^p_{2\pi})} = 1$, $1 < p < \infty$.

4.6. *Perturbation*

Let A be a sectorial operator on a Banach space X. We consider perturbations $A + B$ where $B : D(A) \to X$ is linear. We say that B is *A-bounded* if B is continuous with respect to the graph norm on $D(A)$; i.e., if there exist constants $a \geqslant 0$, $b \geqslant 0$ such that

$$\|Bx\| \leqslant a\|Ax\| + b\|x\|, \quad x \in D(A). \tag{4.36}$$

The infimum over all $a > 0$ such that there exists $b > 0$ such that (4.36) holds is called the *A-bound* of B. A *small perturbation* of A is a linear mapping $B : D(A) \to X$ with A-bound 0. This is equivalent to $\lim_{\lambda\to\infty} \|BR(\lambda, A)\| = 0$ as is easy to see. Here we consider the following two stronger conditions.

$$\left\|BR(\lambda, A)\right\| \leqslant \frac{c}{\lambda^\beta}, \quad \lambda \geqslant 1; \tag{4.37}$$

$$\left\|R(\lambda, A)Bx\right\| \leqslant \frac{c}{\lambda^\beta}\|x\|, \quad \lambda \geqslant 1, x \in D(A). \tag{4.38}$$

THEOREM. *Assume that $B : D(A) \to X$ is linear such that* (4.37) *or* (4.38) *is satisfied for some $\beta > 0$, $c \geqslant 0$. Assume that A and $A + B$ are invertible and sectorial. Then the following holds*:

(a) $A \in H^\infty \Rightarrow A + B \in H^\infty$;
(b) $A \in BIP \Rightarrow A + B \in BIP$.

We refer to [AHS94] or [ABH01] for the proof. Given a sectorial operator A there exists a constant $\varepsilon_A > 0$ such that $A + B + \omega$ is sectorial for some $\omega \in \mathbb{R}$ whenever B has an A-bound smaller than ε_A. However, the properties H^∞ and *BIP* are not preserved.

COUNTEREXAMPLE [MY90]. There exists an invertible operator $A \in H^\infty$ on a Hilbert space X such that, for each $\varepsilon > 0$, there exists an operator $B_\varepsilon : D(A) \to X$ with A-bound less than $\varepsilon > 0$ such that $A + B_\varepsilon + \omega \notin BIP$ for any $\omega \in \mathbb{R}$.

In view of the counterexample one would not expect that the properties H^∞ or *BIP* are preserved under small perturbations. This seems to be unknown, though.

4.7. *Groups and positive contraction semigroups*

In this section we give two classes of examples of operators with bounded H^∞-calculus.

4.7.1. *Groups.* Generators of bounded C_0-groups and their squares yield an interesting class of examples.

THEOREM [HP98]. *Let B be the generator of a bounded C_0-group on a UMD-space X. Then*

(a) *B has a bounded $H^\infty(\Sigma_\varphi)$-calculus for each $\varphi > \pi/2$;*
(b) *$-B^2$ has a bounded $H^\infty(\Sigma_\varphi)$-calculus for each $\varphi > 0$.*

We give an example.

4.7.2. *The abstract Laplace operator.* Let U_j, $j = 1, \ldots, n$, be n commuting C_0-groups on a UMD-space X with generators B_j, $j = 1, \ldots, n$. By Section 4.7.1 the operators $-B_j^2$ have an $H^\infty(\Sigma_\varphi)$-calculus for all $\varphi > 0$, $j = 1, \ldots, n$. It follows from the Dore–Venni theorem (Section 4.4.8) that the operator $A := B_1^2 + \cdots + B_n^2$ is closed with domain

$$D(A) = \bigcap_{j=1}^{n} D\bigl(B_j^2\bigr).$$

Moreover, $-A$ has a bounded $H^\infty(\Sigma_\varphi)$-calculus for all $\varphi > 0$. We may consider A as an abstract Laplace operator. Indeed, a concrete example is given by $(U_j(t)f)(x) = f(x + te_j)$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, where e_j denotes the jth unit vector in $\mathbb{R}^n$. Thus $Af = \Delta f$ with $D(A) = W^{2,p}(\mathbb{R}^n)$. For a more general noncommutative setting on Lie groups see ter Elst and Robinson [tER96a].

4.7.3. *Positive contraction semigroups on L^p.* Next we consider positive semigroups. The following result is due to Duong [Duo89] after a more special result was proved by Cowling [Cow83]. Vector-valued versions are given in [HP98].

THEOREM. *Let $-A$ be the generator of a positive, contractive C_0-semigroup on $L^p(\Omega)$, $1 < p < \infty$, where (Ω, Σ, μ) is a measure space. Then A has a bounded $H^\infty(\Sigma_\varphi)$-calculus for each $\varphi > \pi/2$.*

This result can be extended to semigroups on L^p which are dominated by a positive contraction semigroup. The proof can be reduced to the group case (Section 4.7.1) by an interesting dilation theorem due to Fendler extending the Akcoglu dilation theorem for single operators to semigroups. We give more details.

4.7.4. *r-contractive semigroups on L^p.* Let S be an operator on $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$. We say that S is *regular*, if S is a linear combination of positive operators; or equivalently, if S is dominated by a positive operator T on $L^p(\Omega)$, i.e.,

$$|Sf| \leqslant T|f|, \quad f \in L^p(\Omega),$$

where $|f|(x) = |f(x)|$, $x \in \Omega$. In that case there exists a smallest positive operator $|S|$ which dominates S. We call $\|S\|_r := \||S|\|$ the r-norm of S. If $p = 1$ or ∞, then every bounded operator S on $L^p(\Omega)$ is regular and $\|S\| = \|S\|_r$, but if $1 < p < \infty$ this is false (if $L^p(\Omega)$ has infinite dimension). We refer to [[Sch74], Chapter IV].

THEOREM. *Let $1 < p < \infty$. Let T be a C_0-semigroup on $L^p(\Omega)$ such that $\|T(t)\|_r \leqslant 1$ for all $t \geqslant 0$. Denote by $-A$ the generator of T. Then A has a bounded $H^\infty(\Sigma_\varphi)$-calculus for each $\varphi > \pi/2$.*

PROOF [LeM98b]. By Fendler's dilation theorem [Fen97] there exist a space $L^p(\widehat{\Omega})$, contractions $j : L^p(\Omega) \to L^p(\widehat{\Omega})$, $P : L^p(\widehat{\Omega}) \to L^p(\Omega)$ and a contraction C_0-group U on $L^p(\widehat{\Omega})$ such that $T(t) = PU(t)j$ for all $t \geqslant 0$. Let $-\widehat{A}$ be the generator of U. By Section 4.7.1 the $H^\infty(\Sigma_\varphi)$-calculus for $\widehat{A}$ is bounded ($\varphi > \pi/2$). It is obvious that this carries over to the $H^\infty(\Sigma_\varphi)$-calculus of A, cf. Section 5.2.2. □

4.7.5. *Holomorphic positive contraction semigroups.* The angle obtained in Section 4.7.3 can be improved if the semigroup is bounded holomorphic. In fact, the following result due to Kalton and Weis [KW01], Corollary 5.2, is proved with the help of R-boundedness techniques which will be presented in Section 6.

THEOREM. *Let $1 < p < \infty$. Let $-A$ be the generator of a bounded holomorphic C_0-semigroup T on $L^p(\Omega)$. Assume that*

$$T(t) \geqslant 0, \quad \|T(t)\|_{\mathcal{L}(L^p(\Omega))} \leqslant 1 \quad \textit{for all } t \geqslant 0.$$

Then A has a bounded H^∞-calculus and $\varphi_{H^\infty}(A) < \pi/2$.

5. Form methods and functional calculus

On Hilbert space generators of contraction C_0-semigroups are characterized by m-accretivity. This most convenient criterion also implies boundedness of the H^∞-calculus, as an easy consequence of the Spectral Theorem after a dilation to a unitary group. If the semigroup is holomorphic, then conversely, a bounded H^∞-calculus also implies m-accretivity after rescaling and changing the scalar product. In addition, the generator is associated with a closed form. The interesting interplay of forms, m-accretivity and H^∞-calculus on Hilbert space, this is the subject of the present section.

5.1. *Bounded H^∞-calculus on Hilbert space*

Things are much simpler on Hilbert space than on general Banach spaces. The *BIP* property is equivalent to bounded H^∞-calculus and all angles can be chosen optimal as was shown in Section 4.4.10, (4.26) and (4.34). We reformulate this more formally.

THEOREM (McIntosh). *Let A be an injective, sectorial operator on a Hilbert space. The following are equivalent*:

(i) $A \in \mathit{BIP}$;
(ii) *for all $\varphi > \varphi_{\sec}(A)$, the $H^\infty(\Sigma_\varphi)$-calculus is bounded*;
(iii) *the $H^\infty(\Sigma_\pi)$-calculus is bounded.*

In that case $\varphi_{bip}(A) = \varphi_{\sec}(A)$.

This optimal situation in Hilbert space contrasts the L^p-case, $p \neq 2$, where we had seen that property *BIP* does not imply boundedness of the $H^\infty(\Sigma_\pi)$-calculus (see Section 4.5.3). Also had we seen that it can happen that $\varphi_{\sec}(A) < \pi < \varphi_{bip}(A)$ on L^p, $p \neq 2$.

Also the Dore–Venni theorem needs fewer hypotheses on Hilbert space.

THEOREM (Dore–Venni on Hilbert space [DoVe87]). *Let A and B be two commuting injective, sectorial operators on a Hilbert space such that $\varphi_{\mathrm{sec}}(A) + \varphi_{\mathrm{sec}}(B) < \pi$. Assume that $A \in BIP$. Then $A + B$ is closed.*

We mention that this result does not hold on L^p for $p \neq 2$ ([Lan98], Theorem 2.3).

5.2. *m-accretive operators on Hilbert space*

The simplest unbounded operators are multiplication operators. Let (Ω, Σ, μ) be a measure space and let $m : \Omega \to \mathbb{R}$ be a measurable function. Define A_m on $L^2(\Omega)$ by $A_m u = mu$ with domain

$$D(A_m) = \left\{u \in L^2(\Omega)\colon\ m \cdot u \in L^2(\Omega)\right\}.$$

For the operator A_m one has the best possible spectral calculus. For each $f \in L^\infty(\mathbb{R})$, we may define $f(A) = A_{f \circ m} \in \mathcal{L}(L^2(\Omega))$. It is not difficult to see that this definition is consistent with our previous one whenever $f \in H^\infty(\Sigma_\varphi)$ for some $\varphi > \pi/2$.

5.2.1. *The Spectral theorem.* The spectral theorem says that each self-adjoint operator is unitarily equivalent to a multiplication operator.

SPECTRAL THEOREM. *Let A be a self-adjoint operator on H. Then there exists a measure space (Ω, Σ, μ), a measurable function $m : \Omega \to \mathbb{R}$ and a unitary operator $U : H \to L^2(\Omega)$ such that*

(a) $U D(A) = D(A_m)$;

(b) $A_m U x = U A x$, $x \in D(A)$.

Thus, multiplication operators and self-adjoint operators are just the same thing.

Now recall that an operator A generates a *unitary group* (i.e., a C_0-group of unitary operators) if and only if $\mathrm{i}A$ is self-adjoint. Thus $\mathrm{i}A$ is equivalent to a multiplication operator. This shows in particular that each generator A of a unitary group on H has a bounded $H^\infty(\Sigma_\varphi)$-calculus for all $\varphi > \pi/2$. In fact, it is easy to show that for $f \in H^\infty(\Sigma_\varphi)$ one has

$$f(A) = U^{-1} A_{f \circ m} U.$$

5.2.2. *Bounded H^∞-calculus for m-accretive operators.* Recall that an operator A on a Hilbert space H is called *m-accretive* if

(a) $\mathrm{Re}(Ax|x) \geqslant 0$ for all $x \in D(A)$;

(b) $I + A$ is surjective.

The Lumer–Phillips theorem asserts that an operator A is m-accretive if and only if $-A$ generates a contractive C_0-semigroup. In particular, if A is m-accretive, then A is sectorial and $\varphi_{\mathrm{sec}}(A) \leqslant \frac{\pi}{2}$.

THEOREM. *Let A be an m-accretive operator on H. Then A has a bounded $H^\infty(\Sigma_\varphi)$-calculus for each $\varphi > \varphi_{\mathrm{sec}}(A)$.*

PROOF. Denote by T the semigroup generated by $-A$. By the dilation theorem [[Dav80], p. 157] there exist a Hilbert space $\widehat{H}$ containing H as a closed subspace and a unitary group U on $\widehat{H}$ such that

$$P \circ U(t) \circ i = T(t), \quad t \geqslant 0,$$

where $i : H \to \widehat{H}$ is the injection of H into $\widehat{H}$ and $P : \widehat{H} \to H$ is the orthogonal projection. Denote by $\widehat{A}$ the generator of U. Then $\mathrm{i}\widehat{A}$ is selfadjoint. Thus $\widehat{A}$ has a bounded $H^\infty(\Sigma_\varphi)$-calculus for any $\varphi > \pi/2$. It is easy to see from the definitions that

$$f(A) = P \circ f\big(\widehat{A}\big) \circ i.$$

Thus $f(A) \in \mathcal{L}(H)$ for all $f \in H^\infty(\Sigma_\varphi)$ (and even $\|f(A)\| \leqslant \|f(\widehat{A})\| \leqslant \|f\|_{L^\infty(\Sigma_{\pi/2})}$). Thus A has a bounded $H^\infty(\Sigma_\varphi)$-calculus for all $\varphi > \frac{\pi}{2}$. Now by McIntosh's result in Section 5.1 the $H^\infty(\Sigma_\varphi)$-calculus is also bounded for $\varphi > \varphi_{\mathrm{sec}}(A)$. □

5.2.3. *Equivalence of bounded H^∞-calculus and m-accretivity.* Let $(\cdot|\cdot)_1$ be an equivalent scalar product on H, i.e., there exist $\alpha > 0$, $\beta > 0$ such that

$$\alpha\|u\|_H^2 \leqslant (u|u)_1 \leqslant \beta\|u\|_H^2$$

for all $u \in H$. Of course, having a bounded H^∞-calculus is independent of the equivalent norm on H we choose. Thus, if A is a sectorial operator on H which is m-accretive with respect to some equivalent scalar product, then $A \in H^\infty$. This describes already the class of all operators A with bounded H^∞-calculus.

THEOREM (Le Merdy [LeM98a]). *Let A be an injective, sectorial operator on H such that $\varphi_{\mathrm{sec}}(A) < \pi/2$. The following are equivalent*:

(i) $A \in H^\infty$,

(ii) *there exists an equivalent scalar product* $(\cdot|\cdot)_1$ *on H such that* $\mathrm{Re}(Au|u)_1 \geqslant 0$ *for all $u \in D(A)$*;

(iii) *let* $0 \leqslant \varphi < \varphi_{\mathrm{sec}}(A)$; *then there exists an equivalent scalar product* $(\cdot|\cdot)_1$ *on H such that $(Au|u) \in \Sigma_\varphi$ for all $u \in D(A)$.*

Denote by T the C_0-semigroup generated by $-A$. Then (ii) means that

$$\big\|T(t)u\big\|_1 \leqslant \|u\|_1 := \sqrt{(u|u)_1}$$

for all $u \in H$, $t \geqslant 0$, and (iii) means that

$$\big\|T(z)u\big\|_1 \leqslant \|u\|_1$$

for all $u \in H$ and $z \in \Sigma_{\pi/2-\varphi}$.

The proof of Le Merdy [LeM98a] uses the theory of completely bounded operators and a deep theorem of Paulsen. A more direct proof, based directly on "*quadratic estimates*", is given by Haase [Haa02].

The theorem gives a very satisfying characterization of the boundedness of the H^∞-calculus; see also Section 5.3 for a description by forms. However, it is restricted to sectorial operators of sectorial type smaller than $\pi/2$:

5.2.4. *Counterexample.* There exists an invertible, sectorial operator A on a Hilbert space H such that $-A$ generates a bounded C_0-semigroup T on H. Moreover, A has a bounded $H^\infty(\Sigma_\varphi)$-calculus for any $\varphi > \pi/2$. But for all $\omega > 0$, $A + \omega$ is not accretive with respect to any equivalent scalar product.

The counterexample is due to Le Merdy [LeM98a] (based on a famous example of Pisier solving the Halmos problem) with an additional argument by Haase [Haa02] (taking care of the case where $\omega > 0$).

5.3. *Form methods*

A most efficient way to define generators of holomorphic semigroups on a Hilbert space H is the *form method* (sometimes it is also called "*variational method*").

Let H be a Hilbert space. If V is another Hilbert space we write $V \hookrightarrow H$ if V is a subspace of H such that the embedding is continuous. We write $V \hookrightarrow_d H$ if in addition V is dense in H.

DEFINITION. A *closed form* on H is a sesquilinear form $a : V \times V \to \mathbb{C}$ which is *continuous*, i.e.,

$$\left|a(u, v)\right| \leqslant M\|u\|_V\|v\|_V, \quad u, v \in V, \tag{5.1}$$

for some $M \geqslant 0$ and *elliptic*, i.e.,

$$\operatorname{Re} a(u, u) + \omega\|u\|_H^2 \geqslant \alpha\|u\|_V^2, \quad u \in V, \tag{5.2}$$

for some $\omega \in \mathbb{R}$, $\alpha > 0$. The space V is called the *domain* of a.

Properly speaking, a closed form is a couple (a, V) with the properties stipulated in the above definition. Note that by (5.2),

$$\|u\|_a := \left(\operatorname{Re} a(u, u) + \omega\|u\|_H^2\right)^{1/2}$$

defines an equivalent norm on V. Sometimes we will write $D(a)$ instead of V and will use the norm $\|\cdot\|_a$ on $D(a)$ for which $D(a)$ is complete by hypothesis.

REMARK. Let $a : D(a) \times D(a) \to \mathbb{C}$ be sesquilinear, where $D(a)$ is a subspace of H. Assume that a is *bounded below*, i.e.,

$$\|u\|_a^2 := \operatorname{Re} a(u,u) + \omega \|u\|_H^2 \geqslant \|u\|_H^2$$

for all $u \in D(a)$ and some $\omega \in \mathbb{R}$. Then $\| \cdot \|_a$ defines a norm on $D(a)$. The form is continuous with respect to this norm if and only if $a(u,u) \in \omega_1 + \Sigma_\theta$ for some $\omega_1 \in \mathbb{R}$ and some $\theta \in (0, \pi/2]$, i.e., if and only if the form a is *sectorial* in the terminology of Kato [[Kat66]].

5.3.1. *The operator associated with a closed form.* Let a be a closed form on H with dense domain V. Then we associate an operator A with a, given by

$$D(A) = \{u \in V \colon \exists v \in H \text{ such that } a(u,\varphi) = (v|\varphi)_H \text{ for all } \varphi \in V\},$$
$$Au = v.$$

Note that v is well defined since V is dense in H. We call A *the operator associated with* a. It is not difficult to see that $A + \omega$ is sectorial with angle $\varphi_{\mathrm{sec}}(A+\omega) < \pi/2$. Moreover, $A + \omega$ is m-accretive (where ω is the constant occurring in (5.2)). Thus we obtain the following.

THEOREM. *Let A be an operator associated with a closed form. Then there exists $\omega \in \mathbb{R}$ such that $A + \omega \in H^\infty$.*

5.3.2. *Example*: *self-adjoint operators.* Let a be a densely defined closed form on H with domain V. Assume that a is symmetric; i.e., $a(u,v) = \overline{a(v,u)}$ for all $u, v \in V$. Let A be the operator associated with a. Then A is *bounded below* (i.e., $(Au|u) \geqslant -\omega \|u\|_H^2$ for all $u \in D(A)$ and some $\omega \in \mathbb{R}$) and self-adjoint. Conversely, let A be a self-adjoint operator which is bounded below. Then A is associated with a densely defined symmetric closed form.

We give a classical concrete example right now. Later we will study elliptic operators in more detail.

5.3.3. *Example* (*The Laplacian with Dirichlet and with Neumann boundary conditions*). Let $\Omega \subset \mathbb{R}^n$ be open and $H = L^2(\Omega)$. Let $W^{1,2}(\Omega) := \{u \in L^2(\Omega) \colon D_j u \in L^2(\Omega), j = 1, \ldots, n\}$ be the first Sobolev space and let $W_0^{1,2}(\Omega)$ be the closure of the space $\mathcal{D}(\Omega)$ of all test functions in $W^{1,2}(\Omega)$. Define $a : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$a(u,v) = \int_\Omega \nabla u \overline{\nabla v} \, \mathrm{d}x.$$

(a) Let $V = W_0^{1,2}(\Omega)$. Then a is a closed symmetric form. Denote by $-\Delta_\Omega^D$ the operator associated with a. Then it is easy to see that $D(\Delta_\Omega^D) = \{u \in W_0^{1,2}(\Omega) \colon \Delta u \in L^2(\Omega)\}$, $\Delta_\Omega^D u = \Delta u$, in $\mathcal{D}(\Omega)'$. We call Δ_Ω^D the *Dirichlet Laplacian* on $L^2(\Omega)$.

(b) Let $V = W^{1,2}(\Omega)$. Then a is closed and symmetric. Denote by $-\Delta_\Omega^N$ the operator associated with a. Then Δ_Ω^N is a self-adjoint operator, called the *Neumann Laplacian*. This can be justified if Ω is bounded with Lipschitz boundary. Then $D(\Delta_\Omega^N) = \{u \in W^{1,2}(\Omega)\colon\ \Delta u \in L^2(\Omega);\ \frac{\partial u}{\partial \nu} = 0 \text{ weakly}\}$, where the weak sense has to be explained. We omit the details. And indeed, for many purposes it is not important to know in which sense the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ is realized. The interesting point is that the Neumann–Laplacian can be defined for arbitrary open sets without using the normal derivative.

5.3.4. *Characterization of operators defined by forms.* Let A be an operator on H. We say that A *is induced by a form*, if there exists a densely defined closed form a on H such that A is associated with a. It is easy to describe such operators by an accretivity-condition.

THEOREM [[Kat66]]. *Let A be an operator on H. The following are equivalent*:

(i) *A is induced by a form*;

(ii) *there exist $\omega \in \mathbb{R}$ and $0 < \theta < \frac{\pi}{2}$ such that*

(a) $(\omega + A)D(A) = H$;

(b) $e^{\pm i\theta}(\omega + A)$ *is accretive.*

(iii) *$-A$ generates a holomorphic C_0-semigroup T of angle $\theta \in (0, \frac{\pi}{2})$ such that for some $\omega \in \mathbb{R}$*,

$$\|T(z)\| \leqslant e^{\omega|z|}, \quad z \in \Sigma_\theta;$$

(iv) *the numerical range $W(A) := \{(Ax|x)\colon\ x \in D(A), \|x\| = 1\}$ is contained in $\Sigma_\theta + \omega$ for some $\theta \in (0, \pi/2)$, $\omega \in \mathbb{R}$ and $(-\infty, \omega) \cap \rho(A) \neq \emptyset$.*

Property (ii) is sometimes formulated by saying that A is *strongly quasi-m-accretive* (and *strongly m-accretive* if $\omega = 0$).

5.3.5. *Changing scalar products.* Let $a : V \times V \to \mathbb{C}$ be a closed densely defined form on H. The definition of the associated operator A depends on the scalar product on H. Let $(\cdot|\cdot)_1$ be another equivalent scalar product on H. Then there exists a self-adjoint operator $Q \in \mathcal{L}(H)$ such that

$$(Qu|v)_1 = (u|v) \quad \text{for all } u, v \in H,$$

and Q is *strictly form-positive*. By this we mean that $(Qu|u) \geqslant \varepsilon\|u\|_H^2$ for all $u \in H$ and some $\varepsilon > 0$. Let $u \in D(A)$. Then $u \in V$ and $a(u, \varphi) = (Au|\varphi) = (QAu|\varphi)_1$ for all $\varphi \in V$. This shows that the operator A_1 associated with a on $(H, (\cdot|\cdot)_1)$ is given by $A_1 = QA$. With the help of Le Merdy's theorem in Section 5.2.3 one obtains the following characterization of the class H^∞.

THEOREM [ABH01]. *Let A be an operator on H. The following are equivalent*:

(i) *$A + \omega \in H^\infty$ for some $\omega \geqslant 0$*;

(ii) *there exists an equivalent scalar product* $(\cdot|\cdot)_1$ *on* H *such that* A *is induced by a form on* $(H, (\cdot|\cdot)_1)$;
(iii) *there exists a strictly form-positive self-adjoint operator* $Q \in \mathcal{L}(H)$ *such that* QA *is induced by a form.*

We remark that the rescaling $A + \omega$ is needed to obtain a sectorial operator. From Section 4.6 we know that for an invertible sectorial operator A and $\omega > 0$ one has

$$A \in H^\infty \quad \text{if and only if} \quad A + \omega \in H^\infty.$$

5.3.6. *More on equivalent scalar products.* In the Section 5.3.5 we characterized those operators A which come from a form on $(H, (\cdot|\cdot)_1)$ for some equivalent scalar product. What happens if we replace "*some*" by "*all*"?

THEOREM (Matolcsi [Mat03]). *Let A be an operator on H. The following are equivalent*:
(i) *A is bounded*;
(ii) *for each equivalent scalar product* $(\cdot|\cdot)_1$ *on* H, *the operator* A *is induced by a form on* $(H, (\cdot|\cdot)_1)$.

5.4. *Form sums and Trotter's product formula*

There is a natural way to define the sum of closed forms leading to a closed form again. Here we want to allow also nondense form domains. Let H be a Hilbert space.

5.4.1. *Closed forms with nondense domain.* We define closed forms on H as before but omit the assumption that the form domain be dense in H. Thus, a *closed form* a on H is a sesquilinear form $a : D(a) \times D(a) \to \mathbb{C}$, where $D(a)$ is a subspace of H (the *form domain*) such that, for some $\omega \in \mathbb{R}$, $\alpha \in [0, \pi/2)$:
(a) $a(u, u) + (\omega - 1)\|u\|_H^2 \in \Sigma_\alpha$, $u \in D(a)$;
(b) $(D(a), \|\cdot\|_a)$ is complete, where $\|u\|_a^2 = \operatorname{Re} a(u, u) + \omega\|u\|_H^2$.
Then one can show that a is a continuous sesquilinear form on $V = (D(a), \|\cdot\|_a)$. Thus (5.1) and (5.2) are satisfied.

Denote by A the operator on $\overline{D(a)}$ (the closure of $D(a)$ in H) associated with a. Then $-A$ generates a holomorphic C_0-semigroup $(\mathrm{e}^{-tA})_{t \geqslant 0}$ on $\overline{D(a)}$. We extend this semigroup by 0 to H defining

$$\mathrm{e}^{-ta}x := \begin{cases} \mathrm{e}^{-tA}x & \text{if } x \in \overline{D(a)}, \\ 0 & \text{if } x \in D(a)^\perp, \end{cases} \tag{5.3}$$

for $t > 0$ and letting e^{-0a} the orthogonal projection onto $\overline{D(a)}$. Then $t \mapsto \mathrm{e}^{-ta}$: $[0, \infty) \to \mathcal{L}(H)$ is strongly continuous and satisfies the semigroup property

$$\mathrm{e}^{-ta}\mathrm{e}^{-sa} = \mathrm{e}^{-(t+s)a}, \quad t, s \geqslant 0.$$

We call $(\mathrm{e}^{-ta})_{t\geqslant 0}$ *the semigroup on H associated with the closed form a*. Before giving examples we consider form sums.

5.4.2. *Form sums.* Now let a and b be two closed forms on H. Then $D(a)\cap D(b)$ is a Hilbert space for the norm $(\|u\|_a^2+\|u\|_b^2)^{1/2}$. Hence, $a+b$ with domain $D(a+b)=D(a)\cap D(b)$ is a closed form again.

THEOREM. *Let a and b be two closed forms on H. Then Trotter's formula*

$$\mathrm{e}^{-t(a+b)}x=\lim_{n\to\infty}\left(\mathrm{e}^{-(t/n)a}\mathrm{e}^{-(t/n)b}\right)^n x \tag{5.4}$$

holds for all $x\in H$, $t\geqslant 0$.

This result is due to Kato [Kat78] for the case of symmetric forms and it was generalized to closed forms by Simon (see [Kat78], Theorem and Addendum).

It is interesting that this result can be applied in particular when $\mathrm{e}^{-tb}\equiv P$, where P is an orthogonal projection. This is done in the following section.

5.4.3. *Induced semigroups.* Let a be a closed form on H. Let H_1 be a closed subspace of H. Even if H_1 is not invariant under $(\mathrm{e}^{-ta})_{t\geqslant 0}$ there is a natural induced semigroup on H_1. In fact, let $D(b)=H_1$ and $b\equiv 0$. Then $\mathrm{e}^{-tb}\equiv P$, the orthogonal projection onto H_1. Thus by Trotter's formula (5.4),

$$\mathrm{e}^{-ta_1}x=\lim_{n\to\infty}\left(\mathrm{e}^{-ta/n}P\right)^n x \tag{5.5}$$

converges for all $x\in H$, where $D(a_1)=D(a)\cap H_1$ and $a_1(u,v)=a(u,v)$ for all $u,v\in D(a_1)$. Then $(\mathrm{e}^{-ta_1})_{t\geqslant 0}$ is a C_0-semigroup on H_1 if and only if $D(a)\cap H_1$ is dense in H_1.

5.4.4. *Nonconvergence of Trotter's formula.* Before giving a concrete example we want to point out that this way to induce a semigroup on closed subspaces via (5.5) does not work for arbitrary contraction semigroups.

THEOREM (Matolcsi [Mat03]). *Let $-A$ be the generator of a contraction C_0-semigroup T on H. The following assertion are equivalent*:

(i) *A is induced by a form*;

(ii) $\lim_{n\to\infty}(T(t/n)P)^n x$ *converges for each $t>0$, $x\in H$, and each orthogonal projection P.*

5.4.5. *From the Gaussian semigroup to the Dirichlet Laplacian.* Denote by G the Gaussian semigroup on $L^2(\mathbb{R}^n)$, i.e.,

$$G(t)f(x)=(4\pi t)^{-n/2}\int_{\mathbb{R}^n}\mathrm{e}^{-(x-y)^2/4t}f(y)\,\mathrm{d}y$$

for all $t > 0$, $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$. Then G is associated with the form $a(u,v) = \int_{\mathbb{R}^n} \nabla u \overline{\nabla v}\, \mathrm{d}x$ with domain $D(a) = W^{1,2}(\mathbb{R}^n)$. Now let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$. We identify $L^2(\Omega)$ with the space $\{f \in L^2(\mathbb{R}^n)\colon\ f(x) = 0 \text{ a.e. on } \Omega^c\}$. Then the orthogonal projection P onto $L^2(\Omega)$ is given by $Pf = 1_\Omega f$ for all $f \in L^2(\mathbb{R}^n)$. Now let $W_0^{1,2}(\overline{\Omega}) = W^{1,2}(\mathbb{R}^n) \cap L^2(\Omega) = \{u \in W^{1,2}(\mathbb{R}^n)\colon\ u(x) = 0 \text{ a.e. on } \Omega^c\}$. Consider the form b on $W_0^{1,2}(\overline{\Omega})$ given by $b(u,v) = \int_\Omega \nabla u \nabla v$. Then, by (5.5), we have

$$\mathrm{e}^{-tb} f = \lim_{n\to\infty} \big(G(t/n)1_\Omega\big)^n f \tag{5.6}$$

for all $f \in L^2(\mathbb{R}^n)$. Now we consider the more familiar space $W_0^{1,2}(\Omega)$. For $u \in W_0^{1,2}(\Omega)$ let $\tilde{u}(x) = u(x)$ for $x \in \Omega$ and $\tilde{u}(x) = 0$ for $x \in \Omega^c$. Then $\tilde{u} \in W_0^{1,2}(\overline{\Omega})$ and $D_j\tilde{u} = \widetilde{D_j u}$. Thus we may identify $W_0^{1,2}(\Omega)$ with a subspace of $W_0^{1,2}(\overline{\Omega})$. One says that Ω is *stable*, if $W_0^{1,2}(\Omega) = W_0^{1,2}(\overline{\Omega})$. For example, if Ω has Lipschitz boundary, then Ω is stable. Thus Ω is stable if and only if $(\mathrm{e}^{-tb})_{t\geqslant 0}$ is the semigroup generated by the Dirichlet Laplacian with the canonical extension to $L^2(\mathbb{R}^n)$, i.e.,

$$\big(\mathrm{e}^{-tb} f\big)(x) = \begin{cases} \big(\mathrm{e}^{t\Delta_\Omega^D} f\big|_\Omega\big)(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases}$$

Thus, if Ω is stable, then the semigroup generated by the Dirichlet Laplacian is obtained from the Gaussian semigroup via Trotter's formula (5.6).

Stable open sets can be characterized as follows. Let Ω be an open, bounded set in $\mathbb{R}^n$. We assume that the boundary $\partial\Omega$ of Ω is a null-set (for the n-dimensional Lebesgue measure), and that $\mathring{\overline{\Omega}} = \Omega$. By

$$\operatorname{cap}(A) = \inf\big\{\|u\|_{H^1}^2\colon\ u \in H^1\big(\mathbb{R}^n\big), u \geqslant 1,\ \text{a.e. in a neighborhood of } A\big\}$$

we denote the *capacity* of a subset A of $\mathbb{R}^n$.

THEOREM. *The following assertions are equivalent.*

(i) $W_0^{1,2}(\Omega) = W_0^{1,2}(\overline{\Omega})$;

(ii) $\operatorname{cap}(G \setminus \overline{\Omega}) = \operatorname{cap}(G \setminus \Omega)$ *for every open set* $G \subset \mathbb{R}^n$;

(iii) *for each function* $u \in C(\overline{\Omega})$, *which is harmonic on* Ω, *there exist* u_n *harmonic on an open neighborhood of* $\overline{\Omega}$ *which converge to* u *uniformly on* $\overline{\Omega}$ *as* $n \to \infty$.

It is interesting that these properties do not imply Dirichlet regularity. In fact, if Ω is the Lebesgue cusp, then $W_0^{1,2}(\Omega) = W_0^{1,2}(\overline{\Omega})$. References: [Hed93] and [Kel66].

5.5. *The square root property*

Let H be a Hilbert space. During this section we consider a continuous sesquilinear form $a : V \times V \to \mathbb{C}$ which is coercive with respect to H, i.e.,

$$\operatorname{Re} a(u,u) \geqslant \nu \|u\|_V^2, \quad u \in V,$$

where $V \hookrightarrow_d H$ and $\nu > 0$. This means that the ellipticity condition (5.2) is satisfied for $\omega = 0$. We consider the operator A on H which is associated with a. Thus A is a sectorial operator and $0 \in \rho(A)$.

5.5.1. *The square root property.* We say that the operator A (or the form a) has the *square root property* if $V = D(A^{1/2})$. It is easy to see that this is the case if a is symmetric (see the Example below). But an example of McIntosh [McI82] shows that the square root property does not hold for all closed, coercive forms. Here is an easy characterization of the square root property.

PROPOSITION. *The following are equivalent*:
(i) $D(A^{1/2}) = V$,
(ii) $D(A^{*1/2}) = V$,
(iii) $D(A^{1/2}) = D(A^{*1/2})$.

In that case the three norms $\|\cdot\|_V$, $\|A^{1/2}x\|$ and $\|A^{*1/2}x\|$ are equivalent. This is an immediate consequence of the Closed Graph theorem since all three spaces are continuously embedded into H. Of course one may also ask whether (iii) is valid for other powers α than $\alpha = \frac{1}{2}$. Surprisingly, by a result of Kato [[Kat66]] one always has

$$D\big(A^{\alpha}\big) = D\big(A^{*\alpha}\big)$$

for all $0 < \alpha < \frac{1}{2}$ for each m-accretive operator. Thus $\alpha = \frac{1}{2}$ is the critical power for which things may go wrong.

5.5.2. *Induced operators on V and V'.* Since $V \hookrightarrow_d H$, we may identify H with a subspace of V'. Indeed, given $x \in H$, we define $j_x \in V'$ by $j_x(y) = (x|y)_H$. Then $j : H \to V'$, $x \mapsto j_x$ is linear, continuous and has dense image. We will identify x and j_x so that

$$V \underset{d}{\hookrightarrow} H \underset{d}{\hookrightarrow} V'.$$

By the Lax–Milgram theorem, there is an isomorphism $A_{V'} : V \to V'$ given by $\langle A_{V'}u, v\rangle = a(u,v)$, $u, v \in V$. Here V' denotes the space of all continuous, anti-linear forms on V (i.e., $V' = \{\bar{\varphi}\colon \varphi \in V^*\}$, V^* the dual space of V). The operator $-A_{V'}$ generates a bounded holomorphic C_0-semigroup $T_{V'}$ on V' (see [[Tan79], Section 3.5]). Moreover, $T_{V'}(t)|_H = T(t)$, the semigroup generated by $-A$ on H and, clearly, A is the part of $A_{V'}$ in H. Finally, the part of $A_{V'}$ in V, i.e., the operator A_V given by $A_V u = Au$ with domain $D(A_V) = \{u \in D(A)\colon Au \in V\}$ is similar to $A_{V'}$ and $-A_V$ generates a holomorphic

C_0-semigroup T_V which is the restriction of T to V. Thus, A_V and $A_{V'}$ are always similar; but A_V and A are not, unless A has the square root property:

THEOREM. *The following are equivalent*:
(i) $D(A^{1/2}) = V$;
(ii) $A_V \in H^\infty$;
(iii) $A_{V'} \in H^\infty$;
(iv) A_V *and* A *are similar.*

Thus, one can describe the square root property by the boundedness of the H^∞-calculus of A_V (or $A_{V'}$).

Before giving a proof of the Theorem we determine the diverse notions in the case of self-adjoint operators.

EXAMPLE (Self-adjoint operators). Assume that the form a defined at the beginning of the section is symmetric. Then the associated operator A is self-adjoint. Thus, by the spectral theorem, we may assume that $H = L^2(\Omega, \Sigma, \mu)$ and $Au = m \cdot u$, $D(A) = \{u \in L^2(\Omega)\colon mu \in L^2(\Omega)\}$, where $m\colon \Omega \to [\nu, \infty)$ is measurable. Then $V = L^2(\Omega, m\,\mathrm{d}\mu)$ and $a(u, v) = \int_\omega mu\bar{v}\,\mathrm{d}\mu$. Moreover, $V' = L^2(\Omega, 1/m\,\mathrm{d}\mu)$ and $\langle u, v\rangle = \int_\Omega u\bar{v}\,\mathrm{d}\mu$ for all $u \in V'$, $v \in V$. The operator $A_{V'}$ is given by $D(A_{V'}) = V = L^2(\Omega, m\,\mathrm{d}\mu)$, $A_{V'}u = mu$, and A_V is given by $A_V u = mu$ with domain $D(A_V) = \{u \in V\colon mu \in V\} = L^2(\Omega, m^3\,\mathrm{d}\mu)$. Now, $A^{-\alpha}$ is a bounded operator on H given by $A^{-\alpha}u = m^{-\alpha}u$. Thus $D(A^\alpha) = \{m^{-\alpha}u\colon\ u \in L^2(\Omega)\} = L^2(\Omega, m^{2\alpha}\,\mathrm{d}\mu)$. It is clear that $D(A^\alpha) = [H, D(A)]_\alpha$ for all $0 < \alpha < 1$.

REMARK (Definition of the complex interpolation space). Let $V \hookrightarrow_d H$. It is possible to define the complex interpolation spaces with help of the spectral theorem. Indeed, $a(u, v) = (u|v)_V$ defines a continuous, coercive form on V. Then we can assume that $V = L^2(\Omega, m\,\mathrm{d}\mu)$ and $H = L^2(\Omega, \mathrm{d}\mu)$. Then $[H, V]_\alpha = L^2(\Omega, m^\alpha\,\mathrm{d}\mu)$ for $0 < \alpha < 1$, where $[H, V]_\alpha$ denotes the complex interpolation space.

PROOF OF THE THEOREM. The operator $A^{-1/2}\colon H \to D(A^{1/2})$ is an isomorphism. If $V = D(A^{1/2})$, then $A_V = A^{-1/2}AA^{1/2}$, i.e., A and A_V are similar. This shows that (i) $\Rightarrow$ (iv). Since $A \in H^\infty$, (iv) implies (ii). And (ii) implies (iii) since A_V and $A_{V'}$ are always similar. Finally we show that (iii) implies (i). From the previous remark it is clear that $H = [V', V]_{1/2}$. Now if $A_{V'} \in H^\infty$, then, by Section 4.4.10, $D(A_{V'}^{1/2}) = [V', D(A_{V'})]_{1/2} = [V', V]_{1/2} = H$, i.e., $H = A_{V'}^{-1/2}(V')$. This implies that

$$D\bigl(A^{1/2}\bigr) = A^{-1/2}H = A_{V'}^{-1}\bigl(V'\bigr) = V. \qquad \square$$

5.6. *Groups and cosine functions*

In the preceding sections we have characterized those sectorial operators A on a Hilbert space with sectorial angle $\varphi_{\sec}(A) < \pi/2$ which have a bounded H^∞-calculus. This is equivalent to saying that A is defined by a closed form on $(H, (\cdot|\cdot)_1)$ for some equivalent scalar product $(\cdot|\cdot)_1$. Now we will see that squares of group generators are always of this form.

5.6.1. *Groups on Hilbert spaces.* Let U be a bounded C_0-group on a Hilbert space H. Then, by a classical result of Sz.-Nagy, there exists an equivalent scalar product $(\cdot|\cdot)_1$ on H such that the group is unitary in $(H(\cdot|\cdot)_1)$. Concerning arbitrary C_0-groups the following holds (see, e.g., [Haa03a]).

PROPOSITION. *Let B be the generator of a C_0-group on H. Then there exists a bounded operator C on H such that $B + C$ generates a bounded C_0-group.*

Thus, on Hilbert space, up to equivalent scalar product, each generator of a C_0-group B is of the form

$$B = \mathrm{i}B_0 + C,$$

where B_0 is self-adjoint and C bounded.

5.6.2. *Squares of generators of C_0-groups.* Let B be the generator of a C_0-group. Then B^2 generates a holomorphic C_0-semigroup of angle $\pi/2$. Thus, for each $0 < \theta < \pi/2$, there exists $\omega \in \mathbb{R}$ such that $-B^2 + \omega I$ is sectorial and $\varphi_{\mathrm{sec}}(-B^2 + \omega I) < \theta$.

THEOREM (Haase [Haa03a]). *Let $A = -B^2$, where B generates a C_0-group on a Hilbert space. Then there exists $\omega \in \mathbb{R}$ such that $A + \omega \in H^\infty$.*

Operators of the form $B^2 - \omega$ with B a group generator are the same as generators of cosine functions. This holds in Hilbert spaces, and more generally on UMD-spaces. We describe these facts, starting on arbitrary Banach spaces first and concluding with the square root property.

5.6.3. *Cosine functions.* Let X be a Banach space. A *cosine function* on X is a strongly continuous function $C : \mathbb{R} \to \mathcal{L}(X)$ satisfying

$$2C(t)C(s) = C(t+s) + C(t-s), \quad s, t \in \mathbb{R}, \qquad C(0) = I.$$

In that case there exist $M \geqslant 1$, $\omega \in \mathbb{R}$ such that

$$\|C(t)\| \leqslant M\mathrm{e}^{\omega t}, \quad t \geqslant 0.$$

Moreover, there is a unique operator A such that $(\omega, \infty) \subset \rho(A)$ and

$$\lambda R(\lambda^2, A)x = \int_0^\infty \mathrm{e}^{-\lambda t} C(t)x \,\mathrm{d}t$$

for all $x \in X$, $\lambda > \omega$. We call A the *generator* of C. The cosine function C generated by A governs the second-order Cauchy problem defined by A

$$\begin{cases} \ddot{u}(t) = Au(t), & t \in \mathbb{R}, \\ u(0) = x, & \dot{u}(0) = y. \end{cases} \tag{5.7}$$

We will make this more precise. Introducing the function $v = \dot{u}$, problem (5.7) can be reformulated by

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad u(0) = x, \quad v(0) = y.$$

However, the operator $\left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$ with domain $D(A) \times X$ generates a C_0-semigroup on $X \times X$ if and only if A is bounded (see [[ABHN01], Section 3.14.9]). Thus, another space than $X \times X$ has to be considered.

THEOREM [[ABHN01], Section 3.14.11]. *Let A be an operator on a Banach space X. The following assertions are equivalent*:

(i) *A generates a cosine function*;

(ii) *there exists a Banach space W such that $D(A) \subset W \hookrightarrow X$, and the operator*

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

with domain $D(A) \times W$ generates a C_0-semigroup on $W \times X$.

In that case the Banach space W is uniquely determined by property (ii). We call $W \times X$ the *phase space* of problem (5.7). The phase space is important to produce classical solutions:

PROPOSITION [[ABHN01], Section 3.14.12]. *Let A be the generator of a cosine function, with phase space $W \times X$. Then, for each $x \in D(A)$, $y \in W$, there exists a unique solution $u \in C^2(\mathbb{R}, X) \cap C(\mathbb{R}, D(A))$ of* (5.7).

So far we have described well-posedness of the second-order Cauchy problem (5.7) in arbitrary Banach spaces. Now we come back to the initial subject of this section.

5.6.4. *Squares of group generators and cosine functions.* On UMD-spaces one has the following characterization.

THEOREM (Fattorini [[ABHN01], Section 3.16.8]). *Let A be an operator on a UMD-space X. The following assertions are equivalent*:

(i) *A generates a cosine function*;

(ii) *there exist a generator B of a C_0-group on X and $\omega \geqslant 0$ such that*

$$A = B^2 + \omega I.$$

In that case, the phase space is $D(B) \times X$, where $D(B)$ carries the graph norm.

This theorem explains why L^p-spaces for $p \neq 2$ are not so good as functional framework for hyperbolic problems: There are not many groups operating on L^p for $p \neq 2$. For example, the Laplacian Δ generates a cosine function on $L^p(\mathbb{R}^n)$ if and only if $p = 2$ or $n = 1$. On the other hand, on Hilbert spaces there are many C_0-groups since there are many self-adjoint operators. This explains from an abstract point of view why many examples of well-posed second-order problems exist on Hilbert space (see Section 8.9). In view of the results mentioned so far, they all are related to self-adjoint operators.

5.6.5. *Forms and cosine functions.* Let H be a Hilbert space. If $-A$ generates a cosine function, then it follows from Sections 5.6.4 and 5.6.2 that $A + \omega \in H^\infty$ for ω large enough. Thus, after changing the scalar product on H, the operator A is associated with a closed form. Now let us assume that A is associated with a coercive, continuous form $a : V \times V \to \mathbb{C}$, where $V \hookrightarrow_d H$. Assume in addition that $-A$ generates a cosine function. Then by Fattorini's theorem [[ABHN01], Section 3.16.7] $B = \mathrm{i}A^{1/2}$ generates a C_0-group. Thus, the phase space is $D(A^{1/2}) \times H$. Now it becomes apparent why the square root property is interesting in this context. The given known space is V (for example in applications, V may be a Sobolev space). One would like to know whether $V \times H$ is the phase space of the second-order Cauchy problem.

5.6.6. *Numerical range in a parabola.* If A is an operator on a Hilbert space, then A comes from a form if and only if the numerical range of A

$$W(A) = \big\{(Ax|x)_H\colon\ x \in D(A),\ \|x\| = 1\big\}$$

is contained in a sector $\Sigma_\theta + \omega$ for some $\omega \in \mathbb{R}$, $\theta \in [0, \pi/2)$ and $(-\infty, \omega) \cap \rho(A) \neq \emptyset$ (see Section 5.3.4). Now assume that $-A$ is the generator of a cosine function. Then the spectrum $\sigma(A)$ is contained in a parabola

$$P_\omega = \big\{\xi + \mathrm{i}\eta \in \mathbb{C}\colon\ \xi \leqslant \omega^2 - \eta^2/4\omega^2\big\}$$

for some $\omega \in \mathbb{R}$ [[ABHN01], Section 3.14.18]. Recall that $\sigma(A) \subset \overline{W(A)}$ by [[Kat66], Section V.3.2].

THEOREM (McIntosh [McI82], Theorems A and C). *Let a be a closed densely defined form on H with associated operator A. Assume that the numerical range $W(A)$ of A is contained in a parabola P_ω for some $\omega \in \mathbb{R}$. Then A has the square root property.*

Generators of cosine functions can be characterized by a real condition on the resolvent [[ABHN01], Section 3.15.3] which is however of little practical use. Much more interesting is the following most remarkable new criterion:

THEOREM (Crouzeix [Cro03]). *Let $a : V \times V \to \mathbb{C}$ be a closed densely defined form on a Hilbert space H such that*

$$a(u,u) \in P_\omega, \quad u \in V, \|u\|_H = 1,$$

for some $\omega \in \mathbb{R}$. Then the associated operator generates a cosine function.

Haase [Haa03a], Corollary 5.18, proved that also the converse implication holds: If A generates a cosine function on a Hilbert space, then there exist an equivalent scalar product $(\cdot|\cdot)_1$ on H, a closed form $a : V \times V \to \mathbb{C}$ on H with $V \hookrightarrow_d H$ such that $-A$ is associated with the form a on $(H, (\cdot|\cdot)_1)$ and

$$a(u,u) \in P_\omega, \quad u \in V, \|u\|_H = 1,$$

for some $\omega \in \mathbb{R}$. Thus, in view of Section 5.3.4 we can formulate the following beautiful characterization.

COROLLARY. *Let A be an operator on a Hilbert space H. The following assertions are equivalent*:

(i) *A generates a cosine function*;

(ii) *there exists an equivalent scalar product $(\cdot|\cdot)_1$ on H and $\omega \in \mathbb{R}$ such that*

$$-(Au|u)_1 \in P_\omega, \quad u \in D(A), \|u\|_H = 1,$$

and $\rho(-A) \cap P_\omega \neq \emptyset$.

6. Fourier multipliers and maximal regularity

The Fourier transform is a classical tool to treat differential equations. In order to apply it to partial differential equations, vector-valued functions have to be considered. Problems of regularity, but also of asymptotic behavior, can frequently be reformulated as the question whether a certain operator is an L^p-Fourier multiplier. Michlin's theorem gives a most convenient criterion for this. It was Weis [Wei00a] (after previous work by Clément et al. [CPSW00]) who discovered the right formulation of Michlin's theorem in the vector-valued case.

We describe the situation in the periodic case which is technically easier, and ideas become more transparent in this case. Still, the periodic case leads to the main application, namely characterization of maximal regularity for the nonhomogeneous Cauchy problem.

6.1. *Vector-valued Fourier series and periodic multipliers*

It was the study of the heat equation which lead Fourier to the introduction of one of the most fundamental concepts of Analysis. In the same spirit, vector-valued Fourier series help us to understand arbitrary abstract evolution equations.

Let X be a Banach space. For $1 \leqslant p \leqslant \infty$ denote by $L^p_{2\pi} := L^p_{2\pi}(X)$ the space of all equivalence classes of 2π-periodic measurable functions $f : \mathbb{R} \to X$ such that

$$\|f\|_{L^p_{2\pi}} := \left(\int_0^{2\pi} \|f(t)\|^p \, \mathrm{d}t \right)^{1/p} < \infty,$$

if $1 \leqslant p < \infty$, and such that

$$\|f\|_{L^\infty_{2\pi}} = \operatorname*{ess\,sup}_{t \in \mathbb{R}} \|f(t)\| < \infty,$$

where functions are identified if they coincide a.e. Then $L^p_{2\pi}$ is a Banach space and $L^p_{2\pi} \hookrightarrow L^1_{2\pi}$, $1 \leqslant p \leqslant \infty$. For $f \in L^1_{2\pi}$, denote by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kt} f(t) \, \mathrm{d}t$$

the kth *Fourier coefficient* of f, where $k \in \mathbb{Z}$. The Fourier coefficients determine the function f; i.e.,

$$\hat{f}(k) = 0 \quad \text{for all } k \in \mathbb{Z} \quad \text{if and only if} \quad f(t) = 0 \text{ a.e.} \tag{6.1}$$

DEFINITION. Let $1 \leqslant p \leqslant \infty$. *A* sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ is an $L^p_{2\pi}$*-multiplier* if, for each $f \in L^p_{2\pi}(X)$, there exists a function $g \in L^p_{2\pi}(X)$ such that

$$M_k \hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

In that case, by the closed graph theorem, the mapping $M := (f \mapsto g)$ defines a bounded linear operator from $L^p_{2\pi}(X)$ into $L^p_{2\pi}(X)$, which we call the *operator associated with* $(M_k)_{k \in \mathbb{Z}}$.

Let $e_k(t) = \mathrm{e}^{\mathrm{i}kt}$, $t \in \mathbb{R}$, $k \in \mathbb{Z}$, and for $x \in X$, denote by $e_k \otimes x$ the function $t \mapsto \mathrm{e}^{\mathrm{i}kt} x$. Linear combinations of such functions are called *trigonometric polynomials*. By Fejér's theorem [[ABHN01], Theorem 4.2.19], the space of all trigonometric polynomials is dense in $L^p_{2\pi}(X)$. Note that the associated operator M acts on trigonometric polynomials by

$$M \sum_{k=-m}^{m} e_k \otimes x_k = \sum_{k=-m}^{m} e_k \otimes M_k x_k. \tag{6.2}$$

6.1.1. *Multipliers on Hilbert spaces.* If H is a Hilbert space, then the Fourier transform $f \mapsto \hat{f}$ is an isometric isomorphism of $L^2_{2\pi}(H)$ onto $\ell^2(H)$ (the space of all sequences $(x_k)_{k \in \mathbb{Z}}$ in $\ell^2(H)$ such that $\sum_{k \in \mathbb{Z}} \|x_k\|^2 < \infty$). This follows easily from the scalar case by considering an orthonormal basis of H. As a consequence, if H is a Hilbert space, then a sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(H)$ is an $L^2_{2\pi}$-multiplier if and only if it is bounded. If $1 < p < \infty$ and $p \neq 2$, then multipliers can no longer be characterized in a satisfying way, even in the scalar case. An important subject of harmonic analysis is to find sufficient conditions. In the scalar case the following is a special case of Marcinkiewicz' multiplier theorem. We state it on Hilbert space (where it is a special case of the Theorem in Section 6.1.5).

THEOREM. *Let H be a Hilbert space and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(H)$ be a bounded sequence satisfying*

$$\sup_{k\in\mathbb{Z}} \|k(M_{k+1} - M_k)\| < \infty. \tag{$M1$}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is an $L^p_{2\pi}$-multiplier for $1 < p < \infty$.

6.1.2. *Characterization of Hilbert spaces.* Even if $p = 2$, the Theorem in Section 6.1.1 does no longer hold for Banach spaces other than Hilbert space.

THEOREM. *Let X be a Banach space and let $1 < p < \infty$. Assume that each bounded sequence $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ satisfying $(M1)$ is an $L^p_{2\pi}$-multiplier. Then X is isomorphic to Hilbert space.*

An explicit proof of this theorem is given in [ArBu03b], Proposition 1.17. It is based on a deep characterization of Hilbert spaces due to Kwapien. It was Pisier (unpublished) who had discovered that certain classical operator-valued multiplier theorems hold merely on Hilbert spaces.

6.1.3. *UMD-spaces and the Riesz-projection.* In order to obtain an operator-valued multiplier result one has to replace boundedness in operator norm in $(M1)$ by a stronger assumption (R-boundedness), and a restricted class of Banach spaces has to be considered.

DEFINITION. Let X be a Banach space. For $k \in \mathbb{Z}$, let

$$M_k = \begin{cases} I & \text{if } k \geqslant 0, \\ 0 & \text{if } k < 0. \end{cases} \tag{6.3}$$

We say that X is a *UMD-space* if the sequence $(M_k)_{k\in\mathbb{Z}}$ is an $L^p_{2\pi}(X)$-multiplier for all (equivalently one) $p \in (1, \infty)$. The associated operator R is called the *Riesz-projection*. Note that the sequence (6.3) satisfies condition $(M1)$ of Section 6.1.1.

The letters UMD stay for "*unconditional martingale differences*" and refer to an equivalent property introduced and studied by Burkholder [Bur83]. Here we will not be confronted with martingales and use the term UMD just for saying that the Riesz-projection is bounded.

EXAMPLES. (a) The space $L^q(Y)$ is a UMD-space if $1 < q < \infty$ for any σ-finite measure space (Y, Σ, μ).

(b) Every closed subspace of a UMD-space is a UMD-space.

(c) Each UMD-space is reflexive (even superreflexive).

(d) It follows from (a)–(c) that, for any nonempty open subset Ω of $\mathbb{R}^n$, the Sobolev space $W^{k,p}(\Omega)$ is a UMD-space if and only if $1 < p < \infty$, where $k \in \mathbb{N}_0$.

(e) If $\Omega \subset \mathbb{R}^n$ is a nonempty open bounded set, then $C(\overline{\Omega})$ is not a UMD-space.

6.1.4. *R-boundedness.* In order to formulate an operator-valued version of Marcinkiewicz's theorem we need a new notion of boundedness for sets of operators. Recall that a series $\sum_{k=1}^{\infty} x_k$ in a Banach space is called *unconditionally convergent* if the series $\sum_{k=1}^{\infty} \varepsilon_k x_k$ converges for any choice of signs $(\varepsilon_k)_{k\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$. This implies convergences of $\sum_{k=1}^{\infty} \alpha_k x_k$ for each $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \ell^{\infty}$. But in general, this does not imply that $\sum_{k=1}^{\infty} \|x_k\| < \infty$, unless X is finite dimensional. We introduce the means over all signs

$$\big\|(x_1, \dots, x_n)\big\|_R := \frac{1}{2^n} \sum_{\varepsilon\in\{-1,1\}^n} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|. \tag{6.4}$$

DEFINITION. Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{L}(X, Y)$ is called *R-bounded* if there exists a constant $c \geqslant 0$ such that

$$\big\|(T_1 x_1, \dots, T_n x_n)\big\|_R \leqslant c \big\|(x_1, \dots, x_n)\big\|_R \tag{6.5}$$

for all $T_1, \dots, T_n \in \mathcal{T}$, $x_1, \dots, x_n \in X$, $n \in \mathbb{N}$. The least constant c such that (6.5) is satisfied is called the *R-bound* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$.

The notion of R-boundedness was introduced by Berkson and Gillespie [BG94], where the "R" stands for 'Riesz'. Since the mean

$$\big\|(x_1, \dots, x_n)\big\|_R$$

can be expressed in terms of Rademacher functions some pronounce it "Rademacher bounded". Finally, the Rademacher functions may be replaced by other independent random variables which leads some to use "randomized boundedness". In any case, it is some sort of unconditional boundedness.

R-boundedness clearly implies boundedness. But if $X = Y$, the notion of R-boundedness is strictly stronger than boundedness unless the underlying space is a Hilbert space [ArBu02], Proposition 1.17.

6.1.5. *The operator-valued Marcinkiewicz theorem.* On an arbitrary Banach space, in general it is a difficult task to verify R-boundedness (we will come back to this point later). Nevertheless, for multipliers it is the right notion as the following two results show. First of all, we state that it is a necessary condition.

PROPOSITION ([ArBu02], [CP01]). *Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ be an $L^p_{2\pi}$-multiplier for some $1 < p < \infty$. Then the set $\{M_k\colon k \in \mathbb{Z}\}$ is R-bounded.*

If in Marcinkiewicz's theorem we replace boundedness by R-boundedness, the theorem remains true on UMD-spaces (see [ArBu02], Theorem 1.3).

THEOREM (The operator-valued Marcinkiewicz theorem). *Let X be a UMD-space. Let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ be R-bounded. Assume that*

$$\{k(M_{k+1} - M_k)\colon\ k \in \mathbb{Z}\}$$

is R-bounded. Then $(M_k)_{k\in\mathbb{Z}}$ is an $L^p_{2\pi}$-multiplier whenever $1 < p < \infty$.

We remark that a set of scalar operators (i.e., operators of the form $c \cdot I$, where $c \in \mathbb{C}$) is bounded if and only if it is R-bounded. In that case, the Theorem had been proved by Clément, de Pagter, Sukochev and Witvliet [CPSW00]. It was Weis [Wei00a] who proved the first operator-valued multiplier theorem on UMD-spaces, namely the operator-valued version of Michlin's theorem on the real line.

6.1.6. *The variational Marcinkiewicz' condition.* Finally we state the original, more general version of Marcinkiewicz's theorem. It holds in Hilbert spaces as was shown by Schwartz [Schw61].

THEOREM. *Let X be a Hilbert space and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$ be a bounded sequence satisfying*

$$\sup_{n\in\mathbb{N}} \sum_{2^{n-1}\leqslant|k|<2^n} \|M_{k+1} - M_k\| < \infty. \tag{MV}$$

Then $(M_k)_{k\in\mathbb{Z}}$ is an $L^p_{2\pi}$-multiplier for $1 < p < \infty$.

We call (MV) the *variational Marcinkiewicz condition*. There is a version on UMD-spaces due to Strkalj and Weis [SW00]. But the "R-version" of being of uniform bounded variation on dyadic intervals is more complicated to formulate.

Finally we mention a remarkable difference between the variational Marcinkiewicz condition and the stronger condition $(M1)$ of Section 6.1.1. In the scalar case, if $(M1)$ is satisfied the associated operator is of weak type $(1, 1)$. This does not hold, if merely the more general condition (MV) is satisfied (see Fournier's example [[EG77], Section 7.5]).

6.2. *Maximal regularity via periodic multipliers*

Let A be a closed operator and let $\tau > 0$. For $f \in L^1((0,\tau); X)$ and $x \in X$ we consider the problem

$$P_x(f)\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,\tau],\\ u(0) = x. \end{cases}$$

6.2.1. *Mild and strong L^p-solutions.* Recall the notion of mild solution of $P_x(f)$ given in Section 1.5. We now define a stronger notion of solution. Let $1 \leqslant p < \infty$. By $W^{1,p}((0,\tau);X)$ we denote the space of all $u \in C([0,\tau];X)$ such that there exists $u' \in L^p((0,\tau);X)$ such that

$$u(t) = u(0) + \int_0^t u'(s)\,\mathrm{d}s, \quad t \in [0,\tau]. \tag{6.6}$$

Then $u'(t)$ is the derivative of u a.e. Let $f \in L^p((0,\tau);X)$. A *strong L^p-solution u* of $P_x(f)$ is a function $u \in W^{1,p}((0,\tau);X) \cap L^p((0,\tau);D(A))$ such that $u(0) = x$ and $u'(t) = Au(t) + f(t)$ a.e. on $(0,\tau)$. Each strong solution is also a mild solution. Conversely, since A is closed, a mild solution u of $P_x(f)$ is a strong L^p-solution if and only if $u \in W^{1,p}((0,\tau);X)$. Now assume that A is the generator of a C_0-semigroup T. Then, by Section 1.5, $u = T(\cdot)x + T * f$ is the unique mild solution of $P_x(f)$. Thus, we are lead to investigate under which conditions $T(\cdot)x \in W^{1,p}((0,\tau);X)$. We start to consider the case $f = 0$.

6.2.2. *An interpolation space and strong L^p-solutions of the homogeneous problem.* Let X, Y be Banach spaces such that $Y \hookrightarrow X$. For $1 < p < \infty$ one may define the real interpolation space

$$(X,Y)_{1/p^*,p} = \bigl\{u(0)\colon\ u \in W^{1,p}\bigl((0,\tau);X\bigr) \cap L^p\bigl((0,\tau);Y\bigr)\bigr\},$$

where $1/p^* + 1/p = 1$. In particular, if A generates a holomorphic C_0-semigroup T on X, then

$$\bigl(X, D(A)\bigr)_{1/p^*,p} = \bigl\{x \in X\colon\ AT(\cdot)x \in L^p\bigl((0,\tau);X\bigr)\bigr\}. \tag{6.7}$$

Thus, the mild solution $u = T(\cdot)x$ of $P_x(0)$ is in $W^{1,p}((0,\tau);X))$ if and only if $x \in (D(X),A)_{1/p^*,p}$.

6.2.3. *Strong solutions of periodic problems.* Let A be a closed operator on a Banach space X. For $f \in L^p((0,2\pi);X)$, we consider the periodic problem

$$P_{\mathrm{per}}(f) \begin{cases} u'(t) = Au(t) + f(t), & t \in [0,2\pi], \\ u(0) = u(2\pi). \end{cases}$$

A *strong L^p-solution* of $P_{\mathrm{per}}(f)$ is a function $u \in W^{1,p}((0,\tau);X) \cap L^p((0,\tau);D(A))$ such that (P_{per}) is satisfied t-a.e. We may identify $L^p((0,2\pi);X)$ and $L^p_{2\pi}(\mathbb{R},X)$. Then it is not difficult to see the following.

PROPOSITION. *Let $1 < p < \infty$. The following assertions are equivalent*:

(i) *For each $f \in L^p((0,2\pi);X)$, there exists a unique strong L^p-solution of $P_{\mathrm{per}}(f)$*;
(ii) $\mathrm{i}\mathbb{Z} \subset \rho(A)$ *and $(kR(\mathrm{i}k,A))_{k\in\mathbb{Z}}$ is an L^p-multiplier.*

Let us consider a Hilbert space H. Then we obtain the following theorem.

THEOREM. *Let $1 < p < \infty$. Let A be a closed operator on a Hilbert space H. The following are equivalent*:

(i) $\mathrm{i}\mathbb{Z} \subset \rho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|kR(\mathrm{i}k, A)\| < \infty$;

(ii) *for all $f \in L^p$, there exists a unique strong L^p-solution of $P_{\mathrm{per}}(f)$.*

For the implication (i) $\Rightarrow$ (ii) one considers $M_k = kR(\mathrm{i}k, A)$. The resolvent identity implies that $(k(M_{k+1} - M_k))_{k\in\mathbb{Z}}$ is bounded. Thus assertion (ii) follows from Section 6.1.1. Similarly, one obtains from Section 6.1.5 the following characterization on UMD-spaces.

THEOREM [ArBu02]. *Let A be a closed operator on a UMD-space X and let $1 < p < \infty$. The following are equivalent*:

(i) $\mathrm{i}\mathbb{Z} \subset \rho(A)$ *and* $\{kR(\mathrm{i}k, A)\colon k \in \mathbb{Z}\}$ *is R-bounded*;

(ii) *for all $f \in L^p((0, 2\pi); X)$, there exists a unique strong L^p-solution of* (P_{per}).

Two things are remarkable:

1. Condition (ii), i.e., well-posedness of the periodic problem in the sense of strong L^p-solutions, is p-independent for $1 < p < \infty$.
2. Whereas no characterization of L^p-multipliers seems possible in general (if $1 < p < \infty$, $p \neq 2$), in the context of resolvents, it is.

Finally, we remark that A satisfies the equivalent conditions of the theorem if and only if $-A$ does so. In particular, A need not be the generator of a C_0-semigroup.

6.2.4. *Maximal regularity on Hilbert space.* Let X be a Banach space. Assume that A generates a holomorphic semigroup T. Then, for $f \in L^1((0, \tau); X)$, $x \in X$,

$$u = T(\cdot)x + T * f$$

is the unique mild solution of $P_x(f)$. We want to investigate when the solution is strong. If $f \equiv 0$, this is done in Section 6.2.2. Thus we may consider $x = 0$, i.e., we want to investigate $P_0(f)$.

THEOREM. *Let A be the generator of a holomorphic C_0-semigroup T on a Hilbert space H. Then*

$$T * f \in W^{1,p}\big((0, \tau); X\big) \quad \textit{for all } f \in L^p\big((0, \tau); X\big), \quad 1 < p < \infty. \tag{6.8}$$

PROOF. Let $\tau = 2\pi$. Replacing A by $A - \omega$ we may assume that $\lambda \in \rho(A)$ whenever $\operatorname{Re}\lambda \geqslant 0$ and $\|\lambda R(\lambda, A)\| \leqslant M$. Let $f \in L^p((0, \tau); H)$. Then by the first theorem of Section 6.2.3 there exists a unique strong L^p-solution v of $P_{\mathrm{per}}(f)$. Then $v(0) \in (H, D(A))_{1/p^*,p}$ by Section 6.2.2. Let $u(t) = v - T(t)v(0)$. Then u is a strong L^p-solution of $P_0(f)$. $\square$

Property (6.8) is frequently called *maximal regularity*, or (*MR*) for short. Thus, on Hilbert space, every generator of a holomorphic C_0-semigroup does enjoy this property (*MR*). It is also known that (*MR*) implies holomorphy of the semigroup (see Section 6.2.6).

6.2.5. *Characterization of Hilbert space.* For a long time it has been an open problem whether the preceding theorem holds more generally on UMD-spaces. However, the answer is negative.

THEOREM (Kalton and Lancien [KL00]). *Let X be a Banach space possessing an unconditional basis. Let $1 < p < \infty$. If X is not isomorphic to a Hilbert space, then there exist a holomorphic C_0-semigroup T and $f \in L^p((0,\tau); X)$ such that $T * f \notin W^{1,p}((0,\tau); X)$.*

The semigroup T is not constructed directly, the existence of such semigroup T is proved implicitly using several deep results of geometry of Banach spaces. However one can say that the generator A of T is a block-diagonal operator with respect to some conditional basis in X.

It is an open problem whether the Theorem does also hold in Banach spaces with conditional bases. Also, so far no explicit counterexample, and in particular, no model is known for which (*MR*) fails.

6.2.6. *Maximal regularity on UMD-spaces.* On UMD-spaces, with the same proof as in Section 6.2.4 based on the periodic operator-valued multiplier theorem of Section 6.1.5 one obtains the following characterization.

THEOREM (Weis). *Assume that X is a UMD-space. Let A be a closed operator, $\tau > 0$, $1 < p < \infty$. The following are equivalent*:

(i) *for all $f \in L^p((0,\tau); X)$, there exists a unique $u \in W^{1,p}((0,\tau); X) \cap L^p((0,\tau); D(A))$ such that*

$$\begin{cases} u'(t) = Au(t) + f(t) \quad a.e., \\ u(0) = 0; \end{cases}$$

(ii) *there exist $\omega \in \mathbb{R}$ such that $\lambda \in \rho(A)$ whenever $\operatorname{Re}\lambda > \omega$ and the set $\{\lambda R(\lambda, A): \operatorname{Re}\lambda > \omega\}$ is R-bounded.*

It was Weis [Wei00a] who proved that (ii) implies (i), Clément and Prüss [CP01] showed necessity of the condition. Weis gave a different proof (than we indicated here), based on his operator-valued Michlin's theorem on the real line.

DEFINITION. We say that an operator A satisfies *condition* (*MR*) (for maximal regularity) if condition (i) is satisfied.

It can be seen from the Theorem that property (*MR*) is independent of $p \in (1,\infty)$ and of $\tau > 0$. Moreover, (*MR*) implies that A generates a holomorphic semigroup T. Thus, the solution u of (i) is given by $u = T * f$. If A satisfies (*MR*) then the existence of strong L^p-solutions on the entire half-line $(0, \infty)$ is merely a question of the asymptotic behavior of T as the following result shows. See Section 1.3 for the definition of $\omega(T)$.

THEOREM. *Let A be the generator of a holomorphic C_0-semigroup T on a Banach space X. The following assertions are equivalent*:

(i) *A satisfies (MR) and $\omega(T) < 0$;*

(ii) *$T * f \in W^{1,p}((0, \infty), X) \cap L^p((0, \infty); D(A))$ for all $f \in L^p((0, \infty); X)$.*

PROOF. (ii) $\Rightarrow$ (i) follows from Datko's theorem [[ABHN01], p. 336]. For (ii) $\Rightarrow$ (i), see [Dor00], Theorem 5.2. □

Finally, we mention that the restriction to $p \in (1, \infty)$ is essential. If A generates a C_0-semigroup T on a reflexive space X such that $T * f \in W^{1,1}((0, 1); X)$ for all $f \in L^1((0, 1); X)$, then A is bounded (see [Gue95]).

6.2.7. *Perturbation of* (*MR*). In general it is not at all easy to prove R-boundedness. However, it is not difficult to see that condition (ii) of the preceding theorem is stable under small perturbation (see [KuWe01], [ArBu02]).

THEOREM. *Let A be an operator satisfying (MR) on a UMD-space X. Then there exists $\varepsilon > 0$ such that, for each linear $B : D(A) \to X$,*

$$\|Bx\| \leqslant \varepsilon\|Ax\| + b\|x\|, \quad x \in D(A),$$

for some $b \geqslant 0$, also $A + B$ satisfies (MR).

6.2.8. *BIP and maximal regularity.* An important criterion for maximal regularity is property *BIP*. The following theorem is due to Dore and Venni [DoVe87].

THEOREM. *Let $A \in BIP$ on a UMD-space X such that $\varphi_{bip}(A) < \pi/2$. Then $-A$ has property (MR).*

We sketch two different proofs.

FIRST PROOF. One can show directly that *BIP* implies that for $\varphi_{bip} < \varphi < \pi/2$ one has $\sigma(A) \subset \Sigma_\varphi$ and that the set $\{\lambda R(\lambda, A)\colon \lambda \in \mathbb{C} \setminus \Sigma_\varphi\}$ is R-bounded, [DHP01], Theorem 4.5. Thus the first theorem of Section 6.2.6 can be applied. □

SECOND PROOF [DOVE87]. The space $Y = L^p((0, \tau); X)$ has the UMD-property and the operator $\mathcal{A}$ on $L^p((0, \tau); X)$ given by $(\mathcal{A}u)(t) = Au(t)$ with domain $D(\mathcal{A}) = L^p((0, \tau); D(A))$ obviously inherits *BIP* from A with angle $\varphi_{bip}(\mathcal{A}) \leqslant \varphi_{bip}(A)$. On the other hand, the operator $\mathcal{B}$ on $L^p((0, \tau); X)$ given by $D(\mathcal{B}) = W_0^{1,p}((0, \tau); X)$, $\mathcal{B}u = u'$, has *BIP* with angle $\varphi_{bip}(\mathcal{B}) = \pi/2$ (see [HP97] or [DoVe87]). Assuming that $0 \in \rho(A)$, it follows from the Dore–Venni theorem (Section 4.4.8) that $\mathcal{A} + \mathcal{B}$ is closed. The Theorem of Section 4.2 shows that $\mathcal{A} + \mathcal{B}$ is invertible. This is exactly property (*MR*). □

6.2.9. *Maximal regularity for positive contraction semigroups on $L^p(\Omega)$.* The characterization of maximal regularity by R-sectoriality 6.2.11 (and Section 4.7.5) has the following interesting consequence [Wei00b].

THEOREM (Lamberton and Weis). *Let $1 < p < \infty$. Let $-A$ be the generator of a positive contractive C_0-semigroup T on $L^p(\Omega)$. If T is holomorphic, then A has (MR).*

This result had been first proved by Lamberton [Lam87] in the case where T is also contractive for the L^∞- and L^1-norm.

6.2.10. *Maximal regularity and quasi-linear problems.* The property of maximal regularity is most important in order to solve quasi-linear problems. Here we give a result on local existence.

Let X be a UMD-space and let D be a Banach space such that $D \hookrightarrow_d X$. Let $1 < p < \infty$ and $Y = (X, D)_{1/p^*,p}$, where $1/p + 1/p^* = 1$. Let $A : Y \to \mathcal{L}(D, X)$ be Lipschitz continuous on each bounded subset of Y. Let $u_0 \in Y$ and assume that the operator $-A(u_0)$ on X with domain $D(A(u_0)) = D$ has property (*MR*). Then the following result holds.

THEOREM (Clément and Li [CL94]). *For each $f \in L^p_{\mathrm{loc}}((0,\infty); X)$, there exist $\tau > 0$ and a solution $u \in W^{1,p}((0,\tau); X) \cap L^p((0,\tau); D)$ of*

$$\begin{cases} u'(t) + A\big(u(t)\big)u(t) = f(t), & t \in (0,\tau), \\ u(0) = u_0. \end{cases} \tag{P}$$

Observe that $W^{1,p}((0,\tau); X) \cap L^p((0,\tau); D) \subset C([0,\tau]; Y)$ so that the condition on the initial value of u makes sense.

SKETCH OF THE PROOF OF THE THEOREM. We show how maximal regularity is used to give a fixed point argument. Rewrite the problem as

$$u'(t) + A(u_0)u(t) = f(t) + \big(A(u_0) - A(u)\big)u.$$

Let $MR(\tau) := L^p((0,\tau); D) \cap W^{1,p}((0,\tau); X)$. Then $S_1 v = f + (A(u_0) - A(v))v$ defines a mapping $S_1 : MR(\tau) \to L^p((0,\tau); X)$. Consider $S_2 : L^p((0,\tau); X) \to MR(\tau)$ defined by $S_2 g = u$, where $u \in MR(\tau)$ is the solution of $\dot{u} + A(u_0)u = g$, $u(0) = u_0$. Then $S = S_2 \circ S_1$ is a mapping from $MR(\tau)$ into $MR(\tau)$. One can show that S is a strict contraction if $\tau > 0$ is small enough. Thus the Banach fixed point theorem shows that S has a fixed point which is a solution of (P). $\square$

6.2.11. *R-sectorial operators.* As we mentioned before, on Hilbert space (and only on Hilbert space), boundedness and R-boundedness are the same. On the other hand, it turns out that many results known for Hilbert spaces can be carried over to L^p-spaces, $1 < p < \infty$, if boundedness is replaced by R-boundedness. The operator-valued Marcinkiewicz of Section 6.1.5 and the Michlin multiplier theorem [Wei00a] are of this kind. We mention some further results. We say that an operator A is *R-sectorial* if there exists $0 < \varphi < \pi$ such that $\sigma(A) \subset \Sigma_\varphi$ and such that the set $\{\lambda R(\lambda, A)\colon \lambda \in \mathbb{C} \setminus \Sigma_\varphi\}$ is R-bounded. In that case let $\varphi_{R\sec}(A)$ be the infimum of all angles with these properties.

On a UMD-space an operator A has property (MR) if and only if $A + \omega$ is R-sectorial for some $\omega \in \mathbb{R}$.

In analogy to Section 5.1 one has the following results. Assume that $X = L^p(\Omega)$, $1 < p < \infty$.

If A has a bounded H^∞-calculus, then A is R-sectorial and $\varphi_{H^\infty}(A) = \varphi_{R\sec}(A)$. Moreover, if B is a second R-sectorial operator commuting with A such that $\varphi_{H^\infty}(A) + \varphi_{R\sec}(B) < \pi$, then $A + B$ is closed.

For this and many other interesting properties we refer to [Wei00a], [Wei00b] and [KW01].

7. Gaussian estimates and ultracontractivity

Frequently it is easy to define a semigroup on $L^2(\Omega)$ with the help of form methods for example. In this section we discuss under which conditions such a semigroup can be extended to $L^p(\Omega)$. An important item is to investigate whether semigroup properties (as holomorphy, maximal regularity, H^∞-calculus) extrapolate to L^p. Integral representations by Gaussian kernels play a big role.

7.1. *The Beurling–Deny criteria*

For simplicity we consider real spaces in this section. Let H be a real Hilbert space. Let $V \hookrightarrow_d H$ be another Hilbert space and $a : V \times V \to \mathbb{R}$ a continuous bilinear form which is *H-elliptic*, i.e.,

$$a(u, u) + \omega \|u\|_H^2 \geqslant \alpha \|u\|_V^2, \quad u \in V,$$

for some $\omega \in \mathbb{R}$, $\alpha > 0$. Recall that this is the same as saying that a is a closed form with form domain $D(a) = V$. Denote by A the operator associated with a (see Section 5.3.1). Then $-A$ generates a C_0-semigroup T on H which has a holomorphic extension to $H_{\mathbb{C}}$, the complexification of H. In the following we assume that $H = L^2(\Omega)$ for some measure space (Ω, Σ, μ) so that $H_{\mathbb{C}} = L^2(\Omega, \mathbb{C})$. We let $V_+ = V \cap L^2(\Omega)_+$.

7.1.1. *Theorem* (*First Beurling–Deny criterion*). *The semigroup T is positive if and only if $u \in V$ implies $u^+ \in V$ and*

$$a\big(u^+, u^-\big) \geqslant 0.$$

Here, for $u \in L^2(\Omega)$, we let $u^+(x) = \max\{u(x), 0\}$, $u^- = (-u)^+$. The next condition characterizes L^∞-contractivity. An operator S on $L^2(\Omega)$ is called *submarkovian* if $S \geqslant 0$ and $\|Sf\|_\infty \leqslant \|f\|_\infty$ for all $f \in L^2(\Omega) \cap L^\infty(\Omega)$. A semigroup T on $L^2(\Omega)$ is called *submarkovian* if each $T(t)$ is submarkovian.

7.1.2. *Theorem* (*Second Beurling–Deny criterion*). *Assume that the semigroup T is positive. Then T is submarkovian if and only if $u \wedge 1 \in V$ for each $u \in V_+$ and*

$$a\bigl(u \wedge 1, (u-1)^+\bigr) \geqslant 0.$$

Here we denote by 1 the function identically equal to 1. Note that $u \wedge 1 + (u-1)^+ = u$. We refer to [Ouh92], [[Ouh04]] for a proof of these criteria. Note that T^* is associated with $a^* : V \times V \to \mathbb{R}$ given by $a^*(u, v) = a(v, u)$. If T and T^* are submarkovian, then

$$\bigl\|T(t)f\bigr\|_{L^p} \leqslant \|f\|_{L^p} \tag{7.1}$$

for all $f \in L^p(\Omega) \cap L^2(\Omega)$, $t \geqslant 0$ and $p = 1, \infty$ at first; but by interpolation also for all $p \in (1, \infty)$. The form a is called a *Dirichlet form* if the two criteria of Beurling–Deny are satisfied. If in addition the form a is *symmetric* (i.e., $a = a^*$), then the semigroup T consists of self-adjoint operators satisfying (7.1). Each C_0-semigroup of symmetric submarkovian operators is associated with a symmetric Dirichlet form. The monographs [[FOT94]], [[MR86]], [[BH91]] are devoted to the theory of Dirichlet forms and their important and interesting interplay with stochastic processes. Here we are merely interested in analytical properties.

7.1.3. *Domination* [Ouh96]. *Let a and b be two closed forms on $L^2(\Omega)$ with associated semigroups S and T, respectively. Assume that S and T are positive. The following assertions are equivalent*:

(i) $S(t) \leqslant T(t)$, $t \geqslant 0$;

(ii) (a) $D(a) \subset D(b)$ *and for* $u \in D(b)$, *if* $0 \leqslant u \leqslant v \in D(a)$, *then* $u \in D(a)$ (ideal property), *and*

(b) $a(u, v) \leqslant b(u, v)$ *for all* $0 \leqslant u, v \in D(a)$ (monotonicity).

We mention two prototype examples.

7.1.4. *Examples.* Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. The Dirichlet Laplacian Δ_Ω^D and the Neumann Laplacian Δ_Ω^N generate symmetric submarkovian semigroups satisfying $0 \leqslant \mathrm{e}^{t\Delta_\Omega^D} \leqslant \mathrm{e}^{t\Delta_\Omega^N}$, $t \geqslant 0$.

7.2. *Extrapolating semigroups*

Now we consider complex spaces. Let (Ω, Σ, μ) be a measure space and T a C_0-semigroup on $L^2(\Omega)$. We assume throughout this section that

$$\bigl\|T(t)\bigr\|_{\mathcal{L}(L^p)} \leqslant M, \quad 0 < t \leqslant 1, \tag{7.2}$$

for $p = 1, \infty$ and hence for all $p \in [1, \infty]$ by interpolation. Here, given an operator $S \in \mathcal{L}(L^2(\Omega))$ and $1 \leqslant p, q \leqslant \infty$, we let

$$\|S\|_{\mathcal{L}(L^p, L^q)} = \sup\bigl\{\|Sf\|_{L^q} \colon f \in L^p \cap L^2, \|f\|_{L^p} \leqslant 1\bigr\}. \tag{7.3}$$

Note that T and T^* are submarkovian if and only if $M = 1$. It follows from (7.2) that for suitable constants M_1, ω_1,

$$\bigl\|T(t)\bigr\|_{\mathcal{L}(L^p)} \leqslant M_1 \mathrm{e}^{\omega_1 t}, \quad t \geqslant 0, \tag{7.4}$$

for all $p \in [1, \infty]$. From this we deduce that, for $1 \leqslant p < \infty$, there exist bounded operators $T_p(t) \in \mathcal{L}(L^p(\Omega))$ which are *consistent*, i.e.,

$$T_p(t)f = T_q(t)f, \quad f \in L^p \cap L^q, t > 0, \tag{7.5}$$

and such that $T_2(t) = T(t)$, $t > 0$. It is clear that $T_p(t+s) = T_p(t)T_p(s)$ for $t, s \geqslant 0$, $1 \leqslant p < \infty$. If $1 < p < 2$, we deduce from the interpolation inequality

$$\bigl\|T_p(t)f - f\bigr\|_{L^p} \leqslant \bigl\|T_1(t)f - f\bigr\|_{L^1}^{\theta} \bigl\|T_2(t)f - f\bigr\|_{L^2}^{1-\theta},$$

where $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$, so that $\lim_{t\downarrow 0} T_p(t)f = f$ in L^p for $f \in L^2 \cap L^1$. Thus T_p is a C_0-semigroup for $1 < p \leqslant 2$, and also for $2 < p < \infty$ by a similar argument. It is clear that $T_1 : (0, \infty) \to \mathcal{L}(L^1(\Omega))$ is strongly measurable, and hence strongly continuous [[Dav80], p. 18]. However, it seems to be unknown whether T_1 is a C_0-semigroup, in general. It is if one of the following conditions is satisfied.

7.2.1. *Conditions for T_1 being a C_0-semigroup.* *Assume that one of the following conditions is satisfied*:

(a) $M = 1$;
(b) *Ω has finite measure*;
(c) $T(t) \geqslant 0, t > 0$;
(d) *there exist an open set $\Omega' \supset \Omega$ and a C_0-semigroup S on $L^1(\Omega')$ such that*

$$\bigl|T(t)f\bigr| \leqslant S(t)|f| \quad \text{on } \Omega \text{ for all } f \in L^1(\Omega). \tag{7.6}$$

Then T_1 is a C_0-semigroup.

See [Voi92] for the cases (a)–(c) and [AtE97] for (d).

We assume throughout that one of the four conditions (a)–(d) is satisfied. Then T_1 is a C_0-semigroup. Observe that also T^* satisfies (7.2) and one of these conditions. This allows us to define the extension of T to $L^\infty(\Omega)$. In fact, $(T^*)_1$ is a C_0-semigroup. Now define $T_\infty(t) = (T^*)_1(t)^*$. Then T_∞ is a weak*-continuous semigroup whose generator we define by $A_\infty = (A_1^*)^*$. The consistency property (7.5) remains valid for $p = 1$ and $1 \leqslant q \leqslant \infty$. If Ω has finite measure, then of course, $T_\infty(t) = T(t)|_{L^\infty(\Omega)}$.

We call T_p the *extrapolation semigroup* of T on L^p, $1 \leqslant p \leqslant \infty$.

Now we want to investigate how properties of the semigroup $T = T_2$ are inherited by the extrapolation semigroups T_p. We denote by A_p the generator of T_p. Note that $A_\infty = (A_1^*)^*$.

7.2.2. *The Heritage list*

Property of T_2	Inherited by
(a) bounded generator	T_p for $1 < p < \infty$
(b) holomorphy	T_p for $1 < p < \infty$, not by T_1
(c) norm continuous on (t_0, ∞)	T_p for $1 < p < \infty$, not by T_1
(d) A_2 has compact resolvent	A_p for $1 < p < \infty$, not by A_1
(e) spectrum	$\sigma(A_p) \neq \sigma(A_q)$, $p \neq q$ in general, but $\sigma(A_2) = \sigma(A_p)$ for $1 < p < \infty$ if A_2 has compact resolvent
(f) positivity	T_p for $1 \leqslant p \leqslant \infty$
(g) positivity and irreducibility	T_p for $1 \leqslant p < \infty$
(h) (*MR*) for A_2	unknown
(i) H^∞-calculus for A_2	unknown
(j) *BIP*	unknown

7.2.3. *Proofs and comments.* We comment on the diverse properties. The first three (a)–(c) concern regularity of the semigroup.

(a) Let $1 < p < 2$, $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$. Then by the Riesz–Thorin theorem $\|T_p(t) - I\| \leqslant \|T_1(t) - I\|^\theta \|T_2(t) - I\|^{1-\theta} \to 0$ as $t \downarrow 0$. This implies that A_p is bounded.

(b) This is a consequence of Stein's interpolation theorem (cf. [[Dav90], Theorem 1.4.2]). But a similar proof as for (a) can be given for contraction semigroups using the following result (see [[Paz83], p. 68]).

THEOREM (Kato–Neuberger–Pazy). *Let $1 < p < \infty$ and let T be a C_0-semigroup on $L^p(\Omega)$ such that $\|T(t)\| \leqslant 1$, $t \geqslant 0$. Then T is holomorphic if and only if $\overline{\lim}_{t\downarrow 0}\|T(t) - I\| < 2$.*

Now the argument of (a) also gives a proof of (b).

(c) The argument of (a) also gives a proof of (c).

(d) This is [[Dav90], Theorem 1.6.1 and Corollary 1.6.2]. Properties (f) and (g) are trivial. Here a semigroup is called *positive*, if it leaves the real space invariant and its restriction to the real space is positive.

(i) Example 4.5.3 provides a counterexample where the semigroups extrapolate to L^p merely for $1 < p < \infty$, the operator A_2 has H^∞-calculus, but A_p has not H^∞-calculus for $p \neq 2$. We do not know such an example where the semigroups extrapolate to L^p for $1 \leqslant p \leqslant \infty$.

Next we give some interesting examples showing, in particular, that the positive results in (b), (c) or (d) cannot be extended to $p = 1$.

7.2.4. *The harmonic oscillator* [[Dav90], Section 4.3]. Consider the operator given by

$$A_p u = \frac{1}{2}\left(-u'' + x^2 u - u\right)$$

on $L^p(\mathbb{R}, e^{-x^2}\, dx)$ with maximal domain. Then A_p generates a C_0-semigroup T_p on $L^p(\mathbb{R}, e^{-x^2}\, dx)$, $1 \leqslant p < \infty$. This family is consistent. Moreover, T_p is holomorphic for $1 < p < \infty$, but T_1 is not norm continuous on $[t_0, \infty)$ for any $t_0 > 0$. A_p has a compact resolvent for $1 < p < \infty$. One has $\sigma(A_p) = \mathbb{N}_0$ for $1 < p < \infty$ but $\sigma(A_1) = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \geqslant 0\}$. Thus A_1 has not a compact resolvent.

7.2.5. *Black–Scholes equation.* For $1 \leqslant p < \infty$, let the operator B_p on $L^p(0, \infty)$ be given by

$$(B_p)(x) = x^2 u''(x) + 2xu'(x),$$
$$D(B_p) = \left\{u \in L^p(0,\infty)\colon\ xu' \in L^p(0,\infty), x^2 u'' \in L^p(0,\infty)\right\}.$$

Then B_2 is associated with the symmetric Dirichlet form a given by

$$a(u,v) = \int_0^\infty u'v'x^2\, dx,$$
$$D(a) = \left\{u \in L^2(0,\infty)\colon\ xu' \in L^2(0,\infty)\right\}$$

and thus generates a symmetric Markovian semigroup T_2. The extrapolation semigroup T_p has B_p as generator. The spectrum of B_p is the parabola $\sigma(B_p) = \{(1/p - 1/2 + is)^2 - 1/4\colon\ s \in \mathbb{R}\}$, $1 \leqslant p < \infty$. In particular,

$$\sigma(B_p) \cap \sigma(B_q) = \emptyset \quad \text{if } 1 \leqslant p < q \leqslant 2.$$

Note that T_p governs the Black–Scholes partial differential equation

$$u_t = x^2 u_{xx} + 2xu_x.$$

Reference: [Are94], Section 3, Example 3.

7.2.6. *Symmetric submarkovian semigroups, optimal angles and the Neumann Laplacian on horn domains.* Let A_2 be the generator of a symmetric submarkovian semigroup T_2 on $L^2(\Omega)$ and denote by T_p the C_0-semigroup on $L^p(\Omega)$ with generator A_p extrapolating T_2, $1 \leqslant p < \infty$. Then Liskevich and Perelmuter [LP95] showed that the angle obtained by the Stein interpolation theorem can be improved. In fact, T_p has a holomorphic, contractive extension to the sector Σ_{θ_p}, where $0 \leqslant \theta_p < \frac{\pi}{2}$ such that $\cos\theta_p = |1 - 2/p|$. In particular, the spectrum of A_p is contained in the sector

$$S_p := \left\{re^{i\theta}\colon\ r \geqslant 0, \frac{\pi}{2} + \theta_p \leqslant |\theta| \leqslant 2\pi\right\}, \quad 1 \leqslant p \leqslant \infty.$$

It was Voigt [Voi96] who first showed that this sector is optimal. In fact, modifying Example 7.2.5 he constructed a degenerate elliptic operator A_p on $L^p(0,\infty)^2$ such that the spectrum of A_p is equal to S_p.

Another interesting example in this context is the symmetric Ornstein–Uhlenbeck operator A_p on $L^p(\mathbb{R}^n,\mu)$, where μ is the invariant measure. It generates a symmetric submarkovian semigroup T_p on $L^p(\mathbb{R}^n,\mu)$. Metafune, Pallara and Priola [MPP02] computed explicitly the spectrum and showed that it is independent of $p \in [1,\infty)$. In particular, the spectrum of A_p is real for all $p \in [1,\infty)$. However, Chill, Fašangová, Metafune and Pallara [CFMP03] showed that Σ_{θ_p} is the sector of holomorphy of T_p, i.e., the worst case is realized for the symmetric Ornstein–Uhlenstein semigroup.

Kunstmann [Kun02] constructed an open unbounded domain Ω in $\mathbb{R}^n$ (which is composed by infinitely many "horn domains") such that the Neumann Laplacian has bad interpolation properties. In particular, for a domain Ω composed by infinitely many horn domains the spectrum of the Neumann Laplacian $\Delta^N_{\Omega,p}$ in $L^p(\Omega)$ depends on p. Moreover, the semigroup $(\mathrm{e}^{t\Delta^N_{\Omega,1}})_{t\geqslant 0}$ is not holomorphic in $L^1(\Omega)$. In fact, it is not even eventually norm continuous since the spectrum of $\Delta^N_{\Omega,1}$ is the left-half plane.

Concerning the functional calculus, one can say the following: given a symmetric submarkovian semigroup T_2 with extension semigroups T_p on $L^p(\Omega)$, by Section 4.7.5 the negative generator A_p of T_p has a bounded H^∞-calculus and

$$\varphi_{R\sec}(A_p) = \varphi_{H^\infty}(A_p), \quad 1 < p < \infty.$$

Cowling [Cow83] showed that $\varphi_{H^\infty}(A_p) \leqslant \varphi_s(A_p)$, where $\varphi_s(A_p) = \pi/2 - \pi|1/2 - 1/p|$ is the angle obtained by the Stein interpolation theorem. However Kunstmann and Strkalj [KS03] showed that $\varphi_{H^\infty}(A_p) < \varphi_s(A_p)$ for $1 < p < \infty$, $p \neq 2$. Still, it seems to be open whether always $\varphi_{H^\infty}(A_p) = \theta_p$.

7.3. *Ultracontractivity, kernels and Sobolev embedding*

In this section we consider a semigroup T on $L^2(\Omega)$ which is regularizing in the L^p-sense: We will ask that $T(t)L^2(\Omega) \subset L^q(\Omega)$ for some $q > 2$. This property implies in particular that $T(t)$ is an integral operator. Throughout this section (Ω, Σ, μ) is a measure space.

7.3.1. *The Dunford–Pettis criterion.* Let $K \in L^\infty(\Omega \times \Omega)$. Then

$$(S_K f)(x) = \int_\Omega K(x,y) f(y)\,\mathrm{d}y \tag{7.7}$$

defines a bounded operator $S_K \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$.

THEOREM (see, e.g., [AB94]). *The mapping $K \mapsto S_K$ is an isometric isomorphism from $L^\infty(\Omega \times \Omega)$ onto $\mathcal{L}(L^1(\Omega), L^\infty(\Omega))$. Moreover, $K(x,y) \geqslant 0$ a.e. if and only if $S_K \geqslant 0$.*

7.3.2. *Ultracontractive semigroups.* Let T be a C_0-semigroup on $L^2(\Omega)$. We assume that (7.2) is satisfied and denote by T_p the extrapolation semigroup of T on $L^p(\Omega)$. Thus T_p is a C_0-semigroup for $1 < p < \infty$, we suppose that this be true also for $p = 1$ (cf. Section 7.2.1). Denote by A_p the generator of T_p. It follows from (7.2) that

$$\|T_p(t)\| \leqslant M_1 \mathrm{e}^{\omega t}, \quad t \geqslant 0, \tag{7.8}$$

for some $M_1 \geqslant 1$, $\omega \in \mathbb{R}$ and all $p \in [1, \infty]$. Thus $(\omega - A_p)$ is sectorial for all $1 \leqslant p < \infty$ by Section 4.1.3 and the fractional powers $(\omega - A_p)^\alpha$ are defined for all $\alpha \geqslant 0$. Note that

$$D\big((\omega - A_p)^\alpha\big) = D\big((\omega_1 - A_p)^\alpha\big) = (\omega_1 - A_p)^{-\alpha} L^p(\Omega)$$

for all $\omega_1 > \omega$.

THEOREM. *Let $n > 0$ be a real number. Consider the following conditions.*

(i) *There exist $c > 0$, $1 \leqslant p < q \leqslant \infty$ such that*

$$\|T(t)\|_{\mathcal{L}(L^p, L^q)} \leqslant c t^{-\frac{n}{2}|\frac{1}{p} - \frac{1}{q}|} \quad \textit{for all } 0 < t \leqslant 1.$$

(ii) *There exists a constant $c > 0$ such that for all $1 \leqslant p < q \leqslant \infty$,*

$$\|T(t)\|_{\mathcal{L}(L^p, L^q)} \leqslant c t^{-\frac{n}{2}|\frac{1}{p} - \frac{1}{q}|}, \quad 0 < t \leqslant 1.$$

(iii) *There exist $1 < p < \infty$ and $0 < \alpha < \frac{n}{2p}$ such that $D((\omega - A_p)^\alpha) \subset L^q$, where q is defined by $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$.*

(iv) *For all $0 < \alpha < \frac{n}{2p}$ and all $1 < p < \infty$ one has $D((\omega - A_p)^\alpha) \subset L^q$, where q is defined by $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii). *If T is holomorphic then all four assertions are equivalent.*

If the generator A of T is associated with a closed form with form domain $V \hookrightarrow L^2(\Omega)$ such that $V \cap L^1(\Omega)$ is dense in $L^1(\Omega)$ and if $n > 2$, then these four conditions are also equivalent to

(v) $V \hookrightarrow L^{2n/(n-2)}(\Omega)$.

We call a semigroup *ultracontractive* if the equivalent conditions (i) and (ii) are satisfied. The number $\dim(T) = \inf\{n > 0\colon$ (i) is valid$\}$ is called the *semigroup dimension*. Note that n is not entire, in general. It was Varopoulos who startet a systematic investigation of the dimension of a semigroup, mainly in the framework of symmetric submarkovian semigroups. We refer to [[VSC93]], [[Sal02]], [[Dav90]] for the historical references and further results.

PROOF OF THE THEOREM. (i) $\Leftrightarrow$ (ii): Assertion (ii) for $p = 1, q = \infty$ becomes

$$\text{(ii)}' \quad \|T(t)\|_{\mathcal{L}(L^1, L^\infty)} \leqslant \text{const} \cdot t^{-n/2}, \quad 0 < t \leqslant 1.$$

The Riesz–Thorin theorem allows one to go from (ii)$'$ to (ii). The following trick due to Coulhon [Cou90] allows one to show that (i) implies (ii)$'$. It is done in two steps. We assume (i).

(a) We show that $\|T(t)\|_{\mathcal{L}(L^1,L^q)} \leqslant \text{const} \cdot t^{-\frac{n}{2}(1-\frac{1}{q})}$, $0 \leqslant t \leqslant 1$. Let $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$, $\beta = \frac{n}{2}(1 - \frac{1}{q})$. Choose $0 < \theta < 1$ such that $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{q}$. Then $\frac{1}{p} - \frac{1}{q} = \theta(1 - \frac{1}{q})$, i.e., $\alpha = \theta\beta$. By the hypothesis (i), $\|T(t)\|_{\mathcal{L}(L^p,L^q)} \leqslant \text{const} \cdot t^{-\alpha}$ and we want to show that

$$\left\|T(t)\right\|_{\mathcal{L}(L^1,L^q)} \leqslant \text{const} \cdot t^{-\beta}.$$

Let

$$f \in L^1 \cap L^\infty, \quad \|f\|_{L^1} \leqslant 1,$$
$$c_f := \sup_{0<t\leqslant 1} t^\beta \left\|T(t)f\right\|_{L^q}.$$

Then

$$\begin{aligned}
\left\|T(t)f\right\|_{L^q} &= \left\|T(t/2)T(t/2)f\right\|_{L^q} \\
&\leqslant \text{const} \cdot t^{-\alpha} \left\|T(t/2)f\right\|_{L^p} \\
&\leqslant \text{const} \cdot t^{-\alpha} \left\|T(t/2)f\right\|_{L^1}^{\theta} \left\|T(t/2)f\right\|_{L^q}^{1-\theta} \\
&\leqslant \text{const} \cdot t^{-\alpha} t^{-\beta(1-\theta)} (c_f)^{1-\theta} \\
&\leqslant \text{const} \cdot t^{-\beta} (c_f)^{1-\theta}.
\end{aligned}$$

Hence, $c_f = \sup t^\beta \|T(t)f\|_{L^q} \leqslant \text{const} \cdot (c_f)^{1-\theta}$. Thus $c_f \leqslant \text{const}^{1/\theta}$, where const does not depend of f. This proves the claim.

(b) It follows from (a) by duality that

$$\begin{aligned}
\left\|T(t)\right\|_{\mathcal{L}(L^{q'},L^\infty)} &\leq \text{const} \cdot t^{-\frac{n}{2}(1-\frac{1}{q})} \\
&= \text{const} \cdot t^{-\frac{n}{2}(\frac{1}{q'}-\frac{1}{\infty})}.
\end{aligned}$$

Applying (a) to this gives (ii)$'$.

(i) $\Rightarrow$ (iv). The proof of [[Dav90], Theorem 2.4.2] based on the Marcinkiewicz interpolation theorem carries over to this case.

(iii) $\Rightarrow$ (i). Assume that T is holomorphic. Replacing A by $A - \omega$ we can assume that T is exponentially stable. Let $q = \frac{np}{n-2\alpha p}$. Since $\|t^\alpha A_p^\alpha T_p(t)f\| \leqslant \text{const} \cdot \|f\|_p$, $0 < t \leqslant 1$, and $D(A_p^\alpha) \hookrightarrow L^q$ by hypothesis, we have

$$\left\|T_p(t)f\right\|_q \leq \text{const} \cdot \left\|T_p(t)f\right\|_{D(A_p^\alpha)}$$

$$
\begin{aligned}
&= \text{const} \cdot \left\| A_p^\alpha T_p(t) f \right\|_{L^p} \\
&\leq \text{const} \cdot t^{-\alpha} \| f \|_{L^p}.
\end{aligned}
$$

Since $\alpha = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$, the claim follows.

Now assume that A is associated with a closed form.

(v) $\Rightarrow$ (i). Since $\|T(t)\|_{\mathcal{L}(L^2,V)} \leqslant \text{const} \cdot t^{-1/2}$, $0 < t \leqslant 1$ (see [[Tan79]]), the argument is as in the previous implication for $p = 2$.

(i) $\Rightarrow$ (v). As we know from Section 5.5.1 it may happen that $V \neq D(A^{1/2})$. Thus we cannot use the previous implication (i) $\Rightarrow$ (iv). However, for forms, ultracontractivity can be characterized by Nash's inequality

$$
\|f\|_2^{2+4/n} \leqslant \text{const} \cdot \|f\|_V^2 \|f\|_1^{4/n}, \quad f \in V \cap L^1(\Omega),
$$

by the proof of [[Dav90], Theorem 2.4.6]. Thus, assuming (i), Nash's inequality holds. Consider the symmetric form $b(u,v) = \frac{1}{2}(a(u,v) + \overline{a(v,u)})$ with domain V. Denote by B the operator associated with b and by S the semigroup generated by $-B$. It follows from the characterization via Nash's inequality we just mentioned that S satisfies (i). Hence, $D(B^{1/2}) \subset L^{2n/n-2}$ by the implication (i) $\Rightarrow$ (iv). But $V = D(B^{1/2})$ since B is self-adjoint. □

Now we show some further properties of ultracontractive semigroups and give examples.

7.3.3. *Kernels and compactness.* Assume that T is an ultracontractive semigroup on $L^2(\Omega)$. Then

$$
\left\| T(t) \right\|_{\mathcal{L}(L^1,L^\infty)} \leqslant M_2 t^{-n/2} \mathrm{e}^{\omega_2 t} \quad \text{for all } t > 0 \tag{7.9}
$$

and for some $M_2 \geqslant 1$, $\omega_2 \in \mathbb{R}$. Thus, by the Dunford–Pettis criterion there exists a kernel $K_t \in L^\infty(\Omega \times \Omega)$ such that

$$
\big(T_p(t) f\big)(x) = \int_\Omega K_t(x,y) f(y)\, \mathrm{d}y, \quad x\text{-a.e.} \tag{7.10}
$$

for all $f \in L^p(\Omega) \cap L^1(\Omega)$. Moreover,

$$
\left| K_t(x,y) \right| \leqslant M_2 \mathrm{e}^{\omega_2 t} \cdot t^{-n/2}, \quad t > 0. \tag{7.11}
$$

Now assume that Ω has finite measure. Then $T(t)$ is a Hilbert–Schmidt operator and hence compact. This property extends to the entire L^p scale, $1 \leqslant p \leqslant \infty$, in virtue of ultracontractivity. In fact, given $t > 0$, $1 \leqslant p \leqslant \infty$, we can factorize $T_p(t)$ as follows:

$$
L^p \xrightarrow{T_p(t/3)} L^\infty \hookrightarrow L^2 \xrightarrow{T_2(t/3)} L^2 \xrightarrow{T_2(t/3)} L^\infty \hookrightarrow L^p.
$$

Thus $T_p(t)$ is compact. We have shown the following.

PROPOSITION. *Let T be an ultracontractive semigroup on $L^2(\Omega)$ with extrapolating semigroups T_p, $1 \leqslant p \leqslant \infty$. If Ω has finite measure, then $T_p(t)$ is a compact operator on $L^p(\Omega)$ for all $t > 0$, $1 \leqslant p \leqslant \infty$.*

7.3.4. *The Gaussian semigroup and Sobolev embedding.* As an illustrating example we consider the Gaussian semigroup G_p on $L^p(\mathbb{R}^n)$ given by the kernel

$$K_t^G(x, y) = (4\pi t)^{-n/2} \mathrm{e}^{-|x-y|^2/4t}. \tag{7.12}$$

Thus (i) is satisfied for $p = 1, q = \infty$, and in this case the semigroup dimension n is the dimension of the space $\mathbb{R}^n$. Denote the generator of G_p by Δ_p. Then $D(\Delta_p) = W^{2,p}(\mathbb{R}^n)$, $1 < p < \infty$.

PROOF. The operator $G_p(t)$ is a Fourier multiplier with symbol $\mathrm{e}^{-\xi^2 t}$. Thus $R(1, \Delta_p)$ has symbol $(1 + \xi^2)^{-1}$. Since for

$$f \in \varphi(\mathbb{R}^n)', \quad \mathcal{F}(D_j f) = \mathrm{i}\xi_j \mathcal{F} f,$$

we have to show that $\xi_i \xi_j (1 + \xi^2)^{-1}$ is an L^p-Fourier multiplier (in order to deduce that $D(\Delta_p) = R(1, \Delta_p) L^p(\mathbb{R}^n) \subset W^{2,p}(\mathbb{R}^n)$). This follows from Michlin's theorem. $\square$

Similarly, one sees with the help of Michlin's theorem that

$$D(\Delta_p^m) = W^{2m,p}(\mathbb{R}^n), \quad 1 < p < \infty, m \in \mathbb{N}.$$

Thus, the conclusion (iv) of the Theorem in Section 7.3.2 is the usual Sobolev embedding theorem. We remark that such results are of particular interest if they are applied to the Laplace–Beltrami operator on a Riemannian manifold (see [[VSC93]]).

7.3.5. *The Dirichlet Laplacian.* Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set and consider the Dirichlet Laplacian Δ_Ω^D on $L^2(\Omega)$. Then $(\mathrm{e}^{t\Delta_\Omega^D})$ is self-adjoint, submarkovian and ultracontractive since the form domain $W_0^{1,2}(\Omega)$ is in L^q, where $q = \frac{2n}{n-2}$ if $n > 2$ and arbitrary $q < \infty$ if $n \leqslant 2$. Denote by $\Delta_{\Omega,p}^D$ the generator of the extrapolation semigroup, $1 \leqslant p \leqslant \infty$. Then we deduce from the Theorem in Section 7.3.2 that

$$D\big((-\Delta_{\Omega,p}^D)^\alpha\big) \subset L^{np/(n-2\alpha p)} \tag{7.13}$$

whenever $0 < \alpha < \frac{n}{2p}$. On the other hand, if Ω is irregular, then it can happen that $D(\Delta_{\Omega,p}^D) \not\subset W^{1,p}(\Omega)$ for $p > 2$, see [[Gri92]]. Thus Sobolev embedding results cannot be used here to prove (7.13).

7.3.6. *The extension property.* An open set $\Omega \subset \mathbb{R}^n$ has the *extension property* if the restriction mapping $R : W^{1,2}(\mathbb{R}^n) \to W^{1,2}(\Omega)$ is surjective. Then $R_0 : (\ker R)^\perp \to W^{1,2}(\Omega)$ is an isomorphism and $E := R_0^{-1}$ is a bounded operator from $W^{1,2}(\Omega)$ into $W^{1,2}(\mathbb{R}^n)$ such that $(Eu)|_\Omega = u$ for all $u \in W^{1,2}(\Omega)$. We call E an *extension operator*. If Ω is bounded and has Lipschitz boundary, then Ω has the extension property. If Ω has the extension property then

$$W^{1,2}(\Omega) \subset L^q(\Omega) \tag{7.14}$$

for $q = \frac{2n}{n-2}$ if $n > 2$ and for all $q < \infty$ if $n \leqslant 2$. This follows from the case $\Omega = \mathbb{R}^n$. Property (7.14) implies ultracontractivity in many interesting examples. Here we consider the Neumann Laplacian as prototype:

7.3.7. *The Neumann Laplacian.* Let $\Omega \subset \mathbb{R}^N$ be open. The Neumann Laplacian Δ_Ω^N generates a symmetric submarkovian semigroup on $L^2(\Omega)$. If Ω has the extension property then $(e^{t\Delta_\Omega^N})_{t\geqslant 0}$ is ultracontractive. However, this is not true without this hypothesis. For example, in dimension 1, we may consider $\Omega = (0,1) \setminus \{1/n : n \in \mathbb{N}\}$. Then 0 is an eigenvalue of infinite multiplicity of Δ_Ω^N, hence, the resolvent is not compact. In particular, the semigroup is not ultracontractive.

7.3.8. *Asymptotic behavior of $T_p(t)$ as $t \to \infty$. Let T be an ultracontractive, positive, irreducible C_0-semigroup on $L^2(\Omega,\mu)$, where Ω has finite measure. Then there exist $u, \varphi \in L^\infty(\Omega)$ such that $u(x) > 0$, $\varphi(x) > 0$ a.e., $\int_\Omega u(x)\varphi(x)\,\mathrm{d}\mu = 1$ and $\omega \in \mathbb{R}$, $\varepsilon > 0$, $M \geqslant 0$ such that*

$$\left\| e^{-\omega t} T_p(t) - P \right\|_{\mathcal{L}(L^p(\Omega))} \leqslant M e^{-\varepsilon t}, \quad t \geqslant 0,$$

for $1 \leqslant p \leqslant \infty$, where $Pf = \int_\Omega f(x)\varphi(x)\,\mathrm{d}\mu(x) \cdot u$, $f \in L^1(\Omega)$.

PROOF. By Section 7.3.3 the operators $T_p(t)$ are compact for $1 \leqslant p \leqslant \infty$, $t > 0$. It follows that A_p has compact resolvent. Thus $\sigma(A_p)$ is independent of $p \in [1,\infty]$ (see [[Dav90], p. 36]). In particular, $\omega := s(A_p)$ is independent of $p \in [1,\infty]$. Now the claim follows from Section 3.5.1 for $1 \leqslant p < \infty$. For $p = \infty$ one may use a duality argument. $\square$

A particular example is $T(t) = e^{t\Delta_\Omega^n}$ where Ω is a connected, bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. In this case $Pf = \frac{1}{|\Omega|}\int_\Omega f\,\mathrm{d}x \cdot 1_\Omega$ for all $f \in L^p(\Omega)$.

7.4. *Gaussian estimates*

Let $\Omega \subset \mathbb{R}^n$ be an open set and let T be a C_0-semigroup on $L^2(\Omega)$ (the L^2-space of complex-valued functions). We identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^n)$ extending functions on Ω by 0 on $\mathbb{R}^N \setminus \Omega$. Consider the Gaussian semigroup G on $L^2(\mathbb{R}^n)$.

DEFINITION. The semigroup T admits a *Gaussian estimate* if there exist constants $c > 0$, $b > 0$ such that

$$\left| T(t)f \right| \leqslant cG(bt)|f|, \quad 0 < t \leqslant 1, \tag{7.15}$$

for all $f \in L^2(\Omega)$.

Note that (7.15) is an inequality between measurable functions in the almost everywhere sense. It is not difficult to show that it implies an estimate of the form

$$\left|T(t)f\right| \leqslant M\mathrm{e}^{\omega t}G(bt)|f| \tag{7.16}$$

for all $t > 0$, $f \in L^2(\Omega)$. By the Dunford–Pettis criterion this is equivalent to saying that $T(t) = S_{K_t}$, where $K_t \in L^\infty(\Omega \times \Omega)$, is a kernel satisfying

$$\left|K_t(x,y)\right| \leqslant \mathrm{const} \cdot \mathrm{e}^{\omega t} t^{-n/2} \mathrm{e}^{-|x-y|^2/4bt}, \quad (x,y)\text{-a.e.} \tag{7.17}$$

for all $t > 0$. If T admits a Gaussian estimate then by Section 7.2 (and in particular, Section 7.2.1(d)) there exist consistent extrapolation semigroups T_p on $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, such that $T_2 = T$. For $1 \leqslant p < \infty$, T_p is a C_0-semigroup, T_∞ is w^*-continuous. By A_p we denote the generator of T_p. These notations will be used in the following without further notice. Our aim is to establish consequences of Gaussian estimates for T_p which allow us in particular to replace some of the negative answers in the heritage list of Section 7.2.2 by positive assertions. Later we will see that a large class of elliptic operators generate semigroups which satisfy Gaussian estimates. But before that we give two prototype examples.

7.4.1. *Examples.* (a) *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. The semigroup generated by the Dirichlet Laplacian Δ_Ω^D on $L^2(\Omega)$ satisfies*

$$\left|\mathrm{e}^{t\Delta_\Omega^D} f\right| \leqslant G(t)|f|, \quad t > 0, \tag{7.18}$$

for all $f \in L^2(\Omega)$.

(b) *Assume that Ω has the extension property. Then the semigroup $(\mathrm{e}^{t\Delta_\Omega^N})_{t\geqslant 0}$ generated by the Neumann Δ_Ω^N Laplacian admits a Gaussian estimate.*

PROOF. (a) Let $0 \leqslant f \in L^2(\mathbb{R}^n)$. It suffices to show that $u := R(\lambda, \Delta_\Omega^D)f \leqslant R(\lambda, \Delta)f =: v$ on Ω. By the definition via forms we have

$$\lambda \int_\Omega u\varphi + \int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi, \varphi \in W_0^{1,2}(\Omega),$$
$$\lambda \int_{\mathbb{R}^n} v\varphi + \int_{\mathbb{R}^n} \nabla v \nabla \varphi = \int_{\mathbb{R}^n} f\varphi, \varphi \in W^{1,2}\big(\mathbb{R}^n\big).$$

Hence, $\lambda \int_\Omega (u-v)\varphi + \int_{\mathbb{R}^n} \nabla(u-v)\nabla\varphi = 0$ for all $\varphi \in W_0^{1,2}(\Omega)$. Taking $\varphi = (u-v)^+$ one obtains that $(u-v)^+ = 0$.

(b) In this generality, this is due to Ouhabaz [Ouh03]. For a stronger version of the extension property the result is proved in [[Dav90], p. 90]. □

7.4.2. *Remark* (*Proper domination*). There is a significant difference between the two examples: For the Dirichlet Laplacian we have *proper domination*; i.e., the constant c in (7.15) is equal to 1. But this is the only realization of the Laplacian with this property:

Let $\Omega \subset \mathbb{R}^n$ be open. We assume that Ω is stable (*see Section* 5.4.5). *Let S be a positive*, *symmetric C_0-semigroup on $L^2(\Omega)$ such that $S(t) \leqslant G_2(t)$, $0 < t \leqslant 1$. Assume that the generator A of S satisfies $\mathcal{D}(\Omega) \subset D(A)$ and $Au = \Delta u$ for all $u \in \mathcal{D}(\Omega)$. Then $A = \Delta_\Omega^D$.*

PROOF. (a) We consider $L^2(\Omega)$ as a subspace of $L^2(\mathbb{R}^n)$, extending functions by 0. Then $S(t) = (1_\Omega S(t/n))^n \leqslant (1_\Omega G_2(t/n))^n$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}(1_\Omega G(t/n))^n = \mathrm{e}^{t\Delta_\Omega^D}$ strongly by Section 5.4.5, we conclude that $S(t) \leqslant \mathrm{e}^{t\Delta_\Omega^D}$, $0 < t < \infty$.

(b) Denote by $a : V \times V \to \mathbb{R}$ the symmetric closed form associated with $-A$. It follows from (a) and Section 7.1.3 that $V \subset H_0^1(\Omega)$. For $u, v \in \mathcal{D}(\Omega)$, we have $a(u,v) = -(Au|v)_{L^2} = \int \nabla u \nabla v \,\mathrm{d}x$. Note that $\|u\|_{L^2}^2 + a(u,u)$ defines an equivalent norm on V. Thus $\|\cdot\|_V$ and $\|\cdot\|_{H^1}$ are equivalent. Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that $H_0^1(\Omega) \subset V$. Thus $V = H_0^1(\Omega)$ and $a(u,v) = \int_\Omega \nabla u \nabla v \,\mathrm{d}x$ for all $u, v \in H_0^1(\Omega)$. □

The fact that the constant c in (7.15) might be larger than 1 makes it difficult to prove Gaussian estimates, and no such simple criteria as in Section 7.1.3 are available. We comment on techniques to prove Gaussian estimates in Section 8.7. Now we establish heritage properties made possible by Gaussian estimates.

7.4.3. *Gaussian estimates and extrapolation of holomorphy. Let T be a C_0-semigroup on $L^2(\Omega)$ which admits a Gaussian estimate. Assume that T is holomorphic of angle $\theta \in (0, \pi/2]$. Then also the extrapolation semigroups T_p are holomorphic of the same angle θ for each $p \in [1, \infty]$.*

This result due to Ouhabaz [Ouh95] (see [AtE97], Theorem 5.4, in the nonsymmetric case) contrasts the examples in Sections 7.2.4 and 7.2.6, where, in absence of Gaussian estimates, the angle depends on p and where T_1 is not holomorphic.

7.4.4. *Gaussian estimates and maximal regularity. Let T be a C_0-semigroup on $L^2(\Omega)$ which has Gaussian estimates. If T is holomorphic*, *then T_p has property* (*MR*) *for $1 < p < \infty$.*

This result is most important for applications to nonlinear equations. We refer to Section 6.2.6 for the definition of (*MR*) and to [HP97], [CP01], [CD00], [Wei00b] and [ArBu03a], Corollary 4.5, for proofs of this result.

Recall that holomorphy is a necessary condition for (*MR*). It may happen that a semigroup T on $L^2(\Omega)$ has Gaussian estimates without being holomorphic. In fact, Voigt showed that the operator $\Delta + \mathrm{i}x$ on $L^2(\mathbb{R})$ with suitable domain generates a C_0-semigroup S such that $|S(t)f| \leqslant G(t)|f|$, but S is not holomorphic, cf. [LM97], p. 303. However, so far no example of a nonholomorphic positive semigroup with Gaussian estimates is known.

7.4.5. *Gaussian estimates and H^∞-calculus. Let $-A$ be the generator of a holomorphic C_0-semigroup on $L^2(\Omega)$ admitting a Gaussian estimate. Assume that $(A + \omega)$ has* *a bounded H^∞-calculus for some $\omega \in \mathbb{R}$. Then also $(A_p + \omega)$ has a bounded H^∞-calculus for some $\omega \in \mathbb{R}$, for all $1 < p < \infty$.*

This result is due to Duong and Robinson [DR96]. The restriction on Ω (doubling property) made there was removed later by Duong and McIntosh [DM99]. We do not try to optimize the rescaling constant ω in our formulation, but rather refer to [DR96] and [AtE97], pp. 118 and 122, for this.

We recall that on Hilbert space it suffices that the semigroup T be quasicontractive to insure $(A + \omega)$ to have a bounded H^∞-calculus for some ω.

7.4.6. *Gaussian estimates and p-independence of the spectrum.* *Let T be a C_0-semigroup on $L^p(\Omega)$ admitting a Gaussian estimate. Denote by T_p the extrapolation C_0-semigroup on $L^p(\Omega)$ and by A_p its generator. Then $\sigma(A_p) = \sigma(A_2)$, $1 \leqslant p \leqslant \infty$.*

In [Are94] it was proved that the connected component of $\rho(A_p)$ is p-independent, the general result was obtained by Kunstmann [Kun99]. An example of special interest are Schrödinger operators for which the result had been obtained before by Hempel and Voigt [HV86].

8. Elliptic operators

In this section we apply the results obtained before to semigroups generated by elliptic operators of second order. If the coefficients and the domain in $\mathbb{R}^n$ are smooth, then classical estimates of Agmon–Douglis–Nirenberg give precise information on the domain. Here we consider merely measurable coefficients. Then the domain can no longer be determined. Under some mild conditions, however, Gaussian estimates can still be proved and lead to a variety of semigroup and spectral properties. At the end of this chapter we mention also some results for higher order operators and systems.

Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $a_{ij}, b_i, c_i, a_0 \in L^\infty(\Omega)$, $i, j = 1, \ldots, n$, be complex-valued coefficients. We assume the ellipticity condition

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \geqslant \alpha|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n, x\text{-a.e.},$$

where $\alpha > 0$. Then we consider the elliptic operator

$$L : W^{1,2}_{\mathrm{loc}}(\Omega) \to \mathcal{D}(\Omega)'$$

given by

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + \sum_{i=1}^{n}\bigl(b_iD_iu - D_i(c_iu)\bigr) + a_0u.$$

With the help of forms we will define various realizations of L in $L^2(\Omega)$ corresponding to diverse boundary conditions.

Let V be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. We define the form $a_V : V \times V \to \mathbb{C}$ by

$$a_V(u, v) = \int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) D_i u \overline{D_j v} + \sum_{i=1}^n \bigl(b_i D_i u \bar{v} + c_i u \overline{D_i v} \bigr) + a_0 u \bar{v} \right] \mathrm{d}x.$$

Then a_V is continuous and $L^2(\Omega)$-elliptic. Denote by A_V the operator on $L^2(\Omega)$ associated with a_V. Then $-A_V$ generates a holomorphic C_0-semigroup T_V on $L^2(\Omega)$, and we will investigate various properties of T_V. At first we describe the operator A_V more precisely. It follows from the definition of the associated operator that

$$A_V u = Lu$$

for all $u \in D(A_V)$. We will consider in particular three different boundary conditions which we describe in the following section.

8.1. *Boundary conditions*

(a) Let $V = W_0^{1,2}(\Omega)$. Then we call A_V the elliptic operator with *Dirichlet boundary conditions*. In that case one has $D(A_V) = \{u \in W_0^{1,2}(\Omega)\colon\ Lu \in L^2(\Omega)\}$.

(b) Let $V = W^{1,2}(\Omega)$. Then we call A_V the elliptic operator with *Neumann boundary conditions*. If coefficients and domain are smooth, then for $u \in C^2(\overline{\Omega})$ one has $u \in D(A_V)$ if and only if

$$\sum_{j=1}^n \left(\sum_{i=1}^n a_{ij} D_i u + c_i u \right) \nu_j = 0 \quad \text{on } \partial\Omega,$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal. This is a consequence of Green's formula.

(c) *Mixed boundary conditions* are realized by taking

$$V = \bigl\{ u|_\Omega \colon\ u \in \mathcal{D}\bigl(\mathbb{R}^n \setminus \Gamma\bigr) \bigr\}^{-W^{1,2}(\Omega)}, \quad \text{where } \Gamma \text{ is a closed subset of } \partial\Omega.$$

8.2. *Positivity and irreducibility*

By Section 7.1.1 the semigroup T_V generated by $-A_V$ is positive if and only if all coefficients are real valued and

$$u \in V \quad \text{implies} \quad (\operatorname{Re} u)^+ \in V. \tag{8.1}$$

This is, in particular, the case for all three boundary conditions considered in Section 8.1. Moreover, if Ω is connected, then the semigroup is irreducible.

8.3. *Submarkov property*: *Dirichlet boundary conditions*

Let $V = W_0^{1,2}(\Omega)$. Assume that all coefficients are real. Then T_V is submarkovian if and only if

$$\sum_{j=1}^{n} D_j c_j \leqslant a_0 \quad \text{in } \mathcal{D}(\Omega)'. \tag{8.2}$$

See [ABBO00], Théorème 2.1.

8.4. *Quasicontractivity in L^p*

Assume that all coefficients are real valued. Assume furthermore that (8.1) and

$$u \in V_+ \Rightarrow 1 \wedge u \in V \tag{8.3}$$

hold. Then for each $p \in (1, \infty)$ there exists $\omega_p \in \mathbb{R}$ such that

$$\left\| T_V(t) \right\|_{\mathcal{L}(L^p)} \leqslant \mathrm{e}^{\omega_p t}, \quad t \geqslant 0. \tag{8.4}$$

This assertion is false though for $p = 1$ or $p = \infty$. However, if $b_j,\ c_j \in W^{1,\infty}(\Omega)$, $j = 1, \ldots, n$, then there exists $\omega \in \mathbb{R}$ such that

$$\left\| T_V(t) \right\|_{\mathcal{L}(L^p)} \leqslant \mathrm{e}^{\omega t}, \quad t \geqslant 0, \tag{8.5}$$

for all $p \in [1, \infty]$.

References: [Ouh92], [ABBO00], [Dan00], [[Ouh04]].

8.5. *Gaussian estimates*: *real coefficients*

Assume that all coefficients are real valued. Let V be one of the three spaces considered in Section 8.1. In the case of Neumann or mixed boundary conditions assume that Ω has the extension property (in the weak sense of Section 7.3.6). Then T_V admits Gaussian estimates; see [AtE97], [Dan00], [Ouh03]. In particular, we have the following consequences: Denote by T_p the extrapolation semigroup of T_V in $L^p(\Omega)$ and by $-A_p$ its generator. Then

(a) T_p is a holomorphic C_0-semigroup, $1 \leqslant p < \infty$.
(b) The operator A_p satisfies (*MR*), $1 < p < \infty$.
(c) The operator $(A_p + \omega)$ has a bounded H^∞-calculus for ω large enough and $\varphi_{H^\infty}(A_p + \omega) < \pi/2$, where $1 < p < \infty$.
(d) The spectrum $\sigma(A_p)$ is independent on $p \in [1, \infty]$.

(e) Asymptotic behavior: Assume that Ω is connected and bounded. Then there exist u, $\varphi \in L^\infty(\Omega)$ such that

$$u(x) > 0, \varphi(x) > 0 \text{ a.e.}, \qquad \int_\Omega u(x)\varphi(x)\,\mathrm{d}x = 1,$$

and such that

$$\left\| \mathrm{e}^{-\omega t} T_p(t) - P \right\|_{\mathcal{L}(L^p(\Omega))} \leqslant M \mathrm{e}^{-\varepsilon t}, \quad t \geqslant 0,$$

for all $1 \leqslant p < \infty$ and some $\varepsilon > 0$, $M \geqslant 0$, where $\omega = s(A_p)$ and $Pf = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x \cdot u$.

This follows from Sections 7.4.3 for (a), 7.4.4 for (b), 7.4.5 for (c), 7.4.6 for (d) and 7.3.8 for (e) observing that Gaussian estimates imply ultracontractivity. We remark that assertion (c) can already be deduced from (8.4) by Section 4.7.5.

8.6. *Complex second-order coefficients*

If the coefficients a_{ij} are complex-valued, the situation is more complicated. For $\Omega = \mathbb{R}^n$, there are always Gaussian estimates if the a_{ij} are uniformly continuous [Aus96] or if $n \leqslant 2$; but otherwise there are counterexamples (see Auscher, Coulhon and Tchamitchian [ACT96] and Davies [Dav97]). If Ω is a Lipschitz domain and $V = W_0^{1,2}(\Omega)$ or $W^{1,2}(\Omega)$, then the existence of Gaussian estimates depends on the Lipschitz constant even for constant complex a_{ij}: For small Lipschitz constant Gaussian estimates are valid in that case [AT01a], but a counterexample based on [MNP85] is given for large Lipschitz constant. However, if the imaginary parts of the a_{ij} are symmetric then Gaussian estimates are valid:

THEOREM ([Ouh03], Theorem 5.5). *Assume that the imaginary parts of the coefficients satisfy*

$$\operatorname{Im} a_{ij} \in W^{1,\infty}, \quad \operatorname{Im}(a_{ij} + a_{ji}) = 0.$$

Let $V = W_0^{1,2}(\Omega)$ or $W^{1,2}(\Omega)$ assuming the extension property in the latter case. Then T_V admits Gaussian estimates.

8.7. *Further comments on Gaussian estimates*

Gaussian estimates were first proved by Aronson [Aro67] for real nonsymmetric elliptic operators on $\mathbb{R}^n$ with measurable coefficients. He used Moser's parabolic Harnack inequality [Mos64]. New impetus to the subject came from Davies [Dav87] who introduced a perturbation method ("Davies' trick") which provides an efficient tool to prove Gaussian estimates via ultracontractivity. One of Davies' motivations was to find optimal constants in the estimates, and the results are presented for symmetric operators in his book [[Dav90]].

Here we do not explain how the Gaussian estimates are proved but refer to the corresponding articles mentioned above and to [[Fri64]], [AER94], [AMT98], [DHZ94]. Some of the semigroup properties put together here carry over to estimates valid for higher-order equations, see, for example, [Are97], [Hie96], [HP97], [tER98].

8.8. *The square root property*

Assume that $\Omega \subset \mathbb{R}^n$ is open, bounded with Lipschitz boundary. Consider the operator A_V with

$$V = W_0^{1,2}(\Omega) \quad \text{or} \quad V = W^{1,2}(\Omega),$$

i.e., Dirichlet or Neumann boundary conditions are imposed. Let $\omega \in \mathbb{R}$ be large so that $A_V + \omega$ is sectorial. Then

$$D\big((A_V + \omega)^{1/2}\big) = V.$$

This is an extremely deep result. For $\mathbb{R}^n$, it is the famous *Kato's conjecture* which was solved recently by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [AHLMT02]. The result for the boundary conditions mentioned here is proved in [AT01a]. We refer to these articles as well as [[AT98]] for the sophisticated techniques from harmonic analysis leading to this result and also for various connections of these results with other areas. We mention that Kato's square root problem has also been solved for higher order elliptic operators in $\mathbb{R}^n$ [AHMT01]. Finally, we mention an elegant short proof of the square root property by ter Elst and Robinson [tER96b] (after previous work by McIntosh [McI85]) in the special case where the second-order coefficients are Hölder continuous.

8.9. *The hyperbolic equation*

We keep the notations and assumptions made in the beginning of this section, but we assume in addition that

$$a_{ij} = \overline{a_{ji}}, \quad i, j = 1, \dots, n.$$

Then the following holds.

THEOREM. *The operator $-A_V$ generates a cosine function on $L^2(\Omega)$ with phase space $V \times L^2(\Omega)$.*

The proof [[ABHN01], Theorem 7.1] given for $V = W_0^{1,2}(\Omega)$ carries over to arbitrary closed subspaces V of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. The assertion of the Theorem implies in particular the square root property $D((A_V + \omega)^{1/2}) = V$ (for ω large). However, here A_V is a certain perturbation of a self-adjoint operator [[ABHN01], Corollary 3.14.12] for which the square root property is easy in comparison with the general result mentioned in Section 8.8.

8.10. *Nondivergence form*

Let $a_{ij} \in C(\mathbb{R}^n)$ such that

$$\operatorname{Re} \sum a_{ij}(x)\xi_i \xi_j \geqslant \alpha |\xi|^2$$

for all $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and some $\alpha > 0$. Consider the operator A on $L^2(\mathbb{R}^n)$ given by

$$Af = \sum_{i,j=1}^{n} a_{ij} D_i D_j u$$

with domain $D(A) = H^2(\mathbb{R}^n)$. If the coefficients a_{ij} are Hölder continuous, then A generates a holomorphic C_0-semigroup on $L^2(\mathbb{R}^n)$ admitting Gaussian estimates [[Fri64], Theorem 9.4.2]. However, there are no Gaussian estimates in general, if the coefficients are merely uniformly continuous and bounded, even if they are real [Bau84]. However, by other methods (viz. the $T1$ Theorem) Duong and Simonett [DS97] show that the L^p-realization of the above operator is sectorial and has a bounded H^∞-calculus. Further generalizations to *VMO*-coefficients are given by Duong and Yan [DY01] by wavelet methods and by Heck and Hieber [HH03] via weighted estimates.

8.11. *Elliptic operators with Banach space-valued coefficients*

Elliptic operators with coefficients in a Banach space and the associated parabolic equations were introduced and investigated by Amann [Ama01]. Further regularity results were obtained by Denk, Hieber and Prüss [DHP01] where a comprehensive presentation of the subject is given. A basic result is the following.

THEOREM ([DHP01], Section 5.5). *Let X be a UMD-space, n, $m \in \mathbb{N}$, $1 < p < \infty$, $a_\alpha \in \mathcal{L}(X)$ for multiindices α of order $|\alpha| = m$. Assume the ellipticity condition*

$$\sigma\big(\mathcal{A}(\xi)\big) \in \Sigma_\theta$$

for all $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$, and some $\theta \in [0, \pi/2)$, where we set

$$\mathcal{A}(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

Consider the operator A on $L^p(\mathbb{R}^n, X)$ with domain $D(A) = W^{m,p}(\mathbb{R}^n, X)$ given by

$$Au = \sum_{|\alpha|=m} a_\alpha D^\alpha u.$$

Then A has a bounded $H^\infty(\Sigma_\theta)$-calculus. In particular, A is R-sectorial.

This result can be extended to elliptic operators on $L^p(\Omega, X)$ with diverse boundary conditions. It is remarkable that it is possible to characterize precisely those boundary conditions for which maximal regularity (*MR*) is valid; viz. by the Lopatinskii–Shapiro condition; see [DDHPV04].

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