HOMOGENIZATION TECHNIQUES AND APPLICATIONS TO BIOLOGICAL TISSUES

DOTTORATO DI MODELLI E METODI MATEMATICI
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Contents

Preface

These notes contain the lectures of a 20/25 hours course entitled "Homogenization Techniques and Applications to Biological Tissues", held in 2009/2010 for "Dottorato di Modelli e Metodi Matematici per la Tecnologia e la Società" at Dipartimento Me.Mo.Mat., Facoltà di Ingegneria, "Sapienza Università di Roma".
The aim of the authors is to present an introduction to homogenization techniques based on the asymptotic expansions introduced by Bensoussan-Lions-Papanicolau, jointly with an application to a particular physical problem.
The basic ideas of homogenization techniques are here presented with the purpose of giving to the students a solid knowledge in this field in order to make them able to apply these techniques also in different contexts. To this aim, first we describe the usual homogenization procedure of the Dirichlet problem for a standard elliptic equation with periodic coefficients; afterwards we apply these ideas to study a physical problem, relevant both from the mathematical point of view as well as for the applications. More precisely, we present an application of homogenization techniques to study the behavior of a biological tissue subjected to an electrical current flux. Indeed, it is well known that electrical potentials are crucial for imaging techniques in medical diagnosis, in order to investigate the physical properties of biological tissues.
The model which we present here is described by means of a system of elliptic equations whose solutions are coupled because of the interface conditions, since they have to satisfy the property of flux-continuity and a transmission condition of dynamic type. From the mathematical point of view, the presence of these interface conditions is a non standard problem both for the study of the well-posedness as well as for the homogenization.

These notes are divided into four chapters: in Chapter 1 some preliminary notions of Functional Analysis are very briefly recalled (metric, normed and Hilbert spaces, Sobolev spaces and embedding theorems, space of bounded variation functions); these notions are necessary for a complete understanding of the course itself; in Chapter 2 we study the connections between existence of solutions to minimum problems for integral functionals and well-posedness for the Dirichlet problem for elliptic PDEs in divergence form. To this purpose we introduce the basic ideas of Convex Analysis (preliminary definitions, basic properties of lower semicontinuous and convex functions, Gâteaux and Fréchet derivatives) and we mention preliminary notions of Direct Methods of Calculus of Variations (coercivity, existence theorem for minimizers, applications to integral functionals). We mention also Lax-Milgram Lemma and some of its applications to linear variational PDEs. In Chapter 3 we consider the homogenization of a standard elliptic equation, introducing the technique of asymptotic expansions due to Bensoussan-Lions-Papanicolau and the energy convergence method of Tartar. Finally, in Chapter 4 we apply the previously introduced homogenization techniques to the study of a physical model governing the electrical conduction in biological tissues (*Electric Impedance Tomography*).

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1. **Preliminaries of functional analysis**

1.1. **Metric, normed and Hilbert spaces.**

Definition **1.1**. Let X be a non-empty set. We define a *distance* on $X \times X$, i.e. a functional $d : X \times X \to \mathbb{R}$ such that:
(i) $d(x, y) \geq 0 \ \forall x, y \in X$;
(ii) (*Identity of indiscernibles*) $d(x, y) = 0 \iff x = y$;
(iii) (*Simmetry*) $d(x, y) = d(y, x) \ \forall x, y \in X$;
(iv) (*Triangular inequality*) $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$.
The pair (X, d) is called *metric space*.

Let $x_0 \in X$ and $r > 0$ be given. We call *open ball* of radius r and center x_0 the set

$$B_r(x_0) = \{x \in X \ : \ d(x, x_0) < r\} \ .$$

Definition **1.2**. Let (X, d) be a metric space and $\{x_n\} \subseteq X$ be a sequence of points in X. Let $x_0 \in X$; we say that the sequence $\{x_n\}$ *converges* to x_0 for $n \to +\infty$ (and we write $x_n \to x_0$) if $d(x_n, x_0) \to 0$ for $n \to +\infty$.

Definition **1.3**. Let (X, d) be a metric space. We say that the function $f : X \to \mathbb{R}$ is *continuous* on X if for every $x \in X$ and for each sequence $\{x_n\} \subseteq X$, such that $x_n \to x$, we have

$$\lim_{n\to+\infty} f(x_n) = f(x) \ .$$

Definition **1.4**. Let (X, d) be a metric space. We say that a set $C \subset X$ is *closed* if for each sequence $\{x_n\} \subseteq C$ converging to a point $x \in X$, we have that $x \in C$. We say that a set $A \subset X$ is *open* if its complement A^c is closed.

Definition **1.5**. We say that a set $K \subseteq X$ is *compact* if for each sequence $\{x_n\} \subseteq K$ it is possible to extract a subsequence converging to a point $x \in K$.

Definition **1.6**. Let X be a vector space. We define on X a *norm*, i.e. a functional $\| \cdot \|_X : X \to \mathbb{R}$ such that
(i) (*Identity of indiscernibles*) $\|x\|_X \geq 0 \ \forall x \in X$ and $\|x\|_X = 0 \iff x = 0$;
(ii) (*Positive homogeneity*) $\|\lambda x\|_X = |\lambda| \ \|x\|_X \ \forall x \in X$ and $\forall \lambda \in \mathbb{R}$;
(iii) (*Triangular inequality*) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$ for every $x, y \in X$.
The pair $(X, \| \cdot \|_X)$ (or simply X, if $\| \cdot \|_X$ is assigned) is called *normed space*.

A normed space is a metric space with respect to the distance induced by the norm, i.e. $d(x, y) = \|x - y\|_X$.

Remark 1.7. We observe that in a normed space, as well as in a metric space, the union of any family of open sets and the intersection of a finite family of open sets is still an open set; consequently, the intersection of any family of closed sets and the union of a finite family of closed sets is a closed set.

Definition **1.8**. Let X be a normed space and $\{x_n\} \subseteq X$ be a given sequence. Let $x_0 \in X$; we say that $\{x_n\}$ converges to x_0 for $n \to +\infty$ (and we write $x_n \to x_0$), if $\|x_n - x_0\|_X \to 0$, for $n \to +\infty$. This is the so-called *strong convergence*.
We say that $\{x_n\}$ is a *Cauchy sequence* if

$$\forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} \quad \text{s.t.} \quad \|x_n - x_m\|_X < \varepsilon \qquad \forall n, m \geq n_0 \ .$$

We observe that every convergent sequence is a Cauchy sequence, while, in general, the converse is not true.

Definition 1.9. We say that X is a *complete normed space* if every Cauchy sequence is convergent in X. In this case X is called *Banach space*.

Given a Banach space X (with finite or infinite dimension), we denote by X^* its dual space; i.e., the vector space of all the linear and continuous functionals on X, and by $\langle \cdot, \cdot \rangle$ the canonical duality between X and X^*. For every functional $x^* \in X^*$, $\|x^*\|_{X^*}$ denotes the dual norm in the space X^*; i.e.,

$$\|x^*\|_{X^*} = \sup_{\substack{x \in X \\ \|x\|_X \leq 1}} |\langle x^*, x \rangle| \ .$$

Finally, we denote by X^{**} the bidual space of X (i.e. the dual space of X^*). It is well known that X can be identified with a subspace of X^{**}. We will say that X is a reflexive space if X coincide with its bidual space X^{**}.

Obviously, using the norm defined on X^*, we can introduce on this space, as already done for X, a notion of strong convergence. Indeed, we say that the sequence $\{x_n^*\} \subseteq X^*$ strongly converges to $x^* \in X^*$ for $n \to +\infty$ (and we write $x_n^* \to x^*$), if $\|x_n^* - x^*\|_{X^*} \to 0$, for $n \to +\infty$. The space X^* endowed with this norm is a Banach space.

Definition 1.10. Let $\{x_n\} \subseteq X$ be a given sequence. Let $x \in X$; we say that $\{x_n\}$ *weakly* converges to x for $n \to +\infty$ (and we write $x_n \rightharpoonup x$), if $\langle x^*, x_n \rangle \to \langle x^*, x \rangle$, for $n \to +\infty$ and for every $x^* \in X^*$.
Let $\{x_n^*\} \subseteq X^*$ be a given sequence. We say that x_n^* converges to $x^* \in X^*$ $*$-*weakly* for $n \to +\infty$ (and we write $x_n^* \overset{*}{\rightharpoonup} x^*$), if $\langle x_n^*, x \rangle \to \langle x^*, x \rangle$, for $n \to +\infty$ and $\forall x \in X$.

We remark that in infinite dimensional normed spaces the weak convergence is, in general, strictly weaker than the strong convergence. In particular, if X is a reflexive Banach space of infinite dimension, it is possible to find weakly convergent sequences which do not strongly converge (see Theorem 1.27 and subsequent considerations). For example, the boundary ∂B_1 of the unit sphere is closed in the sense of the Definition 1.4, but we can find sequences $\{x_n\} \subset \partial B_1$ (i.e. $\|x_n\| = 1$) such that $x_n \rightharpoonup x$ with $\|x\| < 1$ (see [17, Chp. III, Remark 4]). On the other hand, there are examples, although rare, of "pathological" infinite-dimensional spaces (for example the space l^1), where all the weakly converging sequences are also strongly converging. On the contrary, these two notions of convergence always coincide in finite dimensional spaces.

We also note that, in general, the weak convergence is not metrizable; however, this is possible on the unit sphere (or on bounded sets), if X^* is separable. Similarly, if X is separable, the same property applies to the $*$-weak convergence in X^*.

The following result is a direct consequence of the well-known Banach-Steinhaus Theorem.

Theorem 1.11. *Let X be a Banach space. Let $\{x_n\} \subseteq X$ be a given sequence and assume that there exists $x \in X$ such that $x_n \rightharpoonup x$. Then there exists a real number $c > 0$ such that $\|x_n\| \leq c$, for every $n \in \mathbb{N}$.*
Let $\{x_n^*\} \subseteq X^*$ be a given sequence and assume that there exists $x^* \in X^*$ such that $x_n^* \overset{*}{\rightharpoonup} x^*$. Then there exists a real number $c > 0$ such that $\|x_n^*\|_{X^*} \leq c$, for every $\in \mathbb{N}$.

Now we will state some relations the strong and the weak or $*$-weak convergence.

Theorem 1.12. *Let X be a Banach space. Let $\{x_n\} \subseteq X$, $\{x_n^*\} \subseteq X^*$, $x \in X$ and $x^* \in X^*$ be given.*

(i) If $x_n \to x$ (strongly), then $x_n \rightharpoonup x$ (weakly).
(ii) If $x_n^* \to x^*$ (strongly), then $x_n^* \overset{*}{\rightharpoonup} x^*$ ($*$-weakly).
(iii) If $x_n \rightharpoonup x$, then $\|x\|_X \leq \liminf_{n\to+\infty} \|x_n\|_X$.
(iv) If $x_n^* \overset{*}{\rightharpoonup} x^*$, then $\|x^*\|_{X^*} \leq \liminf_{n\to+\infty} \|x_n^*\|_{X^*}$.

Theorem 1.13. *Let X be a Banach space. Assume that $\{x_n\} \subseteq X$ is a weakly converging sequence to $x \in X$ and $\{x_n^*\} \subseteq X^*$ is a strongly converging sequence to $x^* \in X^*$. Then $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.*
Similarly, assume that $\{x_n\} \subseteq X$ is a strongly converging sequence to $x \in X$ and that $\{x_n^*\} \subseteq X^*$ is a $*$-weakly converging sequence to $x^* \in X^*$. Then $\langle x_n^*, x_n\rangle \to \langle x^*, x\rangle$.

We recall some results concerning compactness. It is well known that in an infinite-dimensional space, the unit sphere is never compact with respect to the strong convergence; indeed, this is a special property of the finite-dimensional spaces (see [17, Theorem VI.5]). However, as follows by next theorems, the compactness of the unit sphere holds with respect to the weak convergence.

Theorem 1.14. *Let X be a separable Banach space. Then the unit sphere $B_1(0)$ of X^* is compact with respect to the $*$-weak convergence; i.e., if $\{x_n^*\} \subseteq X^*$ and there exists a real number $c > 0$ such that $\|x_n^*\|_{X^*} \leq c$, for every $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_j}^*\}$ from $\{x_n^*\}$ and a point $x^* \in X^*$ such that $x_{n_j}^* \overset{*}{\rightharpoonup} x^*$, for $j \to +\infty$.*

A similar result holds in the space X, as stated in the next theorem.

Theorem 1.15. *Let X be a reflexive Banach space. Let $\{x_n\} \subseteq X$ and assume that there exists a real number $c > 0$ such that $\|x_n\|_X \leq c$, for every $n \in \mathbb{N}$. Then there exists a subsequence $\{x_{n_j}\}$ from $\{x_n\}$ and a vector $x \in X$ such that $x_{n_j} \rightharpoonup x$, for $j \to +\infty$.*

We conclude this first section analyzing the relationship between weak and strong convergence on convex sets.

Definition 1.16. Let X be a vector space. A subset $C \subseteq X$ is called *convex* if for every $x, y \in X$ and every $\alpha \in [0,1]$ we have $\alpha x + (1-\alpha)y \in X$.

Theorem 1.17. (Mazur Lemma) *Let X be a Banach space. Let $\{x_n\} \subseteq X$ and assume that there exists $x \in X$ such that $x_n \rightharpoonup x$. Then there exists a convex combination of elements of the sequence $\{x_n\}$ strongly converging to x, i.e. for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $\alpha_i \in [0,1]$, $i = 1, \ldots, n$, such that*

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad \text{e} \qquad \|\sum_{i=1}^{n} \alpha_i x_i - x\|_X < \varepsilon \ .$$

As a consequence of the previous result, we obtain the following corollary.

Corollary 1.18. *Let X be a Banach space. Let $C \subseteq X$ be a convex set. Then C is closed (with respect to the strong convergence) if and only if for any sequence $\{x_n\} \subset C$ weakly converging to a point $x \in X$, we have that $x \in C$.*

Let us conclude this section by recalling the definition of *Hilbert space*.

Definition 1.19. Let X be a vector space. We define on $X \times X$ a *scalar product*, i.e. a bilinear form $(\cdot,\cdot) : X \times X \to \mathbb{R}$ such that
(i) (*Null property*) $(x,x) \geq 0 \ \forall x \in X$ and $(x,x) = 0 \iff x = 0$;

(ii) (*Symmetry*) $(x,y)=(y,x)$ $\forall x,y\in X$;
(iii) (*Linearity*) $(\lambda x+\mu y,z)=\lambda(x,z)+\mu(y,z)$ for every $x,y,z\in X$ and every $\lambda,\mu\in\mathbb{R}$.

A space with a scalar product is, in particular, a normed space with respect to the norm defined by $\|x\|_X=\sqrt{(x,x)}$. On the contrary, a normed space is not in general a space with scalar product, unless the norm satisfies the *parallelogram identity*; i.e.

$$\|x+y\|_X+\|x-y\|_X=2(\|x\|_X^2+\|y\|_X^2)\qquad\forall x,y\in X\,.$$

If X is also complete with respect to the norm induced by the scalar product, it is called a *Hilbert space*. In a Hilbert space it is possible to introduce the notion of *orthogonality*; i.e., two vectors $x,y\in X$ are orthogonal if $(x,y)=0$. Moreover, the well-known *Cauchy-Schwartz inequality* holds

$$|(x,y)|\le\|x\|_X\|y\|_X\qquad\forall x,y\in X\,.$$

Finally, we recall that every Hilbert space is always reflexive.

1.2. **Lebesgue summable functions and Sobolev spaces.** Let $\Omega\subseteq\mathbb{R}^N$ be an open set, $1\le p<+\infty$ and $f:\Omega\to\mathbb{R}^M$ be a measurable function. We say that f is a Lebesgue p-summable function (and we write $f\in L^p(\Omega;\mathbb{R}^M)$) if

$$\|f\|_p:=\left(\int_\Omega|f(x)|^p\,dx\right)^{1/p}<+\infty\,.$$

Similarly, we say that f is essentially bounded (and we write $f\in L^\infty(\Omega;\mathbb{R}^M)$) if

$$\|f\|_\infty:=\inf\{\alpha\ :\ |f(x)|\le\alpha\text{ q.o. in }\Omega\}<+\infty\,.$$

In particular, we say that $f\in L^p_{loc}(\Omega;\mathbb{R}^M)$, $1\le p\le+\infty$, if $f\in L^p(A;\mathbb{R}^M)$, for any open set $A\subset\subset\Omega$.
The L^p-spaces endowed with the norms defined above are Banach spaces and, for $1\le p<+\infty$, they are also separable. In particular, for $p=2$, $L^2(\Omega;\mathbb{R}^M)$ is a Hilbert space with the scalar product given by

$$(f,g)=\int_\Omega f(x)\,g(x)\,dx\qquad\forall f,g\in L^2(\Omega;\mathbb{R}^M)\,.$$

Obviously, a sequence $\{f_n\}\subseteq L^p(\Omega,\mathbb{R}^M)$ strongly converges to $f\in L^p(\Omega;\mathbb{R}^M)$, i.e. $f_n\to f$, if $\|f_n-f\|_p\to0$, for $n\to+\infty$.
Let $1\le p\le+\infty$ be given; we denote by p' the conjugate exponent of p, i.e. $1/p+1/p'=1$ if $1<p<+\infty$, $p'=+\infty$ if $p=1$ and $p'=1$ if $p=+\infty$. If $1\le p<+\infty$, it is possible to prove that $L^{p'}$ is the dual space of L^p, while L^1 is strictly contained in the dual of L^∞. Moreover, if $1<p<+\infty$, the space L^p is also reflexive.
Let us now specialize in this context the notions of weak and $*$-weak convergence. For the sake of simplicity, we assume $M=1$ (and then write $L^p(\Omega)$ instead of $L^p(\Omega;\mathbb{R})$). The general case can be obtained reasoning by components.
Let, firstly, $1\le p<+\infty$; then $f_n\rightharpoonup f$ in $L^p(\Omega)$ if

$$\int_\Omega f_n(x)g(x)\,dx\to\int_\Omega f(x)g(x)\,dx\qquad\forall g\in L^{p'}(\Omega)\,.$$

Analogously, we have $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$ if

$$\int_\Omega f_n(x)g(x)\ dx \to \int_\Omega f(x)g(x)\ dx \qquad \forall g \in L^1(\Omega)\ .$$

Theorem 1.20. *Let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ strongly converging to $f \in L^p(\Omega)$, $1 \le p \le +\infty$. Then*
(i) $f_n \rightharpoonup f$ in $L^p(\Omega)$ if $1 \le p < +\infty$ and $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$;
(ii) $\|f_n\|_p \to \|f\|_p$, per $1 \le p \le +\infty$.

Proof. Clearly, (i) follows from Theorem 1.12, while (ii) is a direct consequence of the triangular inequality. □

In the case where $1 < p < +\infty$, we can prove also the opposite result, as follows by next theorem.

Theorem 1.21. *Assume $1 < p < +\infty$ and let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$. If $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ in $L^p(\Omega)$.*

Theorem 1.22. *Let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ strongly converging to $f \in L^p(\Omega)$, $1 \le p \le +\infty$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a set $N \subset \Omega$ with zero Lebesgue measure, such that $f_{n_k}(x) \to f(x)$, for every $x \in \Omega \setminus N$; i.e., the subsequence $\{f_{n_k}\}$ pointwise converges to f a.e. in Ω.*

We recall here some relevant results concerning the passage to the limit under the Lebesgue integral.

Theorem 1.23.
(i)(Fatou Lemma) *Let $\{f_n\}$ be a sequence of measurable functions defined on Ω, such that $f_n(x) \ge 0$, for a.e. $x \in \Omega$ and every $n \in \mathbb{N}$. Then*

$$\int_\Omega \left[\liminf_{n\to+\infty} f_n(x)\right]\ dx \le \liminf_{n\to+\infty} \int_\Omega f_n(x)\ dx\ .$$

(ii) (Beppo-Levi Theorem) *Let $\{f_n\}$ be a sequence of measurable functions defined on Ω, such that $0 \le f_n(x) \le f_{n+1}(x)$, for a.e. $x \in \Omega$ and every $n \in \mathbb{N}$. Then*

$$\int_\Omega f_n(x)\ dx \to \int_\Omega f(x)\ dx \qquad \text{where} \qquad f(x) = \sup_{n\in\mathbb{N}} f_n(x)\ .$$

(iii) (Dominated convergence theorem) *Let $\{f_n\}$ be a sequence of measurable functions defined on Ω converging pointwise a.e. to a given function f. Assume that for a.e. $x \in \Omega$ and every $n \in \mathbb{N}$ we have $|f_n(x)| \le g(x)$, where $g \in L^1(\Omega)$. Then, $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$, for $n \to +\infty$.*

From Theorems 1.11, 1.14 e 1.15 we immediately obtain the following result.

Theorem 1.24. *Assume that $\{f_n\}$ is a weakly converging sequence in $L^p(\Omega)$, $1 \le p < +\infty$, (respectively, a $*$-weakly converging sequence in $L^\infty(\Omega)$). Then it is bounded uniformly with respect to $n \in \mathbb{N}$.*
Assume that $\{f_n\}$ is a uniformly bounded sequence in $L^p(\Omega)$, $1 < p \le +\infty$. Then there exists a function $f \in L^p(\Omega)$ and a subsequence of $\{f_n\}$ weakly converging to f, for $1 < p < +\infty$ (respectively $*$-weakly converging to f for $p = +\infty$).

We observe that, since L^2 is a Hilbert space, by the Cauchy-Schwarz inequality we get

$$|(f,g)| = \left|\int_\Omega f(x)g(x)\ dx\right| \le \int_\Omega |f(x)g(x)|\ dx \le \|f\|_2\|g\|_2 \qquad \forall f,g \in L^2(\Omega)\ ,$$

which implies, in particular, that the product of two L^2-functions gives an L^1-function. This result can be generalized to the case of the product of two functions belonging to Lebesgue spaces in duality, as follows from next theorems.

Theorem 1.25. (Hölder inequality) *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $1 \le p \le +\infty$ and p' be the conjugate exponent of p. Then for every $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, we have that $uv \in L^1(\Omega)$ and*

$$\left|\int_\Omega u(x)v(x)\ dx\right| \le \int_\Omega |u(x)v(x)|\ dx \le \|u\|_p\|v\|_{p'}\ .$$

Theorem 1.26. (Young inequality) *Let $\Omega \subset \mathbb{R}^N$ be an open set, let $1 < p < +\infty$ and p' be the conjugate exponent of p. Then, for every $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, we have that $uv \in L^1(\Omega)$ and*

$$\left|\int_\Omega u(x)v(x)\ dx\right| \le \int_\Omega |u(x)v(x)|\ dx \le \frac{1}{p\,\delta^p}\|u\|_p^p + \frac{\delta^{p'}}{p'}\|v\|_{p'}^{p'} \qquad \forall \delta > 0\ .$$

Now let us recall a useful result concerning the weak convergence of periodic functions. To this purpose, denote by Y the unit N-dimensional open square $(0,1)^N$. Assume for the sake of simplicity that f is a Y-periodic function (though the result holds for general periodic functions, up to read the symbol $\overline{f}$ as the mean value of the function f on its period).

Theorem 1.27. *Let $f \in L^p(Y)$, $1 \le p \le +\infty$, be a function which is periodically extended on the whole of $\mathbb{R}^N$. Set $f_\varepsilon(x) = f(x/\varepsilon)$; then, for $\varepsilon \to 0^+$, we have*

$$f_\varepsilon \rightharpoonup \overline{f} := \int_Y f(x)\ dx \qquad \text{se } 1 \le p < +\infty\ ;$$

$$f_\varepsilon \overset{*}{\rightharpoonup} \overline{f} := \int_Y f(x)\ dx \qquad \text{se } p = +\infty\ .$$

When $f(x) = \sin x$, we obtain that $\sin(x/\varepsilon) \overset{*}{\rightharpoonup} 0$ in $L^\infty(0,2\pi)$, which is the well-known Riemann-Lebesgue Lemma. This implies, in particular, that the sequence $\{\sin(nx)\}$ is an example of a weakly, but obviously not strongly, converging sequence.

Lemma 1.28. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $f \in L^1_{loc}(\Omega)$. Assume that*

$$\int_\Omega f(x)\phi(x)\ dx = 0 \qquad \forall \phi \in \mathcal{C}_0^\infty(\Omega)\ .$$

Then $f = 0$ a.e. in Ω.

Corollary 1.29. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and $f \in L^1_{loc}(\Omega)$. Assume that*

$$\int_\Omega f(x)\phi(x)\ dx = 0 \tag{1.1}$$

for all functions $\phi \in \mathcal{C}_0^\infty(\Omega)$ *such that* $⨍_\Omega \phi \, dx = \frac{1}{|\Omega|}\int_\Omega \phi \, dx = 0$. *Then there exists a constant* $c \in \mathbb{R}$ *such that* $f = c$ *a.e. in* Ω.

Proof. Let $f \in L^1_{loc}(\Omega)$, $\psi \in \mathcal{C}_0^\infty(\Omega)$ and $\Psi \in \mathcal{C}_0^\infty(\Omega)$ with $⨍_\Omega \Psi \, dx = 1$. Then

$$\begin{aligned}\int\limits_\Omega \left[f - ⨍_\Omega f\Psi \, dy \right] \psi \, dx &= \int\limits_\Omega f\psi \, dx - \left(⨍_\Omega f\Psi \, dy \right) \left(\int\limits_\Omega \psi \, dx \right) \\ &= \int\limits_\Omega f \left[\psi - \Psi ⨍_\Omega \psi \, dy \right] \, dx = 0\end{aligned}$$

since, setting $\phi = \psi - \Psi ⨍_\Omega \psi \, dy$, we have that $\phi \in C_0^\infty(\Omega)$ and $⨍_\Omega \phi \, dx = 0$. Therefore, by Lemma 1.28 we obtain $f - ⨍_\Omega f\Psi \, dy = 0$ a.e. in Ω, i.e. the thesis is achieved with $c = ⨍_\Omega f\Psi \, dy$. $\square$

We conclude this chapter by recalling the main properties of functions which admit derivatives in a weak sense. First, given $k \in \mathbb{N}$ and $1 \le p \le +\infty$, we denote by $W^{k,p}(\Omega;\mathbb{R}^M)$ the space of measurable functions whose distributional derivatives up to order k belongs to the space L^p. This is a Banach space if it is equipped with the norm

$$\|f\|_{k,p} := \left(\sum_{\alpha=0}^{k} \|\nabla^\alpha f\|_p^p \right)^{1/p} \qquad \text{se } 1 \le p < +\infty \; ;$$

$$\|f\|_{k,\infty} := \max_{0 \le \alpha \le k} \|\nabla^\alpha f\|_\infty \qquad \text{se } p = +\infty \; ;$$

where $\nabla^\alpha f$ is the matrix of the α-th weak derivatives.
As usual, we denote by $H^k(\Omega;\mathbb{R}^M)$ the space $W^{k,2}(\Omega;\mathbb{R}^M)$, which is a Hilbert space. In particular, for $k = 1$ and $M = 1$, on $H^1(\Omega)$ it is defined the scalar product

$$(f, g) = \int\limits_\Omega f(x)\, g(x) \, dx + \int\limits_\Omega \nabla f(x) \, \nabla g(x) \, dx \qquad \forall f, g \in H^1(\Omega) \, .$$

Moreover, for $1 \le p < +\infty$, $W_0^{k,p}(\Omega;\mathbb{R}^M)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega;\mathbb{R}^M)$ with respect to the norm of $W^{k,p}$ and $H_0^k(\Omega;\mathbb{R}^M) = W_0^{k,2}(\Omega;\mathbb{R}^M)$. The space $W_0^{k,p}(\Omega;\mathbb{R}^M)$, equipped with the norm of $W^{k,p}$, is a Banach space. In the case where $\Omega = (a, b)$ is a real interval, $W^{1,1}(a, b)$ is identified with the space of absolutely continuous functions on (a, b).
We recall that, if p' is the conjugate exponent of p, $W^{-k,p'}(\Omega;\mathbb{R}^M)$ is the dual space of $W_0^{k,p}(\Omega;\mathbb{R}^M)$; in particular, for $k = 1$ and $M = 1$, $W^{-1,p'}(\Omega)$ is the dual space of $W_0^{1,p}(\Omega)$ (as usual, when $p = 2$, $H^{-1}(\Omega)$ denotes the dual space $W_0^{-1,2}(\Omega)$) and each element $\overline{g} \in W^{-1,p'}(\Omega)$ can be represented as follows

$$\overline{g} = g - \sum_{j=1}^{N} \partial_j g_j \, ,$$

where $g, g_j \in L^{p'}(\Omega)$ and the derivatives are taken in the distributional sense. Moreover, if Ω is bounded, $h \le k$ and $q < p$, then $W^{k,p}(\Omega;\mathbb{R}^M) \subset W^{h,q}(\Omega;\mathbb{R}^M)$.
We recall also that for $1 \le p < +\infty$, $W^{k,p}(\Omega;\mathbb{R}^M)$ is separable and for $1 < p < +\infty$ it is also a reflexive space. Finally, if Ω is regular (for example $\partial\Omega \in \mathcal{C}^1$) and $1 \le p < +\infty$, we have that $\mathcal{C}^\infty(\Omega;\mathbb{R}^M)$ is dense in $W^{k,p}(\Omega;\mathbb{R}^M)$, with respect to the norm defined above and, if Ω is bounded, $W^{1,\infty}(\Omega;\mathbb{R}^M)$ is identified with the space of Lipschitz functions.

Theorem 1.30. (Embedding Theorem) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Then the following assertion hold.*
(i) If $1 \le p < N$, $W_0^{1,p}(\Omega;\mathbb{R}^M) \subset L^q(\Omega;\mathbb{R}^M)$, for every $1 \le q \le \frac{Np}{N-p}$ and the embedding is compact for $1 \le q < \frac{Np}{N-p}$.
(ii) If $p = N$, $W_0^{1,p}(\Omega;\mathbb{R}^M) \subset L^q(\Omega;\mathbb{R}^M)$, for every $1 \le q < +\infty$ and the embedding is compact.
(iii) If $p > N$, $W_0^{1,p}(\Omega;\mathbb{R}^M) \subset \mathcal{C}_0(\overline{\Omega};\mathbb{R}^M)$ and the embedding is compact.

The previous result still hold if we replace $W_0^{1,p}(\Omega;\mathbb{R}^M)$ with $W^{1,p}(\Omega;\mathbb{R}^M)$, but assuming suitable regularity properties of the boundary of Ω, for example $\partial\Omega$ of class $\mathcal{C}^1$.
We observe that Theorem 1.30, in particular, implies that, if $1 \le p \le +\infty$ and $f_n \rightharpoonup f$ in $W_0^{1,p}(\Omega;\mathbb{R}^M)$, then $f_n \to f$ in $L^p(\Omega;\mathbb{R}^M)$. Moreover, if $p > 1$, we have the following crucial compactness property of the space $W^{1,p}(\Omega;\mathbb{R}^M)$.

Theorem 1.31. *Let $p > 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded open set. Assume that $\{f_n\} \subseteq W^{1,p}(\Omega;\mathbb{R}^M)$ is a bounded sequence; i.e., there exists a constant $c > 0$ such that*

$$\|f_n\|_{1,p} \le c \qquad \forall n \in \mathbb{N}\,.$$

Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in W^{1,p}(\Omega;\mathbb{R}^M)$ such that $f_{n_k} \rightharpoonup f$ in $W^{1,p}(\Omega;\mathbb{R}^M)$, if $p \ne +\infty$, or $f_{n_k} \overset{*}{\rightharpoonup} f$ in $W^{1,\infty}(\Omega;\mathbb{R}^M)$, and hence $f_{n_k} \to f$ strongly in $L^p(\Omega;\mathbb{R}^M)$.

Unfortunately, when $p = 1$, the space $W^{1,1}(\Omega;\mathbb{R}^M)$ fails to be compact with respect to the weak convergence, since it is not a reflexive space. This problem will be overcome with the introduction of the space BV (see Section 1.3 and Theorem 1.38).

Theorem 1.32. (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded and connected set, with boundary of class $\mathcal{C}^1$ and $1 \le p < +\infty$. Then,*
(i) there exists a constant $c > 0$ such that, for every $f \in W_0^{1,p}(\Omega;\mathbb{R}^M)$, we have

$$\|f\|_p \le c\|\nabla f\|_p\ ;$$

(ii) there exists a constant $c > 0$ such that, for every $f \in W^{1,p}(\Omega;\mathbb{R}^M)$, we have

$$\|f - \overline{f}\|_p \le c\|\nabla f\|_p\ ,$$

where $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx$.

Remark 1.33. We note that property (i) holds, in general, for any bounded open set (not necessarily connected or with boundary of class $\mathcal{C}^1$), while property (ii) holds also for $p = +\infty$.

For the sake of simplicity, we give the proof only for $p > 1$.

Proof.
(i) We assume by contradiction that the thesis does not hold, i.e. for every $n > 0$ there exists $f_n \in W_0^{1,p}(\Omega;\mathbb{R}^M)$ such that

$$\|f_n\|_p > n\|\nabla f_n\|_p\ . \tag{1.2}$$

We may suppose, without loss of generality, that $\|f_n\|_p = 1$; therefore, from inequality (1.2) we obtain that

$$\|f_n\|_{1,p} \le 2 \qquad \text{e} \qquad \|\nabla f_n\|_p \to 0 \quad \text{per } n \to +\infty\ . \tag{1.3}$$

From Theorem 1.15 it follows that there exists a function $f \in W_0^{1,p}(\Omega;\mathbb{R}^M)$ and a subsequence, still denoted $\{f_n\}$, such that $f_n \rightharpoonup f$ in $W_0^{1,p}(\Omega;\mathbb{R}^M)$. From Theorem 1.30 we also

obtain that $f_n \to f$ in $L^p(\Omega;\mathbb{R}^M)$. Moreover, since $\|f_n\|_p = 1$, it follows $\|f\|_p = 1$. On the other hand, from (1.3) and Theorem 1.12, we obtain that $\|\nabla f\|_p \leq \liminf \|\nabla f_n\|_p = 0$, i.e. $f = constant$ in $W_0^{1,p}(\Omega;\mathbb{R}^M)$ and then $f = 0$ a.e. in Ω. This contradicts the fact that $\|f\|_p = 1$ and so the thesis follows.
(ii) In order to prove the second inequality, we proceed in a similar way, replacing f_n with $g_n = f_n - \overline{f}_n$. We obtain that $g_n \to g$ in $L^p(\Omega;\mathbb{R}^M)$, with $\|g\|_p = 1$ and $g = constant$ in $W^{1,p}(\Omega;\mathbb{R}^M)$. On the other hand, from the strong convergence we also obtain

$$0 = \frac{1}{|\Omega|}\int_\Omega [f_n(x) - \overline{f}_n]\, dx = \frac{1}{|\Omega|}\int_\Omega g_n(x)\, dx \to \frac{1}{|\Omega|}\int_\Omega g(x)\, dx$$

which implies, since g is a constant vector of $\mathbb{R}^M$, $g = 0$ a.e. in Ω, in contraddiction with the fact that $\|g\|_p = 1$. □

We observe that the Poincaré inequality implies, in particular, that on $W_0^{1,p}(\Omega;\mathbb{R}^M)$, the L^p norm of the gradient is equivalent to the standard norm of $W^{1,p}$. Then this space endowed with the L^p-norm of the gradient is still a Banach space.
Finally, we denote by $W^{1,p}_{loc}(\Omega)$ the space of functions belonging to $W^{1,p}(A)$, for every open set $A \subset\subset \Omega$.

1.3. **The space** BV.

Definition **1.34**. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. The space $\mathrm{BV}(\Omega)$ is defined as the space of all functions $f : \Omega \to \mathbb{R}$ belonging to $L^1(\Omega)$ whose distributional gradient Df is an $\mathbb{R}^N$-valued Radon measure (i.e., $Df \in \mathcal{M}(\Omega;\mathbb{R}^N)$) with total variation $|Df|$ bounded in Ω. We recall that

$$|Df|(\Omega) = \sup\left\{ \int_\Omega f(x)\,\mathrm{div}\phi(x)\,dx \ : \ \phi \in C^1_c(\Omega,\mathbb{R}^N),\ \sup_{x\in\Omega}|\phi(x)| = 1 \right\}.$$

We denote by $D^a f$ and $D^s f$ the *absolutely continuous* and the *singular part* of the measure Df with respect to the Lebesgue measure. We recall that $D^a f$ and $D^s f$ are mutually singular, moreover we can write

$$Df = D^a f + D^s f \qquad \text{and} \qquad D^a f = \nabla f\ \mathcal{L}^N,$$

where ∇f is the *Radon-Nikodým derivative* of $D^a f$ with respect to the Lebesgue measure. In particular,

$$D^s f = D^c f + (f^+ - f^-)\nu_f\ \mathcal{H}^{N-1}\lfloor J_f$$

where J_f is a *countably* $\mathcal{H}^{N-1}$*-rectifiable* Borel set (see [9, Definition 2.57]) and ν_f is the *approximate normal vector* to J_f. The set J_f is known as the set of the *approximate jump points* of f and ν_f is the direction of the jump of f. The remaining part $D^c u$ is called the *Cantor part* of Du.

We recall the main properties of the space $\mathrm{BV}(\Omega)$:
(i) $\mathrm{BV}(\Omega)$ is a Banach space endowed with the norm

$$\|f\|_{\mathrm{BV}} = \|f\|_1 + |Df|(\Omega)\,;$$

(ii) $\mathrm{BV}(\Omega) \subset L^1(\Omega)$ and the inclusion is strict;
(iii) $\mathrm{BV}(\Omega)$ is not a separable space.

Definition **1.35**. ($*$-weak convergence) Let $\{f_n\} \subseteq \mathrm{BV}(\Omega)$ and $f \in \mathrm{BV}(\Omega)$ be given. We say that the sequence $\{f_n\}$ $*$-weakly converges to $f \in \mathrm{BV}(\Omega)$ if

$$f_n \to f \quad \text{strongly in } L^1(\Omega)\,;$$
$$\int_\Omega \phi\, Df_n \to \int_\Omega \phi\, Df \quad \forall \phi \in C_0(\Omega)\,.$$

Definition **1.36**. Let $\{f_n\} \subseteq \mathrm{BV}(\Omega)$ and $f \in \mathrm{BV}(\Omega)$ be given. We say that the sequence $\{f_n\}$ strictly converges to $f \in \mathrm{BV}(\Omega)$ if

$$f_n \to f \quad \text{strongly in } L^1(\Omega)\,;$$
$$|Df_n|(\Omega) \to |Df|(\Omega)\,.$$

Theorem 1.37. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with regular boundary. Let $f \in \mathrm{BV}(\Omega)$. Then there exists a sequence of functions $\{f_n\} \subseteq \mathcal{C}^\infty(\Omega)$ which strictly converges to f.*

Theorem 1.38. (Rellich Theorem) *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with regular boundary. Let $\{f_n\} \subseteq \mathrm{BV}(\Omega)$ be such that there exists a constant $c > 0$ satisfying*

$$\|f_n\|_{\mathrm{BV}} \leq c \qquad \forall n \in \mathbb{N}\,.$$

Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function $f \in \mathrm{BV}(\Omega)$ such that

$$f_{n_k} \to f \quad \text{strongly in } L^1(\Omega)\,.$$

Remark 1.39. Clearly $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ and, thanks to Theorems 1.38 and 1.40, $\mathrm{BV}(\Omega)$ has the compactness property which fails to hold in $W^{1,1}(\Omega)$.

As a consequence of Theorems 1.11 and 1.38, we obtain the following criterion for the $*$-weak convergence.

Theorem 1.40. *Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with regular boundary and $\{f_n\} \subset \mathrm{BV}(\Omega)$ be a given sequence. Then $\{f_n\}$ $*$-weakly converges to $f \in \mathrm{BV}(\Omega)$ if and only if $\{f_n\}$ is bounded in $\mathrm{BV}(\Omega)$ and converges to f strongly in $L^1(\Omega)$.*

Theorem 1.41. (Poincaré inequality) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded and connected set with regular boundary. Let $f \in \mathrm{BV}(\Omega)$ be such that $supp(f) \subset\subset \Omega$. Then there exists a constant $c > 0$, depending only on the dimension N and the set Ω such that*

$$\int_\Omega |f|\, dx \leq c|Df|(\Omega)\,.$$

For many applications, it is useful to introduce some special subspaces of BV, such as the space SBV and the space SBV^2.

Definition **1.42**. The space $\mathrm{SBV}(\Omega)$ is defined as the subspace of all $f \in \mathrm{BV}(\Omega)$ such that

$$Df = \nabla f\, \mathcal{L}^N + (f^+ - f^-)\nu_f\, \mathcal{H}^{N-1} \lfloor J_f\,;$$

i.e. $D_c f = 0$.

Definition **1.43**. The space $\mathrm{SBV}^2(\Omega)$ is defined as the subspace of all $f \in \mathrm{SBV}(\Omega)$ such that

$$\int_\Omega |\nabla f|^2\, dx < +\infty\,, \qquad \text{and} \qquad \mathcal{H}^{N-1}(J_f) < +\infty\,.$$

We point out that $\mathrm{SBV}(\Omega)$ is not a closed space; indeed, there exists sequences of SBV-functions $*$-weakly converging to functions belonging to $\mathrm{BV}(\Omega) \setminus \mathrm{SBV}(\Omega)$. In particular, this means that sequences of functions whose derivatives have only the absolutely continuous and the jump part, can display in the limit also the Cantor part. On the contrary, the space $\mathrm{SBV}^2(\Omega)$ has an essential compactness property. Indeed, it is possible to prove that, if $\Omega \subset \mathbb{R}^N$ is a bounded set with Lipschitz boundary and $\{f_n\} \subseteq \mathrm{SBV}^2(\Omega)$ is a sequence of function such that

$$\sup_{n \in \mathbb{N}} \left(\|f_n\|_\infty \, dx + \int_\Omega |\nabla f_n|^2 \, dx + \mathcal{H}^{N-1}(J_{f_n}) \right) \leq c \tag{1.4}$$

for a proper constant $c > 0$, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function f, still belonging to $\mathrm{SBV}^2(\Omega)$, such that $f_{n_k} \to f$, strongly in $L^1(\Omega)$, $\nabla f_{n_k} \rightharpoonup \nabla f$, weakly in $L^2(\Omega)$, and $\mathcal{H}^{N-1}(J_f) \leq \liminf_{n \to +\infty} \mathcal{H}^{N-1}(J_{f_{n_k}})$. In this case, the limit of sequences in $\mathrm{SBV}^2(\Omega)$ satisfying (1.4), has derivatives which cannot display the Cantor part. Moreover, the absolutely continuous part and the jump part of the derivatives of f_{n_k} converge separately to the absolutely continuous part and the jump part of the derivatives of f, so that no mixing effects between the two mutually singular parts of the derivatives can happen.

For a general survey on the topics covered in this chapter, we refer to [1], [9], [17].

2. **Partial derivatives equations and minimum problems**

2.1. **Direct Methods in Calculus of Variations.** In the following, X denotes a general Banach space which, if it is not differently specified, will be endowed with the strong convergence; i.e., the one induced by the norm. We also set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Definition **2.1**. Let $f : X \to \overline{\mathbb{R}}$. It is *convex* if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \qquad \forall \alpha \in [0,1]$$

and for each x, y such that $f(x), f(y) < +\infty$. The function f is *strictly convex* if it isn't identically $+\infty$ and if

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y) \qquad \forall \alpha \in (0,1)$$

and for each $x \neq y$ such that $f(x), f(y) < +\infty$.

Definition **2.2**. Let $f : X \to \overline{\mathbb{R}}$. It is *lower semicontinuity* (respect. *weakly sequentially lower semicontinuity*), if, for each $x \in X$,

$$f(x) \leq \liminf_{x_n \to x} f(x_n) \qquad \forall x_n \to x \text{ in } X \qquad (\text{respect. } \forall x_n \rightharpoonup x \text{ in } X).$$

In this case, we simply write that f is l.s.c. (respect. weakly l.s.c.).

Remark 2.3. We observe that, if f is convex and takes value $-\infty$ in one extremal point of a segment, it an assume finite value only in one point of this segment; then if $x_0 \in X$, f is upper bounded in a neighborhood of x_0 and $f(x_0) = -\infty$, then it is identically equal to $-\infty$ in the whole neighborhood. If f is convex and l.s.c. and it takes value $-\infty$ in a point, then it cannot be bounded.

As a consequence of the previous remark, we shall consider only functions that never assume the value $-\infty$. We recall that a function f is said *proper* if $f(x) > -\infty$ for each $x \in X$ and if there exists at least a point $\overline{x} \in X$ such that $f(\overline{x}) < +\infty$; i.e., $f : X \to \mathbb{R} \cup \{+\infty\}$ is not identically equal to $+\infty$.

We call *domain* of f the set defined by

$$dom(f) = \{x \in X \ : \ f(x) < +\infty\} \ .$$

If f is convex, its domain is a convex set. Also, if f is proper, its domain is non empty and coincides with the set of points where f is finite.
Finally, we call *epigraph* of f the set

$$epi(f) = \{(x,t) \in X \times \mathbb{R} \ : \ t \geq f(x)\} \ .$$

Theorem 2.4. *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. Then it is l.s.c. if and only if it is weakly l.s.c.*

In particular, previous theorem implies the result already stated in Theorem 1.12 (iii). In fact, the function $x \in X \mapsto \|x\|_X$ is convex and continuous (then l.s.c.), therefore it is also weakly l.s.c. Hence, for each sequence $\{x_n\} \subseteq X$ such that $x_n \rightharpoonup x$ weakly in X we have $\|x\|_X \leq \liminf \|x_n\|_X$.

We are now in the position to solve the problem

$$\min\{F(u) \ : \ u \in X\}\,, \tag{2.1}$$

where X is a Banach space and $F : X \to \overline{\mathbb{R}}$ is a given function.
We want to study under which assumptions we can obtain the existence of a (possibly unique) solution of (2.1). For our purposes, we will have that in general X is a functional

space with appropriate boundary conditions and F is an integral functional. Some classic examples are:

Example 2.5. [Dirichlet integral] Let $\Omega \subset \mathbb{R}^N$ be a bounded open set;

$$X = H^1_0(\Omega) \qquad F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx \; .$$

Example 2.6. [Area functional] Let $\Omega \subset \mathbb{R}^N$ be a bounded open set;

$$X = W^{1,1}_0(\Omega) \qquad F(u) = \int_\Omega \sqrt{1+|\nabla u|^2} \, dx \; .$$

Example 2.7. [Functional of the geometric optics] Let $I = (a,b) \subset \mathbb{R}$ be a bounded real interval;

$$X = \{u \in \mathcal{C}^1(I) \; : \; u(a) = u_0 \; , u(b) = u_1\} \qquad F(u) = \int_I g(t,u)\sqrt{1+(u')^2} \, dt \; .$$

A first approach to such problems can be done through the so-called classical methods (whose pioneers were Bernoulli and Euler). They consist in determining the critical points $u \in X$ for the functional F, i.e. the points such that $F'(u) = 0$, and then to study the successive derivatives in order to determine the nature of the critical points. These methods have the defect that they need to assume some regularity of the functional and the critical points that, in general, may not be present. A second and more recent approach has been developed from the beginning of the twentieth century by Hilbert and Lebesgue, in connection with the study of the Dirichlet integral. These methods were then generalized by Tonelli and are known as *Direct Methods of Calculus of Variations.* The approach with the direct methods is essentially based on the classical Weierstrass theorem. The main idea is, namely, to find minimizing compact sequences (from which it is then possible to extract convergent subsequences) and then exploit the continuity (or rather the l.s.c.) of the functional in order to show that the limit points are in fact minimum points for the functional itself. We observe that compactness and l.s.c. are topological properties in competition with each other; indeed, compactness properties are easily obtained in weak topologies while properties of continuity are obtained more easily in finer topologies. For this reason, it is essential, once assigned F and X, to choose the appropriate topology in which the two requirements can be balanced.
These ideas lead to the statement of the next theorem; however, we need before some definitions.

Definition 2.8. Let $F : X \to \overline{\mathbb{R}}$. We say that a sequence $\{u_n\} \subseteq X$ is a *minimizing* sequence for the functional F on X if

$$\lim_{n\to+\infty} F(u_n) = \inf_X F \; .$$

Definition 2.9. Let $F : X \to \overline{\mathbb{R}}$ and Y be an unbounded subset of X. We say that F is *coercive on* Y (or simply *coercive*, if $Y = X$), if there exists $\alpha > 0$ such that

$$\lim_{\substack{\|u\|_X\to+\infty \\ u\in Y}} \frac{F(u)}{\|u\|_X} \geq \alpha > 0 \; . \tag{2.2}$$

Theorem 2.10. *Let X be a reflexive Banach space and $F : X \to \overline{\mathbb{R}}$ be a coercive and weakly l.s.c functional on X. Then problem (2.1) admits at least one solution.*

Proof. If $F \equiv +\infty$, every point $u \in X$ is a point of minimum. Otherwise we have $\inf_X F(u) < +\infty$. Then, if $\{u_n\} \subseteq X$ is a minimizing sequence, there is $c > 0$ such that $F(u_n) \leq c$ and thus, from the coercivity, there exists $c' > 0$ such that $\|u_n\|_X \leq c'$. Therefore, every minimizing sequence is bounded. Since the space X is reflexive, we can extract a subsequence, still denoted by $\{u_n\}$, weakly converging to a point $\overline{u} \in X$ (see Teorema 1.15). From the weakly l.s.c. of F it follows

$$\inf_X F(u) \leq F(\overline{u}) \leq \liminf_{n\to+\infty} F(u_n) = \inf_X F(u) \ .$$

Then $\overline{u}$ is a minimum point. □

Corollary 2.11. *Let X be a reflexive Banach space, Y be a convex and closed subset of X and $F : Y \to \overline{\mathbb{R}}$ be a coercive functional on Y and weakly l.s.c. on X. Then the problem*

$$\min\{F(u) \ : \ u \in Y\} \ ,$$

admits at least one solution.

Proof. We proceed as in the proof of previous theorem, noting that, as $\overline{u}$ is the weakly limit of a sequence in Y, then $\overline{u} \in Y$, as Y is convex and closed (see Corollary 1.18). □

As for the uniqueness of the solution, we state the following result.

Theorem 2.12. *Let $F : X \to \overline{\mathbb{R}}$ be a strictly convex functional. Then the problem (2.1) admits at most one solution.*

Proof. By contradiction, assume that there exist $u, v \in X$, with $u \neq v$, such that

$$F(u) = F(v) = \min_{w\in X} F(w) \,.$$

Then, from the strict convexity of F it follows

$$F\left(\frac{1}{2}u + \frac{1}{2}v\right) < \frac{1}{2}F(u) + \frac{1}{2}F(v) = \min_{w\in X} F(w) \,,$$

i.e. $w = \frac{1}{2}u + \frac{1}{2}v$ is a point where F reaches a value strictly less than its minimum. Since this is a contradiction, the thesis is proved. □

Next, using Theorem 2.10 (or Corollary 2.11), we will prove the existence of solutions for the minimum problem in some particular case, which is interesting in the applications. To this end, we consider the case of integral functionals defined on spaces of summable functions or on Sobolev spaces. In the following, although not explicitly mentioned, we always assume that Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary.

Definition 2.13. Given $f : \Omega \times \mathbb{R}^M \to \mathbb{R}$, we say that it is a *Carathéodory function* if

$$\begin{array}{lll} (i) & f(\cdot, \xi) & \text{is Lebesgue measurable for every } \xi \in \mathbb{R}^M \ ; \\ (ii) & f(x, \cdot) & \text{is continuous for almost every } x \in \Omega \ . \end{array}$$

Theorem 2.14. *Let $1 < p < +\infty$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. Assume that there exist a nonnegative function $b \in L^1(\Omega)$ and a real number $\lambda > 0$ such that*

$$f(x,\xi) \geq -b(x) + \lambda|\xi|^p \qquad \text{for a.e. } x \in \Omega \ \text{ and for every } \xi \in \mathbb{R}^N \ . \tag{2.3}$$

Assume also that the function $\xi \mapsto f(x, \xi)$ is convex, for a.e. $x \in \Omega$, and that $F : W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$ is the functional defined by

$$F(u) = \int_\Omega f(x, \nabla u) \ dx \ . \tag{2.4}$$

Then, there exists $u_0 \in W_0^{1,p}(\Omega)$ such that

$$F(u_0) = \min_{u \in W_0^{1,p}(\Omega)} F(u) \ .$$

Proof. If $F \equiv +\infty$, all points of $W_0^{1,p}(\Omega)$ are minimum points and the result is trivial. Otherwise, F is a "proper" functional which is convex and, thanks to Fatou's Lemma, also l.s.c. with respect to strong convergence. Indeed, if $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ is a sequence of functions strongly converging to $u \in W_0^{1,p}(\Omega)$, by Theorem 1.22, we have that, up to a subsequence, $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω and hence, since f is a Carathéodory function (i.e. $f(x, \cdot)$ is continuous on $\mathbb{R}^N$ for a.e. $x \in \Omega$), $f(x, \nabla u_n(x)) \to f(x, \nabla u(x))$ a.e. in Ω. Moreover, by (2.3) the function $(x, \xi) \mapsto f(x, \xi) + b(x)$ is nonnegative, so that applying Fatou's Lemma it follows

$$\begin{aligned}
\liminf_{n\to+\infty} F(u_n) &= \liminf_{n\to+\infty} \int_\Omega f(x, \nabla u_n(x)) \ dx \\
&= \liminf_{n\to+\infty} \int_\Omega [f(x, \nabla u_n(x)) + b(x)] \ dx - \int_\Omega b(x) \ dx \\
&\geq \int_\Omega [f(x, \nabla u(x)) + b(x)] \ dx - \int_\Omega b(x) \ dx = \int_\Omega f(x, \nabla u(x)) \ dx = F(u) \,.
\end{aligned}$$

Then, by Theorem 2.4 F is also weakly l.s.c. on $W_0^{1,p}(\Omega)$, which is a reflexive Banach space, since $1 < p < +\infty$. In order to obtain the existence of a minimum point, it is sufficient to prove the coercivity of F on $W_0^{1,p}(\Omega)$ and then to apply Theorem 2.10. We observe that by (2.3) and by the Poicaré inequality (see Teorema 1.32) we obtain

$$\begin{aligned}
\lim_{\|u\|_{1,p}\to+\infty} \frac{\int_\Omega f(x, \nabla u) \ dx}{\|u\|_{1,p}} &\geq \lim_{\|u\|_{1,p}\to+\infty} \frac{-\int_\Omega b(x) \ dx + \lambda \|\nabla u\|_p^p}{\|u\|_{1,p}} \\
&\geq c \lim_{\|u\|_{1,p}\to+\infty} \frac{\|u\|_{1,p}^p}{\|u\|_{1,p}} = +\infty \,;
\end{aligned} \tag{2.5}$$

where the last equality is due to the fact that $p > 1$. Then F is coercive and the proof is accomplished. ☐

The same result as in Theorem 2.14 applies also to functionals $F : W_0^{1,p}(\Omega) \to \mathbb{R}$ $(1 < p < +\infty)$ of the form

$$F(u) = \int_\Omega f(x, \nabla u) \ dx - \int_\Omega gu \ dx \,, \tag{2.6}$$

where $g \in L^{p'}(\Omega)$ $(1/p + 1/p' = 1)$ and f is as in the statment of Theorem 2.14. Indeed, the second term in (2.6) is a linear continuous functional on $W_0^{1,p}(\Omega)$, hence F is still a weakly l.s.c. functional on $W_0^{1,p}(\Omega)$; moreover, recalling (2.5) and Hölder inequality, we obtain that F is also coercive. Indeed,

$$\begin{aligned}
\lim_{\|u\|_{1,p}\to+\infty} \frac{F(u)}{\|u\|_{1,p}} &\geq \lim_{\|u\|_{1,p}\to+\infty} \left[c\frac{\|u\|_{1,p}^p}{\|u\|_{1,p}} - \frac{\|g\|_{p'}\|u\|_p}{\|u\|_{1,p}} \right] = \lim_{\|u\|_{1,p}\to+\infty} \left[c\frac{\|u\|_{1,p}^p}{\|u\|_{1,p}} - \frac{\|g\|_{p'}\|u\|_{1,p}}{\|u\|_{1,p}} \right] \\
&= c \lim_{\|u\|_{1,p}\to+\infty} \frac{\|u\|_{1,p}^p}{\|u\|_{1,p}} - \|g\|_{p'} = +\infty \,.
\end{aligned}$$

Hence, Theorem 2.10 assures the existence of a minimizer. In particular, setting $p = p' = 2$ and $a_{ij} \in L^\infty(\Omega)$, $i, j = 1, \ldots, N$, we obtain the existence of a minimizer in $H_0^1(\Omega)$ for the functional

$$F(u) = \frac{1}{2}\int_\Omega a(x)\nabla u\nabla u\, dx - \int_\Omega gu\, dx\,,$$

where a is the matrix of the coefficients a_{ij}. In this last case, being the functional also strictly convex, the minimizer is unique.

The general case is treated in the next theorem, whose proof is a direct consequence of the previous arguments.

Theorem 2.15. *Let $1 < p < +\infty$, $1/p + 1/p' = 1$, $g \in L^{p'}(\Omega)$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function. Assume that there exist a nonnegative function $b \in L^1(\Omega)$ and a real number $\lambda > 0$ such that*

$$f(x,\xi) \geq -b(x) + \lambda|\xi|^p \qquad \text{for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^N\,. \tag{2.7}$$

Assume also that the function $\xi \mapsto f(x,\xi)$ is convex, for a.e. $x \in \Omega$, and that $F : W^{1,p}(\Omega) \to \overline{\mathbb{R}}$ is the functional defined by

$$F(u) = \int_\Omega f(x,\nabla u)\, dx - \int_\Omega g\, u\, dx\,. \tag{2.8}$$

Then, there exists $u_0 \in W^{1,p}(\Omega)$ such that

$$F(u_0) = \min_{u\in Y} F(u)\,,$$

where $Y = w + W_0^{1,p}(\Omega)$ and $w \in W^{1,p}(\Omega)$.

Remark 2.16. These existence results cannot be easily extended to the case $p = 1$, since the space $X = W^{1,1}(\Omega)$ (or $X = W_0^{1,1}(\Omega)$) is neither reflexive nor the dual of a separable Banach space, hence it is not true, in general, that a minimizing sequence in X is convergent, even up to a subsequence, in the same space.

2.2. **Linear elliptic equations in divergence forms.** In the following X denotes a general Hilbert space. We say that a form $a : X \times X \to \mathbb{R}$ is *bilinear* if, for every $x, y, z \in X$ and every $\lambda, \mu \in \mathbb{R}$, we have

$$a(\lambda x + \mu y, z) = \lambda a(x,z) + \mu a(y,z) \qquad \text{and} \qquad a(z, \lambda x + \mu y) = \lambda a(z,x) + \mu a(z,y)\,;$$

i.e. if it is linear in each entry. We say that a is symmetric if for every $x, y \in X$ we have $a(x,y) = a(y,x)$. Moreover, we say that the bilinear form a is continuous if there exists a constant $\Lambda > 0$ such that

$$a(x,y) \leq \Lambda\|x\|\,\|y\| \qquad \forall x,y \in X\,.$$

Definition **2.17**. Let $a : X \times X \to \mathbb{R}$ be a bilinear form. We say that it is *coercive* if there exists $\lambda > 0$ such that

$$a(x,x) \geq \lambda\|x\|^2 \qquad \forall x \in X\,.$$

A first very simple example of bilinear continuous symmetric and coercive form on X is the scalar product itself.

Let us recall the well-known Lax-Milgram Lemma, which is a crucial result in the framework of variational equations (see, for instance, [28, Section 5]).

Lemma 2.18. *Let $a : X \times X \to \mathbb{R}$ be a bilinear continuous and coercive form. Then, for every $x^* \in X^*$ there exists a unique solution $x \in X$ such that*

$$a(x,y) = \langle x^*, y\rangle \qquad \forall y \in X\,.$$

A special application of the previous result arises in the treatment of linear elliptic equations of variational type. To this purpose, set $X = H_0^1(\Omega)$, with Ω an open bounded subset of $\mathbb{R}^N$. Assume that $A = [a_{ij}]$ is a symmetric matrix with $a_{ij} \in L^\infty(\Omega)$, for every $i, j = 1, \dots, N$, and satisfying

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \qquad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N\,, \tag{2.9}$$

for two suitable constants $0 < \lambda \le \Lambda < +\infty$. Assume in addition that $f \in H^{-1}(\Omega)$ and consider the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u_0) = f(x) & \text{in } \Omega, \\ u_0 \in H_0^1(\Omega)\,. \end{cases} \tag{2.10}$$

Taking into account the weak formulation of (2.10), i.e.

$$\int_\Omega A(x)\nabla u_0(x)\nabla\phi(x)\,dx = \int_\Omega f(x)\phi(x)\,dx \qquad \forall \phi \in H_0^1(\Omega)\,,$$

and defining the bilinear symmetric form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ given by

$$a(u, v) = \int_\Omega A(x)\nabla u\nabla v\,dx\,,$$

problem (2.10) can be rewritten as

$$\begin{aligned} &\text{find } u_0 \in H_0^1(\Omega) \text{ such that} \\ &a(u_0, \phi) = \langle f, \phi\rangle \qquad \forall \phi \in H_0^1(\Omega)\,. \end{aligned}$$

By (2.9), the bilinear form is continuous and coercive on $H_0^1(\Omega)$, since

$$a(u, v) \le c\int_\Omega \nabla u\nabla v\,dx \le c\|\nabla u\|_2\|\nabla v\|_2 \le c\|u\|_{1,2}\|v\|_{1,2}$$

thanks to Hölder inequality (see Theorem 1.25), and

$$a(u, u) \ge \lambda\int_\Omega |\nabla u|^2\,dx \ge C\|u\|_{1,2} \qquad \forall u \in H_0^1(\Omega)\,,$$

thanks to Poincaré inequality (see Theorem 1.32). Hence, as a consequence of Lax-Milgram Lemma, there exists a unique solution $u_0 \in H_0^1(\Omega)$ of (2.10).

It is worthwhile to remark that the same result can be obtained in a completely different way, starting from classical well-posedeness results stated in the regular case and then using a standard regularization procedure, to cover the case of weak solutions. A similar approach can be followed also in the nonlinear case. Indeed, in order to obtain well-posedness results for nonlinear elliptic equations of variational type, one can use the Schauder fixed point theory (see for instance [28]) which assures, under classical regularity assumptions on the data of the problem, existence and uniqueness of a classical solution of the problem

$$\begin{cases} -\operatorname{div} g(x, \nabla u_0) = f & \text{in } \Omega, \\ \qquad\qquad u_0 = \varphi & \text{on } \partial\Omega. \end{cases}$$

Then, weak solutions can be obtained under weaker assumptions via a regularization procedure.

2.3. **Minimum problems and variational PdEs.** In Section 2.1 we stated some results on the existence of minimizers for integral functionals while in Section 2.2 we stated some results on the well-posedness of some variational elliptic equations. In the present section we emphasize how to connect these two problems; more precisely, we will prove that, under suitable conditions, existence of minimizers can be obtained by proving the well-posedness for a proper elliptic equation as well as well-posedness of an elliptic equation can be obtained by proving existence of minimizers for a suitable functional.

Definition **2.19**. Let X be a Banach space. Let $f : X \to \mathbb{R}$. We say that f is *Gâteaux differentiable* at $x \in X$, if for each $y \in X$ the following limit

$$\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t} =: df(x,y)\,,$$

exists and it is finite. Moreover, the functional $y \in X \mapsto df(x,y)$ has to be linear and continuous; i.e., there exists a map $df : X \to X^*$ such that $df(x,y) = \langle df(x), y\rangle$. In this case, the continuous linear functional $df(x)$ is said *differential or Gâteaux derivative* of f at the point x.

Definition **2.20**. Let $f : X \to \mathbb{R}$. We say that it is *Fréchet differentiable* at $x \in X$, if there exists a linear and continuous functional $D \in X^*$ such that

$$\lim_{y\to x} \frac{f(y)-f(x)-\langle D, y-x\rangle}{\|y-x\|_X} = 0\,.$$

We note that previous definitions are local, then it is possible to to state the notion of Gâteaux or Fréchet differentiability at a point $x \in X$ also for an extended real-valued function, up to suppose that it is finite in a neighborhood of the point x.

Remark 2.21.
(i) If f is Fréchet differentiable, then it is also Gâteaux differentiable. In this case, $D = df(x)$.
(ii) If $X = \mathbb{R}^N$ $(N > 1)$, the Gâteaux differentiability of f corresponds to the derivability of f along each direction, together with the request that the directional derivative depends linearly on the direction. In turn, the Fréchet differentiability corresponds to the classical differentiability of functions of several variables.

It is well known that, for functions defined on $\mathbb{R}^N$, the derivability does not imply, in general, the differentiability, unless some continuity assumptions on the partial derivatives are considered. Similarly, the Gâteaux differentiability, in general, does not imply the Fréchet differentiability, unless the continuity of the Gâteaux differential is required, as follows from the next theorem.

Theorem 2.22. *Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable function in X. We assume that the application $df : X \to X^*$ is continuous. Then f is Fréchet differentiable at X.*

We recall that if f is a Gâteaux differentiable function and if x_0 is a minimum point, then $df(x_0) = 0$ in X^*. If f is convex, the converse is also true; i.e., each point x_0 such that $df(x_0) = 0$ in X^* is a minimum point of f.

Example **2.23**. Let $g \in L^2(\Omega)$ and $X = H^1_0(\Omega)$ and consider the functional $F : X \to \mathbb{R}$ defined by

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega gu\,dx\,.$$

Clearly, F satisfies all the hypotheses of Theorem 2.15, hence it admits at least a minimizer $u_0 \in X$. Moreover, F is also Gâteaux differentiable at every point of X. Indeed, for every $v \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\lim_{t\to 0} \frac{F(u+tv)-F(u)}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}\int_\Omega |\nabla u + t\nabla v|^2\, dx - \int_\Omega g(u+tv)\, dx - \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega gu\, dx}{t} \\
&= \lim_{t\to 0} \frac{t\int_\Omega \nabla u \nabla v\, dx + \frac{1}{2}t^2 \int_\Omega |\nabla v|^2\, dx - t\int_\Omega gv\, dx}{t} \\
&= \int_\Omega \nabla u \nabla v\, dx - \int_\Omega gv\, dx = -\langle \Delta u + g, v\rangle_{H^{-1},H_0^1}\,.
\end{aligned}$$

Hence, if $u_0 \in H_0^1(\Omega)$ is a minimum point for F in X, we obtain that u_0 satisfies the equation $-\Delta u_0 = g$ in $H^{-1}(\Omega)$. Finally, by the convexity of F, also the converse holds; i.e., $u_0 \in X$ is a minimizer for F if and only if $-\Delta u_0 = g$ in $H^{-1}(\Omega)$. Analogously, we obtain that, if $A = [a_{ij}]$ is a symmetric matrix such that $a_{ij} \in L^\infty(\Omega)$ for $i, j = 1, \dots, N$, and $F : X \to \mathbb{R}$ is the functional defined by

$$F(u) = \frac{1}{2}\int_\Omega A(x)\nabla u \nabla u\, dx - \int_\Omega gu\, dx\,,$$

then F is Gâteaux differentiable on $H_0^1(\Omega)$ and $u_0 \in H_0^1(\Omega)$ is a minimum point for F if and only if u_0 satisfies the equation $-\operatorname{div}(A(x)\nabla u_0) = g$ in H^{-1}.

Example 2.24. Let $1 < p < +\infty$, $1/p + 1/p' = 1$, $g \in L^{p'}(\Omega)$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying (2.7). Assume also that the function $\xi \mapsto f(x,\xi)$ is convex, for a.e. $x \in \Omega$, there exists $\partial_\xi f$, it is a Carathéodory function and $|\partial_\xi f(x,\xi)| \le \gamma(x)$ for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$, where $\gamma \in L^1(\Omega)$. Let $F : W_0^{1,p}(\Omega) \to \overline{\mathbb{R}}$ be the functional defined by

$$F(u) = \int_\Omega f(x, \nabla u)\, dx - \int_\Omega g\, u\, dx\,. \tag{2.11}$$

Then, F is Gateaux differentiable on $H_0^1(\Omega)$ and for every $v \in H_0^1(\Omega)$ we have

$$\begin{aligned}
&\lim_{t\to 0} \frac{F(u+tv)-F(u)}{t} \\
&= \lim_{t\to 0} \frac{\int_\Omega f(x, \nabla u + t\nabla v)\, dx - \int_\Omega g(u+tv)\, dx - \int_\Omega f(x,\nabla u)\, dx + \int_\Omega gu\, dx}{t} \\
&= \int_\Omega \partial_\xi f(x,\nabla u)\nabla v\, dx - \int_\Omega gv\, dx = -\langle \operatorname{div}\big(\partial_\xi f(\cdot, \nabla u)\big) + g, v\rangle_{H^{-1},H_0^1}\,.
\end{aligned}$$

Hence, $u_0 \in X$ is a minimizer for F if and only if $-\operatorname{div}\big(\partial_\xi f(x,\nabla u_0)\big) = g$ in $H^{-1}(\Omega)$.

Let $Y = (0,1)^N$ and let us denote by $H^1_\#(Y)$ the space of the functions belonging to $H^1_{loc}(\mathbb{R}^N)$ which are Y-periodic and denote by $\widetilde{H}^1_\#(Y)$ the subspace of the functions belonging to $H^1_\#(Y)$ having null mean average on Y. As a consequence of Poincaré inequality (Theorem 1.32), we have that $\widetilde{H}^1_\#(Y)$ endowed with the L^2-norm of the gradient is a Banach space.

We conclude this section with the following lemma (see, for instance, [32, Lemma 2.1]).

Lemma 2.25. *Let $g \in L^2_{loc}(\mathbb{R}^N)$ be a Y-periodic function. Let $A = [a_{ij}]$ be a symmetric and Y-periodic matrix with $a_{ij} \in L^\infty(\mathbb{R}^N)$ and assume that there exist two positive constants $0 < \lambda \leq \Lambda < +\infty$ such that*

$$\lambda|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for a.e. } y \in Y,\ \forall \xi \in \mathbb{R}^N . \tag{2.12}$$

Then the equation

$$\begin{cases} -\operatorname{div}(A(y)\nabla v) = g & \text{in } Y; \\ \qquad\qquad v \in H^1_\#(Y); \end{cases} \tag{2.13}$$

has a unique solution, up to an additive constant, if and only if

$$\int_Y g\,dy = 0 . \tag{2.14}$$

Proof. First we note that, if $v \in H^1_\#(Y)$ is a solution of (2.13), for every $C \in \mathbb{R}$ the function $v + C$ is a solution, too. Then it is enough to prove that (2.13) has a unique solution in $\widetilde{H}^1_\#(Y)$. To this end, we may follow different approaches. We propose here two different proofs: the first one based on the Lax-Milgram Lemma and the second one based on the Direct Methods of Calculus of Variations.

- We remark that problem (2.13) is equivalent to

$$\int_Y A(y)\nabla v \nabla w\,dy = \int_Y g\,w\,dy \qquad \forall w \in H^1_\#(Y) . \tag{2.15}$$

Hence, let v be a solution of (2.15) and take $w \equiv 1$; clearly, (2.14) follows. Next, let us assume that (2.14) holds and prove that there exists a unique solution $v \in \widetilde{H}^1_\#(Y)$ satisfying (2.15). On $\widetilde{H}^1_\#(Y)$, let us consider the bilinear and symmetric form given by

$$a(\widetilde{w}_1, \widetilde{w}_2) = \int_Y A(y)\nabla \widetilde{w}_1 \cdot \nabla \widetilde{w}_2\,dy , \qquad \forall \widetilde{w}_1, \widetilde{w}_2 \in \widetilde{H}^1_\#(Y) .$$

Clearly, by (2.12), it is continuous and coercive on $\widetilde{H}^1_\#(Y)$; i.e., for every $\widetilde{w}_1, \widetilde{w}_2 \in \widetilde{H}^1_\#(Y)$ we have

$$a(\widetilde{w}_1, \widetilde{w}_2) \leq c \int_Y \nabla \widetilde{w}_1\, \nabla \widetilde{w}_2\,dy \leq c\|\widetilde{w}_1\|_{\widetilde{H}^1_\#(Y)} \|\widetilde{w}_2\|_{\widetilde{H}^1_\#(Y)} ,$$

and

$$a(\widetilde{w}_1, \widetilde{w}_1) \geq \lambda \int_Y |\nabla \widetilde{w}_1|^2 dy \geq C\|\widetilde{w}_1\|^2_{\widetilde{H}^1_\#(Y)} ;$$

where in the last inequality we use the Poincaré inequality (see Theorem 1.32). Moreover, the map $\widetilde{w} \in \widetilde{H}^1_\#(Y) \mapsto \int_y f(y)\widetilde{w}(y)\,dy$ is a linear and continuous functional. Hence, by Lax-Milgram Lemma (see Lemma 2.18), there exists a unique function $v \in \widetilde{H}^1_\#(\mathbb{R}^N)$ such that

$$a(v, \widetilde{w}) = \int_Y g\,\widetilde{w}\,dy \qquad \forall \widetilde{w} \in \widetilde{H}^1_\#(Y) . \tag{2.16}$$

In order to prove that the previous equality holds for every $w \in H^1_\#(Y)$, let us fix $w \in H^1_\#(Y)$ and consider the function $\widetilde{w} = w - \int_Y w(y)\,dy \in \widetilde{H}^1_\#(Y)$. It follows that

$$\int_Y A(y)\nabla v \cdot \nabla w\,dy = \int_Y A(y)\nabla v \cdot \nabla \widetilde{w}\,dy = a(v,\widetilde{w})$$
$$= \int_Y g\,\widetilde{w}\,dy = \int_Y g\,w\,dy - \int_Y g\,dy \int_Y w\,dy = \int_Y g\,w\,dy\,,$$

where in the third equality we use (2.16) and in the last equality we use (2.14). Hence (2.15) holds and the thesis is accomplished.

- Assume that $v \in H^1_\#(Y)$ is a solution of (2.13). Integrating on Y both the sides of the first equation in (2.13) and taking into account the Y-periodicity of the solution v and of the matrix $A = [a_{ij}]$, it easily follows

$$0 = \int_{\partial Y} A(y)\nabla v \cdot n\,d\sigma(y) = -\int_Y \operatorname{div}(A(y)\nabla v)\,dy = \int_Y g\,dy\,,$$

where n denotes the outward unit normal vector to ∂Y and $d\sigma$ denotes the surface measure on ∂Y. Hence, (2.14) holds. Next, let us assume (2.14) and prove that (2.13) has a unique solution with null mean average on Y. To this purpose let us define on $\widetilde{H}^1_\#(Y)$ the functional

$$F(\widetilde{w}) = \frac{1}{2}\int_Y A(y)\nabla\widetilde{w}\nabla\widetilde{w}\,dy - \int_Y g\,\widetilde{w}\,dy\,, \qquad \forall \widetilde{w} \in \widetilde{H}^1_\#(Y)\,.$$

Clearly F is strictly convex, weakly l.s.c. and coercive; indeed, using Poincaré inequality (see Theorem 1.32) and Young inequality (see Theorem 1.26), it follows

$$F(\widetilde{w}) \geq \frac{\lambda}{2}\int_Y |\nabla\widetilde{w}|^2\,dy - \frac{1}{2\delta}\int_Y g^2\,dy - \frac{\delta}{2}\int_Y \widetilde{w}^2\,dy$$
$$\geq \left(\frac{\lambda - \delta C}{2}\right)\int_Y |\nabla\widetilde{w}|^2\,dy - \frac{1}{2\delta}\int_Y g^2\,dy\,,$$

which is the required coercivity property for δ small enough. By Corollary 2.11, with $X = H^1_\#(Y)$ and $Y = \widetilde{H}^1_\#(Y)$, and Theorem 2.12 we obtain that the functional F admits a unique minimum $v \in \widetilde{H}^1_\#(Y)$ (note that we could also appeal directly to Theorem 2.15).

Moreover, since F is Gateaux differentiable and v is a minimum, according to analogous calculations as in Example 2.23, it follows

$$\lim_{t\to 0}\frac{F(v+t\widetilde{w}) - F(v)}{t} =$$
$$\int_Y A(y)\nabla v\nabla\widetilde{w}\,dy - \int_Y g\,\widetilde{w}\,dy = 0\,, \qquad \forall \widetilde{w} \in \widetilde{H}^1_\#(Y)\,.$$

Taking into account (2.14) and replacing $\widetilde{w} = w - \int_Y w\, dy$, where $w \in H^1_\#(Y)$, we have also

$$\begin{aligned}\int\limits_Y A(y)\nabla v\nabla w\, dy - \int\limits_Y g\, w\, dy &= \int\limits_Y A(y)\nabla v\nabla \widetilde{w}\, dy - \int\limits_Y g\, \widetilde{w}\, dy - \int\limits_Y g\, dy \int\limits_Y w\, dy \\ &= \int\limits_Y A(y)\nabla v\nabla \widetilde{w}\, dy - \int\limits_Y g\widetilde{w} = 0\,, \end{aligned} \tag{2.17}$$

for every $w \in H^1_\#(Y)$. Equality (2.17) implies that $v \in H^1_\#(Y)$ satisfies the first equation in (2.13) and hence the thesis is accomplished.

$\square$

3. **Homogenization of the standard Dirichlet problem**

In this chapter we present the classical homogenization method, due to Bensoussan-Lions-Papanicolau (see [12], [32]), based on the asymptotic expansion.
Let Ω be a bounded open subset of $\mathbb{R}^N$ with $n \geq 1$. Let $Y = (0,1)^n$ be the unit cell in $\mathbb{R}^N$. A function $f(x)$, defined on $\mathbb{R}^N$, is said to be Y-periodic if it is periodic of period 1 with respect to each variable x_i, with $1 \leq i \leq n$. We denote by $L^2_{\#}(Y)$ and $H^1_{\#}(Y)$ the spaces of functions in $L^2_{loc}(\mathbb{R}^N)$ and $H^1_{loc}(\mathbb{R}^N)$, respectively, which are Y-periodic.
Let $A(y)$ be a symmetric matrix of order n with entries $a_{ij}(y)$ which are measurable Y-periodic functions. We assume that there exist two constants $0 < \lambda < \Lambda < +\infty$ such that, for a.e. $y \in Y$,

$$\lambda|\xi|^2 \leq a_{ij}(y)\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \text{for a.e. } y \in Y, \, \forall \xi \in \mathbb{R}^N. \tag{3.1}$$

Let $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$ be a periodically oscillating matrix of coefficients. For a given function $f \in L^2(\Omega)$ we consider the following problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

which admits a unique solution in $H^1_0(\Omega)$, in the sense that, given $\varepsilon > 0$, there exists a unique function $u_\varepsilon \in H^1_0(\Omega)$ such that

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \nabla \varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x \qquad \forall \varphi \in H^1_0(\Omega)\,, \tag{3.3}$$

(see Lemma 2.18). The homogenization of equation (3.2) is by now a classical matter (see e.g. [10], [11], [12], [37]). We recall the main ingredients of this process that we shall use later (we mainly follow the exposition of Ch. I §2 in [12]). Firstly, we establish the usual energy estimate (obtained multiplying the first equation in (3.2) by u_ε and integrating by parts):

$$\begin{aligned} \|u_\varepsilon\|^2_{1,2} &\leq c \int_\Omega |\nabla u_\varepsilon|^2 \, \mathrm{d}x \leq \frac{c}{\lambda} \int_\Omega A_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon \, \mathrm{d}x \\ &= \frac{c}{\lambda} \int_\Omega f\, u_\varepsilon \, \mathrm{d}x \leq \frac{c}{\lambda} \|f\|_2 \|u_\varepsilon\|_2 \leq C \|u_\varepsilon\|_{1,2}\,, \end{aligned} \tag{3.4}$$

where, in the first inequality, we used the Poincaré inequality (see Theorem 1.32). From (3.4), it follows

$$\|u_\varepsilon\|_{1,2} \leq C\,, \tag{3.5}$$

where the constant C depends on the Poincaré constant c, on the ellipticity constant λ and on the L^2-norm of f, but not on ε. Therefore, from (3.5), it can be easily proved that there exists a function $u \in H^1_0(\Omega)$ such that, up to a subsequence,

$$u_\varepsilon \rightharpoonup u \qquad \text{weakly in } H^1_0(\Omega). \tag{3.6}$$

It remains to identify the limit function u. This will be done in a formal way in the next Section.

3.1. **Formal Expansion.** We assume that the solution u_ε admits the following ansatz (or asymptotic expansion)

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \dots \tag{3.7}$$

where each function $u_i(x, y)$ is Y-periodic with respect to the fast variable y. Plugging this ansatz in equation (3.2) and identifying different powers of ε yields a cascade of equations. Defining an operator L_ε by $L_\varepsilon\phi = -\operatorname{div} A_\varepsilon\nabla\phi$, we may write $L_\varepsilon = \varepsilon^{-2}L_0 + \varepsilon^{-1}L_1 + L_2$, where

$$\begin{aligned}
L_0 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)\\
L_1 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)\\
L_2 &= -\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right).
\end{aligned}$$

The two space variables x and y are taken as independent, and at the end of the computation y is replaced by $\frac{x}{\varepsilon}$. Equation (3.2) is therefore equivalent to the following system

$$\begin{aligned}
&L_0 u_0 = 0\\
&L_0 u_1 + L_1 u_0 = 0\\
&L_0 u_2 + L_1 u_1 + L_2 u_0 = f\\
&L_0 u_3 + L_1 u_2 + L_2 u_1 = 0\\
&\dots\dots
\end{aligned} \tag{3.8}$$

the solutions of which are easily computed. The first equation in (3.8); i.e.,

$$\begin{cases}
-\dfrac{\partial}{\partial y_i}\left(a_{ij}(y)\dfrac{\partial u_0(x,y)}{\partial y_j}\right) = 0 & \text{in } Y\\
y \mapsto u_0(x, y) & Y\text{-periodic}
\end{cases}$$

where y is the independent variable, while x plays the role of a parameter, has a constant solution (where the constant clearly depends on x), due to Lemma 2.25; so that $u_0(x, y) \equiv u_0(x)$, which does not depend on y. The second equation of (3.8) can be explicitly solved in term of u_0; indeed, it can be easily verified that the solution u_1 is given by

$$u_1\left(x, \frac{x}{\varepsilon}\right) = -\chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0}{\partial x_j}(x) + \tilde{u}_1(x) \tag{3.9}$$

where, thanks to Lemma 2.25, $\chi^j(y)$, $j = 1, \dots, n$, are the unique solutions in $H^1_\#(Y)$ with zero average of the cell equation

$$\begin{cases}
L_0\chi^j = -\dfrac{\partial a_{ij}}{\partial y_i}(y) & \text{in } Y;\\
\displaystyle\int_Y \chi^j(y)dy = 0 & y \to \chi^j(y)\ Y\text{-periodic}.
\end{cases} \tag{3.10}$$

By Lemma 2.25, the third equation in (3.8), which can be written as the system

$$\begin{cases}
L_0 u_2 = f - L_1 u_1 - L_2 u_0 & \text{in } Y\\
y \mapsto u_2(x, y) & Y\text{-periodic},
\end{cases}$$

where u_0 and u_1 play the role of known functions, is solvable if and only if

$$\int_Y [f(x) - (L_1 u_1)(x,y) - (L_2 u_0)(x,y)]\ dy = 0\,.$$

Inserting (3.9) in the last equality, by easy calculations it follows

$$\begin{aligned}
0 &= \int_Y f(x)\ dy - \int_Y (L_1 u_1)(x,y)\ dy - \int_Y (L_2 u_0)(x,y)\ dy \\
&= f(x) - \int_Y \partial_{x_i}[a_{ik}(y)\partial_{y_k}(\chi^j(y)\partial_{x_j}u_0(x))]\ dy - \int_Y \partial_{y_i}[a_{ik}(y)\partial_{x_k}(\chi^j(y)\partial_{x_j}u_0(x))]\ dy \\
&\qquad + \int_Y \partial_{x_i}[a_{ij}(y)\partial_{x_j}u_0(x)]\ dy \\
&= f(x) - \partial_{x_i}\left[\Big(\int_Y a_{ik}(y)\partial_{y_k}\chi^j(y)\ dy\Big)\partial_{x_j}u_0(x)\right] - \Big(\int_Y \partial_{y_i}a_{ik}(y)\chi^j(y)\ dy\Big)[\partial_{x_k}\partial_{x_j}u_0(x)] \\
&\qquad + \partial_{x_i}\left[\Big(\int_Y a_{ij}(y)\ dy\Big)\partial_{x_j}u_0(x)\right] \\
&= f(x) - \Big(\int_Y a_{ik}(y)\partial_{y_k}\chi^j(y)\ dy\Big)\partial_{x_i}\partial_{x_j}u_0(x) - \Big(\int_{\partial Y} a_{ik}(y)\chi^j(y)\nu_i\ d\sigma\Big)[\partial_{x_k}\partial_{x_j}u_0(x)]\ dy \\
&\qquad + \Big(\int_Y a_{ij}(y)\ dy\Big)\partial_{x_i}\partial_{x_j}u_0(x) \\
&= f(x) - \Big(\int_Y a_{ik}(y)\partial_{y_k}\chi^j(y)\ dy\Big)\partial_{x_i}\partial_{x_j}u_0(x) + \Big(\int_Y a_{ij}(y)\ dy\Big)\partial_{x_i}\partial_{x_j}u_0(x)\,,
\end{aligned}$$

where, in the last equality, we use the periodicity of the function $y \mapsto a_{ik}(y)\chi^j(y)$. This implies

$$-\partial_i\Big(\int_Y [a_{ij}(y) - a_{ik}(y)\partial_{y_k}\chi^j(y)]\partial_k u_0\Big) = f\,,$$

which can be rewritten

$$-\operatorname{div}(A^*\nabla u_0) = f\,,$$

where the homogenized matrix A^* is defined by its constant entries a^*_{ij} given by

$$a^*_{ij} = \int_Y \left[a_{ij}(y) - a_{ik}(y)\frac{\partial\chi^j}{\partial y_k}(y)\right]\ dy. \tag{3.11}$$

The homogenized problem for $u_0(x)$ is just the previous compatibility condition, complemented with the natural boundary condition obtained by (3.2); i.e.,

$$\begin{cases} -\operatorname{div}(A^*\nabla u_0) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

It is important to observe that the homogenized matrix A^* it is not obtained simply by averaging the original one, even if this could be considered more natural. Obviously, this implies that the convergence of the solutions $\{u_\varepsilon\}$ cannot take place in $H^1_0(\Omega)$ strongly. As a consequence of Lax-Milgram Lemma (see 2.18) we easily obtain that problem (3.12) is well-posed in $H^1_0(\Omega)$, since it can be proved that A^* is symmetric and coercive (see Remark 2.6 in [12]). As remarked in Section 2.3, existence and uniqueness for problem (3.12) can be obtained also passing to the corresponding minimum problem; i.e.,

$$\min_{u\in H^1_0(\Omega)} \int_\Omega A^*\nabla u\nabla u\,dx - \int_\Omega fu\,dx\,.$$

Moreover, also the solution u_2 can be explicitly given in terms of u_0; in fact,

$$u_2\left(x,\frac{x}{\varepsilon}\right) = \chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x) - \chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial \tilde{u}_1}{\partial x_j}(x) + \tilde{u}_2(x) \tag{3.13}$$

where $\chi^{ij} \in H^1_\#(Y)$, for $i,j = 1,\dots,n$, are the solutions of another cell problem (see (2.42) and (2.39) in [12])

$$\begin{cases} L_0\chi^{ij} = b_{ij} - \displaystyle\int_Y b_{ij}(y)\,dy & \text{in } Y; \\ \displaystyle\int_Y \chi^{ij}(y)dy = 0 & y \to \chi^{ij}(y)\ Y\text{-periodic.} \end{cases} \tag{3.14}$$

with

$$b_{ij}(y) = a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k} - \frac{\partial}{\partial y_k}(a_{ki}(y)\chi^j).$$

Remark that, so far (i.e. if we do not look at higher order equations in (3.8)), the functions $\tilde{u}_1$ in (3.9) and $\tilde{u}_2$ in (3.13) are non-oscillating functions that are not determined. As pointed out in [12], if we stop expansion (3.7) at the first order, the function $\tilde{u}_1$ (and a fortiori $\tilde{u}_2$) does not play any role, and so we may choose $\tilde{u}_1 \equiv 0$. However, if higher order terms are considered, then $\tilde{u}_1$ must satisfy some equation. More precisely, the compatibility condition of the fourth equation of (3.8) leads to (see [12], equation (2.45))

$$-\operatorname{div} A^*\nabla\tilde{u}_1 = c_{ijk}\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \tag{3.15}$$

with

$$c_{ijk} = \int_Y \left[a_{kl}(y)\frac{\partial \chi^{ij}}{\partial y_l}(y) - a_{ij}(y)\chi^k(y)\right]\,dy.$$

Similar considerations hold for $\tilde{u}_2$, but we shall not need it in the sequel. At this point we emphasize that the above method of two-scale asymptotic expansion is formal. However, a well-known theorem states that the two first terms of (3.7) are correct.

3.2. Tartar's convergence Theorem.

Theorem 3.1. *For every $\varepsilon > 0$, let u_ε be the solution of (3.2) and u_0 be the solution of (3.12). Then $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$.*

Proof. Set

$$\xi^j_\varepsilon := \sum_{k=1}^N a^\varepsilon_{kj}\frac{\partial u_\varepsilon}{\partial x_k} \qquad \text{for } j = 1,\dots,N\,.$$

Clearly, by the energy estimate (3.5), it follows that

$$\|\xi_\varepsilon^j\|_2 \leq C \qquad \forall j = 1, \dots, N\,,$$

where C does not depend on ε. Then, up to a subsequence, for every $j = 1, \dots, N$, there exists $\xi_0^j \in L^2(\Omega)$, such that $\xi_\varepsilon^j \rightharpoonup \xi_0^j$, weakly in $L^2(\Omega)$. Moreover, by (3.6), again up to a subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Passing to the limit in (3.3), we obtain

$$\int_\Omega \xi_0 \nabla\varphi \,\mathrm{d}x = \int_\Omega f\varphi \qquad \Longleftrightarrow \qquad -\operatorname{div}\xi_0 = f\,. \tag{3.16}$$

Next we have to identify ξ_0^j. To this purpose, for $j = 1, \dots, N$, let w_ε^j be the function defined by

$$w_\varepsilon^j(x) = x_j - \varepsilon\chi^j\left(\frac{x}{\varepsilon}\right)\,,$$

where χ^j is given in (3.10). Clearly,

$$-\operatorname{div}(A_\varepsilon \nabla w_\varepsilon^j) = 0\,. \tag{3.17}$$

Fix $j \in \{1, \dots, N\}$ and take φw_ε^j, with $\varphi \in \mathcal{C}_c^\infty(\Omega)$, as a test function in (3.3) and φu_ε as a test function in the weak formulation of (3.17). Taking the difference, we obtain

$$\int_\Omega \xi_\varepsilon \nabla\varphi\, w_\varepsilon^j \,\mathrm{d}x - \int_\Omega A_\varepsilon \nabla w_\varepsilon^j \nabla\varphi\, u_\varepsilon = \int_\Omega f w_\varepsilon^j \varphi \,\mathrm{d}x\,. \tag{3.18}$$

Passing to the limit in (3.18), it follows

$$\int_\Omega \xi_0 \nabla\varphi\, x_j \,\mathrm{d}x - \int_\Omega \left(\int_Y [A\mathrm{e}_j - A\nabla\chi^j(y)]\,\mathrm{d}y\right) \nabla\varphi\, u = \int_\Omega f x_j \varphi \,\mathrm{d}x\,, \tag{3.19}$$

where e_j is the j^{th}-vector of the canonical base of $\mathbb{R}^N$. Next, let us take $\phi\, x_j$ as a test function in (3.3); passing to the limit it follows

$$\int_\Omega \xi_0 \nabla\varphi\, x_j \,\mathrm{d}x + \int_\Omega \xi_0\, \mathrm{e}_j\, \varphi \,\mathrm{d}x = \int_\Omega f\, x_j\, \varphi \,\mathrm{d}x\,,$$

so that (3.19) becomes

$$-\int_\Omega \left(\int_Y [A\mathrm{e}_j - A\nabla\chi^j(y)]\,\mathrm{d}y\right) \nabla\varphi\, u = \int_\Omega \xi_0\, \mathrm{e}_j\, \varphi \,\mathrm{d}x\,, \tag{3.20}$$

which implies

$$\left(\int_Y [A\mathrm{e}_j - A\nabla\chi^j(y)]\,\mathrm{d}y\right) \nabla u = \xi_0^j\,. \tag{3.21}$$

Then, recalling (3.11), we have that (3.16) and (3.21) imply the first equation in (3.12). Finally, since $u \in H_0^1(\Omega)$ and problem (3.12) admits a unique solution, it follows that u coincides with u_0, so that the whole sequence $\{u_\varepsilon\}$ converges to u_0 and the thesis is accomplished. □

As consequence of previous theorem, we have been able to identify the limit function u in (3.6), which is equal to u_0, solution of (3.12). This identification can be obtained also in different ways, as shown below.

3.3. L^∞**-convergence and error estimate.** In this section we will prove that, under additional assumptions, the convergence of the sequence $\{u_\varepsilon\}$ to the homogenized solution u_0 can be improved. Firstly we concentrate on the L^∞-convergence.

Theorem 3.2. *Assume that, for $i, j = 1, \dots, N$, the coefficients $a_{ij} \in \mathcal{C}^\infty(Y)$ are Y-periodic functions satisfying* (3.1). *For every $\varepsilon > 0$, let u_ε be the solution of* (3.2) *and u_0 be the solution of* (3.12), *with $f \in \mathcal{C}^\infty(\Omega)$. Assume, in addition, that $u_0 \in W^{4,\infty}(\Omega)$. Then $u_\varepsilon \to u_0$ strongly in $L^\infty(\Omega)$.*

Proof. Let u_1, u_2 be defined as in (3.9) and (3.13), respectively, with $\widetilde{u}_1, \widetilde{u}_2 \equiv 0$ and set

$$r_\varepsilon(x) = u_\varepsilon(x) - \left[(u_0(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right)\right] .$$

Taking into account (3.8), an easy calculation gives

$$\begin{aligned} L_\varepsilon r_\varepsilon = f &- \frac{1}{\varepsilon^2} L_0 u_0 - \frac{1}{\varepsilon} L_1 u_0 - L_2 u_0 \\ &- \varepsilon\left(\frac{1}{\varepsilon^2} L_0 u_1 + \frac{1}{\varepsilon} L_1 u_1 + L_2 u_1\right) - \varepsilon^2\left(\frac{1}{\varepsilon^2} L_0 u_2 + \frac{1}{\varepsilon} L_1 u_2 + L_2 u_2\right) \\ &= -\frac{1}{\varepsilon^2} L_0 u_0 - \frac{1}{\varepsilon}(L_1 u_0 - L_0 u_1) + [f - (L_0 u_2 + L_1 u_1 + L_2 u_0)] \\ &\qquad - \varepsilon(L_2 u_1 + L_1 u_2) - \varepsilon^2 L_2 u_2 = -\varepsilon(L_1 u_2 + L_2 u_1 + \varepsilon L_2 u_2) =: \varepsilon g_\varepsilon . \end{aligned}$$

Moreover, by the regularity assumptions, it follows that

$$\|g_\varepsilon\|_{L^\infty(\Omega)} = \|L_1 u_2 + L_2 u_1 + \varepsilon L_2 u_2\|_{L^\infty(\Omega\times Y)} \le C ,$$

with C independent on ε. On the other hand, on $\partial\Omega$, $r_\varepsilon = -(\varepsilon u_1 + \varepsilon^2 u_2)$; hence

$$\sup_{x\in\partial\Omega} |r_\varepsilon(x)| \le \sup_{(x,y)\in(\partial\Omega\times Y)} \left|\varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right)\right| \le C\varepsilon ,$$

with C independent on ε. Finally, by the maximum principle, we obtain that $\|r_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon$, which implies $\|u_\varepsilon - u_0\|_{L^\infty(\Omega)} \le C\varepsilon$, so that the thesis follows. □

Next, let us prove the error estimate.

Theorem 3.3. *Assume that, for $i, j = 1, \dots, N$, the coefficients $a_{ij} \in \mathcal{C}^\infty(Y)$ are Y-periodic functions satisfying* (3.1). *For every $\varepsilon > 0$, let u_ε be the solution of* (3.2) *and u_0 be the solution of* (3.12), *with $f \in \mathcal{C}^\infty(\Omega)$. Assume in addition that $u_0 \in W^{2,\infty}(\Omega)$. Then*

$$\|u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, \frac{x}{\varepsilon})\|_{H^1(\Omega)} \le C\sqrt{\varepsilon}$$

where u_1 is given by (3.9).

Proof. Defining $r_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, x/\varepsilon)$, it satisfies

$$-\operatorname{div} A_\varepsilon \nabla r_\varepsilon = \varepsilon^{-1}(f + \operatorname{div} A_\varepsilon \nabla u_0) + \operatorname{div} A_\varepsilon \nabla u_1 .$$

Taking into account system (3.8), for any $\phi \in H_0^1(\Omega)$, we have

$$\begin{aligned}
&\left| \int_\Omega \left[\frac{1}{\varepsilon}(f + \operatorname{div} A_\varepsilon \nabla u_0) + \operatorname{div} A_\varepsilon \nabla u_1 \right] \phi \, dx \right| \\
&\quad = \left| \int_\Omega \left[-\frac{1}{\varepsilon} \operatorname{div}_y A_\varepsilon \nabla_y u_2 \left(x, \frac{x}{\varepsilon}\right) + \operatorname{div}_x A_\varepsilon \nabla_x u_1 \left(x, \frac{x}{\varepsilon}\right) \right] \phi \, dx \right| \\
&\quad \leq \left| \int_\Omega - \left[\operatorname{div}_x A_\varepsilon \nabla_y u_2 \left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon} \operatorname{div}_y A_\varepsilon \nabla_y u_2 \left(x, \frac{x}{\varepsilon}\right) \right] \phi \, dx \right| \\
&\qquad + \left| \int_\Omega \left[\operatorname{div}_x A_\varepsilon \nabla_y u_2 \left(x, \frac{x}{\varepsilon}\right) + \operatorname{div}_x A_\varepsilon \nabla_x u_1 \left(x, \frac{x}{\varepsilon}\right) \right] \phi \, dx \right| \\
&\quad \leq \left| \int_\Omega A_\varepsilon \nabla_y u_2 \left(x, \frac{x}{\varepsilon}\right) \nabla \phi \, dx \right| + C\|\phi\|_{H_0^1(\Omega)} \leq C\|\phi\|_{H_0^1(\Omega)}.
\end{aligned}$$

Passing to the supremum when $\|\phi\|_{H_0^1(\Omega)} = 1$, we obtain that

$$\| - \operatorname{div} A_\varepsilon \nabla r_\varepsilon \|_{H^{-1}(\Omega)} = \| \frac{1}{\varepsilon}(f + \operatorname{div} A_\varepsilon \nabla u_0) + \operatorname{div} A_\varepsilon \nabla u_1 \|_{H^{-1}(\Omega)} \leq C\,. \tag{3.22}$$

Next, let $\varphi_\varepsilon \in \mathcal{C}^\infty(\Omega)$ be a cut-off function such that $0 \leq \varphi_\varepsilon(x) \leq 1$ and $|\nabla \varphi_\varepsilon(x)| \leq \frac{c}{\varepsilon}$ in Ω; $\varphi_\varepsilon(x) = 1$ on $\partial\Omega$ and $\varphi_\varepsilon(x) = 0$ for $x \in \Omega$ with $dist(x, \partial\Omega) > \varepsilon$. Then define

$$\widetilde{r}_\varepsilon = \varepsilon^{-1}(u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, x/\varepsilon)) + u_1(x, x/\varepsilon)\varphi_\varepsilon(x) = r_\varepsilon(x) + u_1(x, x/\varepsilon)\varphi_\varepsilon(x)\,.$$

Clearly, $\widetilde{r}_\varepsilon \in H_0^1(\Omega)$ and

$$\|r_\varepsilon\|_{H^1(\Omega)} \leq \|\widetilde{r}_\varepsilon\|_{H_0^1(\Omega)} + \|u_1 \varphi_\varepsilon\|_{H^1(\Omega)}\,; \tag{3.23}$$

moreover,

$$\begin{aligned}
\|u_1\varphi_\varepsilon\|^2_{H^1(\Omega)} &\leq \left(\int_\Omega |u_1\varphi_\varepsilon|^2 \, dx + \int_\Omega |u_1|^2 |\nabla \varphi_\varepsilon|^2 \, dx + \int_\Omega |\varphi_\varepsilon|^2 |\nabla u_1|^2 \, dx \right) \\
&\leq C \left(1 + \int_{\Omega_\varepsilon} |\nabla \varphi_\varepsilon|^2 \, dx + \int_{\Omega_\varepsilon} |\nabla_x u_1| + \varepsilon^{-1} \nabla_y u_1|^2 \, dx \right) \\
&\leq C \left(1 + \frac{1}{\varepsilon^2} |\Omega_\varepsilon| + \frac{1}{\varepsilon^2} |\Omega_\varepsilon| \right) \leq \frac{C}{\varepsilon}\,;
\end{aligned}$$

where we set $\Omega_\varepsilon = \{x \in \Omega \; : \; dist(x, \partial\Omega) \leq \varepsilon\}$ and we take into account that there exists $C > 0$ such that $|\Omega_\varepsilon| \leq C\varepsilon$. This implies

$$\|u_1\varphi_\varepsilon\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}\,. \tag{3.24}$$

Taking into account (3.1) and (3.22), it follows

$$
\begin{aligned}
\|\widetilde{r}_\varepsilon\|^2_{H^1_0(\Omega)} &\leq \frac{1}{\lambda}\int_\Omega A_\varepsilon\nabla\widetilde{r}_\varepsilon\nabla\widetilde{r}_\varepsilon\,dx = \frac{1}{\lambda}\left(\int_\Omega A_\varepsilon\nabla r_\varepsilon\nabla\widetilde{r}_\varepsilon\,dx + \int_\Omega A_\varepsilon\nabla(u_1\varphi_\varepsilon)\nabla\widetilde{r}_\varepsilon\,dx\right) \\
&\leq \frac{1}{\lambda}\|-\operatorname{div} A_\varepsilon\nabla r_\varepsilon\|_{H^{-1}(\Omega)}\|\widetilde{r}_\varepsilon\|_{H^1_0(\Omega)} \\
&\quad + \frac{1}{\lambda}\left[\Big(\int_{\Omega_\varepsilon}|A_\varepsilon\varphi_\varepsilon|^2|\nabla u_1|^2\,dx\Big)^{1/2} + \Big(\int_{\Omega_\varepsilon}|A_\varepsilon u_1|^2|\nabla\varphi_\varepsilon|^2\,dx\Big)^{1/2}\right]\|\widetilde{r}_\varepsilon\|_{H^1_0(\Omega)} \\
&\leq C\|\widetilde{r}_\varepsilon\|_{H^1_0(\Omega)}\left[1 + \Big(\frac{1}{\varepsilon^2}|\Omega_\varepsilon|\Big)^{1/2} + \Big(\frac{1}{\varepsilon^2}|\Omega_\varepsilon|\Big)^{1/2}\right] \leq \frac{C}{\sqrt{\varepsilon}}\|\widetilde{r}_\varepsilon\|_{H^1_0(\Omega)}\,.
\end{aligned}
\tag{3.25}
$$

This last inequality, together with (3.24) and (3.23), gives the thesis. □

An immediate consequence of previous theorem is the strong $H^1(\Omega)$-convergence of the sequence $\{u_\varepsilon - \varepsilon u_1\}$ to the homogenized limit u_0.
The results stated in Theorems 3.2 and 3.3 (as well as Theorem 3.1 proved in the previous section) are classical (see e.g. [12] and [32]). Remark that they hold whatever the choice of $\tilde{u}_1$ is. In particular, note that in Theorem 3.3 the term $\varepsilon\tilde{u}_1(x)$ is smaller than $\sqrt{\varepsilon}$ in the $H^1(\Omega)$-norm. However, the error estimate of order $\sqrt{\varepsilon}$, although generically optimal, is a little surprising since one could expect to get ε if the next order term in the ansatz was truly $\varepsilon^2 u_2(x, \frac{x}{\varepsilon})$. As it is well known, this worse-than-expected result is due to the appearance of boundary layers (see [11], [13], [32]). Indeed, the expected result is obtained in Theorem 3.5, where the boundary data "is corrected" by means of the function z_ε, the so-called *boundary layer*, defined in (3.26). Next lemma provides an estimate of the rate of divergence for $\varepsilon \to 0$ of the boundary layer.

Lemma 3.4. *Assume that, for $i, j = 1, \ldots, N$, the coefficients $a_{ij} \in \mathcal{C}^\infty(Y)$ are Y-periodic functions satisfying* (3.1)*. Let u_0 be the unique solutions of (3.12), with $f \in \mathcal{C}^\infty(\Omega)$. Assume, in addition, that $u_0 \in W^{2,\infty}(\Omega)$. For every $\varepsilon > 0$, let $z_\varepsilon \in H^1(\Omega)$ be the unique solution of*

$$
\begin{cases}
-\operatorname{div} A_\varepsilon\nabla z_\varepsilon = 0 & \text{in } \Omega, \\
z_\varepsilon = -u_1\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \partial\Omega.
\end{cases}
\tag{3.26}
$$

Then

$$
\|z_\varepsilon\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}\,.
$$

By assumptions it follows

$$
\|u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\|_{L^\infty(\Omega)} = \sup_{x\in\Omega}|-\chi^j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(x) + \widetilde{u}_1(x)| \leq C \tag{3.27}
$$

and

$$
\|\nabla u_1\left(\cdot, \frac{\cdot}{\varepsilon}\right)\|_{L^\infty(\Omega)} \leq \sup_{\substack{y\in Y\\ x\in\Omega}}\left[|\frac{1}{\varepsilon}\nabla_y\chi^j(y)\frac{\partial u_0}{\partial x_j}(x)| + |\chi^j(y)\nabla\frac{\partial u_0}{\partial x_j}(x)| + |\nabla\widetilde{u}_1(x)|\right] \leq \frac{C}{\varepsilon}. \tag{3.28}
$$

These two estimates will be crucial in the proof of the lemma. Note also that, by the definition of z_ε, the function $u_1 + z_\varepsilon$ belongs to $H^1_0(\Omega)$; i.e., it satisfies the homogeneous boundary condition.

Proof. Let us define $\tilde{z}_\varepsilon(x) = -u_1\left(x, \frac{x}{\varepsilon}\right) \exp\left[-\frac{d(x,\partial\Omega)}{\varepsilon}\right]$. By (3.27) and (3.28), we have that $\|\tilde{z}_\varepsilon\|_{H^1(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}$. Set $\delta_\varepsilon = z_\varepsilon - \tilde{z}_\varepsilon$. Clearly, $\delta_\varepsilon \in H^1_0(\Omega)$; hence,

$$\int_\Omega A_\varepsilon \nabla z_\varepsilon \nabla \delta_\varepsilon \, dx = 0. \tag{3.29}$$

By (3.1) and (3.29), it follows

$$\lambda \|\nabla \delta_\varepsilon\|^2_{L^2(\Omega)} \le \int_\Omega A_\varepsilon \nabla \delta_\varepsilon \nabla \delta_\varepsilon \, dx = -\int_\Omega A_\varepsilon \nabla \tilde{z}_\varepsilon \nabla \delta_\varepsilon \, dx \le C \|\nabla \tilde{z}_\varepsilon\|_{L^2(\Omega)} \|\nabla \delta_\varepsilon\|_{L^2(\Omega)}.$$

This implies that

$$\|\nabla \delta_\varepsilon\|_{L^2(\Omega)} \le C \|\nabla \tilde{z}_\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\sqrt{\varepsilon}}$$

and hence

$$\|z_\varepsilon\|_{H^1(\Omega)} \le \|\delta_\varepsilon\|_{H^1_0(\Omega)} + \|\tilde{z}_\varepsilon\|_{H^1(\Omega)} \le C\left[\|\nabla \delta_\varepsilon\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varepsilon}}\right] \le \frac{C}{\sqrt{\varepsilon}}.$$

□

Taking into account the boundary layer, it is possible to improve the estimate stated in Theorem 3.3.

Theorem 3.5. *Assume that, for $i, j = 1, \dots, N$, the coefficients $a_{ij} \in \mathcal{C}^\infty(Y)$ are Y-periodic functions satisfying* (3.1)*. For every $\varepsilon > 0$, let u_ε be the solution of* (3.2) *and u_0 be the solution of* (3.12)*, with $f \in \mathcal{C}^\infty(\Omega)$. Assume in addition that $u_0 \in W^{2,\infty}(\Omega)$. Let u_1, z_ε be defined by* (3.9) *and* (3.26)*. Then*

$$\|u_\varepsilon(x) - u_0(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon z_\varepsilon(x)\|_{H^1_0(\Omega)} \le C\varepsilon.$$

Proof. As in [34], defining $r_\varepsilon(x) = \varepsilon^{-1}(u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, x/\varepsilon) - \varepsilon z_\varepsilon(x))$, it satisfies

$$\begin{cases} -\operatorname{div} A_\varepsilon \nabla r_\varepsilon = \varepsilon^{-1}(f + \operatorname{div} A_\varepsilon \nabla u_0) + \operatorname{div} A_\varepsilon \nabla u_1 & \text{in } \Omega, \\ r_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

since by (3.26) $\operatorname{div} A_\varepsilon \nabla z_\varepsilon = 0$. Hence, repeating the same calculations as in the proof of Theorem 3.3, it follows that

$$\| - \operatorname{div} A_\varepsilon \nabla r_\varepsilon\|_{H^{-1}(\Omega)} = \|\frac{1}{\varepsilon}(f + \operatorname{div} A_\varepsilon \nabla u_0) + \operatorname{div} A_\varepsilon \nabla u_1\|_{H^{-1}(\Omega)} \le C\,,$$

so that

$$\|r_\varepsilon\|_{H^1_0(\Omega)} \le \frac{1}{\lambda}\|\frac{1}{\varepsilon}(f + \operatorname{div} A_\varepsilon \nabla u_0) + \operatorname{div} A_\varepsilon \nabla u_1\|_{H^{-1}(\Omega)} \le C,$$

which implies the desired result. □

From Theorem 3.5 and taking into account that by Lemma 3.4 the boundary layer z_ε blows up as $1/\sqrt{\varepsilon}$, when $\varepsilon \to 0$, we obtain again the error estimate given in Theorem 3.3, since

$$\begin{aligned} \|u_\varepsilon - u_0 - \varepsilon u_1\|_{H^1(\Omega)} &\le \|u_\varepsilon - u_0 - \varepsilon(u_1 + z_\varepsilon)\|_{H^1_0(\Omega)} + \varepsilon\|z_\varepsilon\|_{H^1(\Omega)} \\ &\le C\varepsilon + \varepsilon\frac{C}{\sqrt{\varepsilon}} \le C\sqrt{\varepsilon}\,. \end{aligned}$$

Remark 3.6. For the sake of simplicity, we assume in this section strong regularity hypotheses on the coefficients a_{ij} and the function f, though the result holds true even in a more general context. However, it is out of our interest here to make any effort in order to consider the best possible regularity assumptions which assure the uniform convergence and the error estimate in the homogenization procedure.

3.4. **Non homogeneous Dirichlet boundary conditions.** In this section we consider the case of non homogeneous boundary conditions; i.e.,

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = g & \text{on } \partial\Omega, \end{cases} \tag{3.30}$$

for a given function $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. We denote by $g_0 \in H^1(\Omega)$ the unique solution of

$$\begin{cases} -\operatorname{div}(A^* \nabla g_0) = 0 & \text{in } \Omega, \\ g_0 = g & \text{on } \partial\Omega, \end{cases} \tag{3.31}$$

where A^* is the homogenized matrix defined in (3.11). Hence, we set $H^1_0(\Omega) \ni v_\varepsilon = u_\varepsilon - g_0$, where $u_\varepsilon \in H^1(\Omega)$ is the unique solution of (3.30). Clearly v_ε satisfies

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) = f + \operatorname{div}(A_\varepsilon \nabla g_0) & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.32}$$

and

$$\int_\Omega |\nabla v_\varepsilon|^2\,dx \le C\left(\int_\Omega |f|^2\,dx + \int_\Omega |\nabla g_0|^2\,dx\right) \le C\,, \tag{3.33}$$

where C depends on $\lambda, \Lambda, \|f\|_2$ and $\|g_0\|_{1,2}$. By (3.33), up to a subsequence, still denoted by $\{v_\varepsilon\}$, we have that there exists a function $v_0 \in H^1_0(\Omega)$ such that $v_\varepsilon \to v_0$ strongly in $L^2(\Omega)$ and $\nabla v_\varepsilon \rightharpoonup \nabla v_0$ weakly in $L^2(\Omega)$ (see Theorems 1.32 and 1.31). Let us choose w^k_ε, $k = 1, \dots, N$, as in the proof of Theorem 3.1 and define

$$\xi_0 := (w - L^2(\Omega)) - \lim_{\varepsilon \to 0} A_\varepsilon \nabla v_\varepsilon\,. \tag{3.34}$$

Then using $\varphi\, w^k_\varepsilon$, with $\varphi \in C^\infty_0(\Omega)$, as a test function in the weak formulation of (3.32) and φv_ε as a test function in the weak formulation of (3.17), integrating by parts and subtracting the two results, as in the proof of Theorem 3.1, it follows

$$\begin{aligned} &\int_\Omega A_\varepsilon \nabla v_\varepsilon \nabla\varphi\, w^k_\varepsilon\,dx - \int_\Omega A_\varepsilon \nabla w^k_\varepsilon \nabla\varphi\, v_\varepsilon\,dx = \\ &\qquad \int_\Omega f\, w^k_\varepsilon\, \varphi\,dx - \int_\Omega A_\varepsilon \nabla g_0 \nabla w^k_\varepsilon\, \varphi\,dx - \int_\Omega A_\varepsilon \nabla g_0 \nabla\varphi\, w^k_\varepsilon\,dx\,. \end{aligned} \tag{3.35}$$

Then, passing to the limit for $\varepsilon \to 0$ and taking into account (3.34) and the fact that $A_\varepsilon \nabla w^k_\varepsilon \rightharpoonup A^* \mathrm{e}_k$ as in (3.19), we obtain

$$\begin{aligned} &\int_\Omega \xi_0 \nabla\varphi\, x^k\,dx - \int_\Omega a^*_{kj} \partial_j \varphi\, v_0\,dx = \\ &\qquad \int_\Omega f\, x^k\, \varphi\,dx - \int_\Omega a^*_{kj} \partial_j g_0\, \varphi\,dx - \int_\Omega \overline{A} \nabla g_0 \nabla\varphi\, x^k\,dx\,, \end{aligned} \tag{3.36}$$

where the matrix $\overline{A}$ is defined by its constant entries $\overline{a}_{ij}$ given by

$$\overline{a}_{ij} = \int_Y a_{ij}(y)\, dy\,.$$

Moreover, using the function $x^k\varphi$ as a test function in the weak formulation of (3.32) and passing again to the limit, we obtain

$$\begin{aligned}\int_\Omega \xi_0 \nabla\varphi\, x^k\, dx + \int_\Omega \xi_0\, \mathrm{e}_k \varphi\, dx &= \int_\Omega f\, x^k\, \varphi\, dx - \int_\Omega \overline{A}\nabla g_0 \nabla\varphi\, x^k\, dx \\ - \int_\Omega \overline{A}\nabla g_0\, \mathrm{e}^k\, \varphi\, dx &= \int_\Omega f\, x^k\, \varphi\, dx + \int_\Omega \operatorname{div}(\overline{A}\nabla g_0)\varphi\, x^k\, dx\,. \end{aligned} \tag{3.37}$$

Replacing (3.37) in (3.36) and simplifying, it follows

$$\begin{aligned} - \int_\Omega \xi_0\, \mathrm{e}_k\varphi\, dx &= \int_\Omega a^*_{kj}\partial_j\varphi\, v_0\, dx - \int_\Omega a^*_{kj}\partial_j g_0\, \varphi\, dx - \int_\Omega \overline{A}\nabla g_0\nabla\varphi\, x^k\, dx \\ -\int \operatorname{div}(\overline{A}\nabla g_0)\varphi\, x^k\, dx &= -\int_\Omega a^*_{kj}(\partial_j v_0 + \partial_j g_0)\varphi\, dx + \int_\Omega \operatorname{div}(\overline{A}\nabla g_0)\varphi\, x^k\, dx + \int_\Omega \overline{A}\nabla g_0\, \mathrm{e}_k\, \varphi\, dx \\ - \int_\Omega \operatorname{div}(\overline{A}\nabla g_0)\varphi\, x^k\, dx &= - \int_\Omega a^*_{kj}(\partial_j v_0 + \partial_j g_0)\varphi\, dx + \int_\Omega \overline{a}_{kj}\partial_j g_0\, \varphi\, dx\,, \end{aligned} \tag{3.38}$$

which implies

$$\xi_0^k = a^*_{kj}(\partial_j v_0 + \partial_j g_0) - \overline{a}_{kj}\partial_j g_0\,. \tag{3.39}$$

Finally, passing to the limit in the weak formulation of (3.32) and taking into account (3.39), we obtain

$$\int_\Omega A^*(\nabla v_0 + \nabla g_0)\nabla\psi\, dx - \int_\Omega \overline{A}\nabla g_0\nabla\psi\, dx = \int_\Omega f\,\psi\, dx - \int_\Omega \overline{A}\nabla g_0\nabla\psi\, dx\,, \qquad \forall\psi\in C_0^\infty(\Omega)\,,$$

which implies

$$\int_\Omega A^*(\nabla v_0 + \nabla g_0)\nabla\psi\, dx = \int_\Omega f\,\psi\, dx\,.$$

Setting $u_0 := v_0 + g_0$, it follows that $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$ and u_0 satisfies

$$\begin{cases} -\operatorname{div}(A^*\nabla u_0) = f & \text{in } \Omega\,; \\ \qquad\qquad u_0 = g & \text{on } \partial\Omega\,; \end{cases} \tag{3.40}$$

so that problem (3.40) is exactly the expected homogenized problem, where the homogenized matrix A^* does not depend on the boundary data.

3.5. **Concluding remarks.** The method presented above can be easily generalized to the case in which the matrix A depends also on the "slow variable" x; i.e., $A = A(x, y)$ and $A_\varepsilon(x) = A(x, x/\varepsilon)$, provided that it continuously depends on x (see, for instance, [12] and [32]).

Similarly, with the previous technique and without relevant changes, we can prove the same results for the case in which the source term depends on ε; i.e., $f = f_\varepsilon$, provided that the sequence $\{f_\varepsilon\}$ converges strongly in $L^2(\Omega)$.

For the sake of completeness, we recall that homogenization is often performed for integral functionals. This leads to an interesting and useful theory (the Γ-convergence theory, see for instance [15], [16], [22]), which however is out of the aims of these Notes.

4. **A model for the electrical conduction in biological tissues**

It is well known that electric potentials can be used in diagnostic devices to investigate the properties of biological tissues. Besides the well-known diagnostic techniques such as magnetic resonance, X-rays and so on, it plays an important role a more recent, cheap and noninvasive technique called electric impedance tomography (EIT). Such a technique is essentially based on the possibility of determining the physiological properties of a living body by means of the knowledge of its electrical behavior.
This leads to an inverse problem for an elliptic equation, usually the Laplacian, which is the equation satisfied by the electrical potential, when the body is assumed to display only a resistive behavior. However, it has been observed that, applying high frequency potentials to the body, a capacitive behavior appears, due to the electric polarization at the interface of the cell membranes, which act as capacitors. This phenomenon (known in physics as Maxwell-Wagner effect) is studied modeling the biological tissue as a composite medium with a periodic microscopic structure of characteristic length ε, where two finely mixed conductive phases (the intra- and the extra-cellular phase) are separated by a dielectric interface (the cellular membrane). From the mathematical point of view, the electrical current flow through the tissue is described by means of a system of decoupled elliptic equations in the two conductive phases (obtained from the Maxwell equations, under the quasi-static assumption; i.e., we assume that the magnetic effects are negligible). The solutions of this system are coupled because of the interface conditions at the membrane, whose physical behavior is described by means of a dynamical boundary condition, together with the flux-continuity assumption. Because of the complex geometry of the domain, these models are not easily handled, for example from the numerical point of view. This justifies the need of the homogenization approach, with the aim of producing macroscopic models for the whole medium as $\varepsilon \to 0$, since the typical scale ε of the microstructure is very small with respect to the tissue macroscopic scale analyzed in the experiments. The macroscopic equation obtained with this approach is an elliptic equation with memory, as it could be expected in any electrical circuit in which a capacitor is present.

The results presented in this section are contained in [3]–[7].

4.1. **Setting of the problem.** Let Ω be an open connected bounded subset of $\mathbb{R}^N$. Let us introduce a periodic open subset E of $\mathbb{R}^N$, so that $E + z = E$ for all $z \in \boldsymbol{Z}^N$. For all $\varepsilon > 0$ define $\Omega^\varepsilon_{\text{int}} = \Omega \cap \varepsilon E$, $\Omega^\varepsilon_{\text{out}} = \Omega \setminus \overline{\varepsilon E}$. We assume that Ω, E have regular boundary, say of class C^∞ for the sake of simplicity. Moreover, we set $\Omega = \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{out}} \cup \Gamma^\varepsilon$, where $\Gamma^\varepsilon = \partial\Omega^\varepsilon_{\text{int}} \cap \Omega = \partial\Omega^\varepsilon_{\text{out}} \cap \Omega$. We also employ the notation $Y = (0,1)^N$, and $E_{\text{int}} = E \cap Y$, $E_{\text{out}} = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that E_{int} is a connected smooth subset of Y such that $\operatorname{dist}(\overline{E_{\text{int}}}, \partial Y) > 0$. Some generalizations may be possible, but we do not dwell on this point here. Finally, we assume that $\operatorname{dist}(\Gamma^\varepsilon, \partial\Omega) > \gamma\varepsilon$ for some constant $\gamma > 0$ independent of ε, by dropping the inclusions contained in the cells $\varepsilon(Y + z)$, $z \in \boldsymbol{Z}^N$ which intersect $\partial\Omega$ (see Figure 1). Finally, let $T > 0$ be a given time.

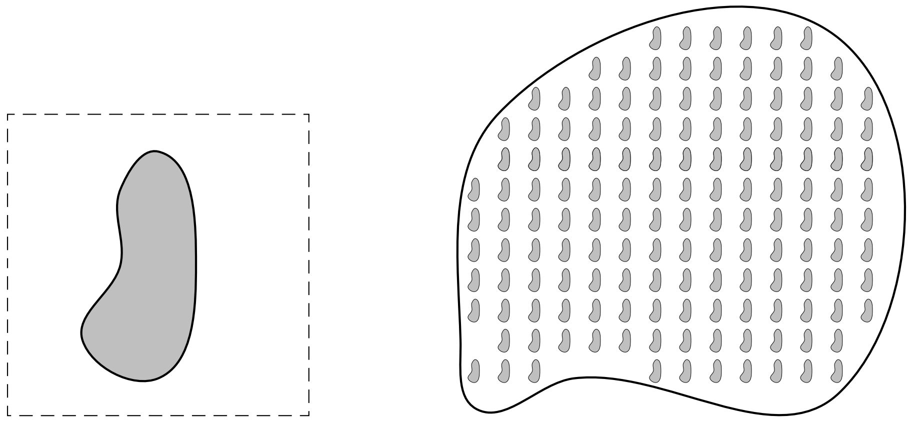

FIGURE 1. An examples of admissible periodic structures in $\mathbb{R}^2$. Left: Y is the dashed square, and $E \cap Y$ is the shaded region. Right: the domain Ω.

We are interested in the homogenization limit as $\varepsilon \searrow 0$ of the problem for $u_\varepsilon(x,t)$ (here the operators div and ∇ act only with respect to the space variable x)

$$-\operatorname{div}(\sigma_{\rm int}\nabla u_\varepsilon) = 0\,, \qquad \text{in } \Omega^\varepsilon_{\rm int}; \tag{4.1}$$

$$-\operatorname{div}(\sigma_{\rm out}\nabla u_\varepsilon) = 0\,, \qquad \text{in } \Omega^\varepsilon_{\rm out}; \tag{4.2}$$

$$\sigma_{\rm int}\nabla u_\varepsilon^{(\rm int)}\cdot\nu = \sigma_{\rm out}\nabla u_\varepsilon^{(\rm out)}\cdot\nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.3}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + \frac{\beta}{\varepsilon}[u_\varepsilon] = \sigma_{\rm out}\nabla u_\varepsilon^{(\rm out)}\cdot\nu\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.4}$$

$$[u_\varepsilon](x,0) = S_\varepsilon(x)\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.5}$$

$$u_\varepsilon(x,t) = 0\,, \qquad \text{on } \partial\Omega. \tag{4.6}$$

The notation in (4.1)–(4.4), (4.6), means that the indicated equations are in force in the relevant spatial domain for $0 < t < T$.
Here $\sigma_{\rm int}$, $\sigma_{\rm out}$ and α are positive constants, $\beta \geq 0$, and ν is the normal unit vector to Γ^ε pointing into $\Omega^\varepsilon_{\rm out}$. Since u_ε is not in general continuous across Γ^ε we have set

$$u_\varepsilon^{(\rm int)} := \text{ trace of } u_{\varepsilon|\Omega^\varepsilon_{\rm int}} \text{ on } \Gamma^\varepsilon; \quad u_\varepsilon^{(\rm out)} := \text{ trace of } u_{\varepsilon|\Omega^\varepsilon_{\rm out}} \text{ on } \Gamma^\varepsilon.$$

Indeed we refer conventionally to $\Omega^\varepsilon_{\rm int}$ as to the *interior domain*, and to $\Omega^\varepsilon_{\rm out}$ as to the *outer domain.* We also denote

$$[u_\varepsilon] := u_\varepsilon^{(\rm out)} - u_\varepsilon^{(\rm int)}\,.$$

Similar conventions are employed for other quantities; for example (4.3) can be rewritten as

$$[\sigma\nabla u_\varepsilon\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon,$$

where

$$\sigma = \sigma_{\rm int} \quad \text{in } \Omega^\varepsilon_{\rm int}, \qquad \sigma = \sigma_{\rm out} \quad \text{in } \Omega^\varepsilon_{\rm out}.$$

The initial data S_ε will be discussed below.
In Section 4.4, under the assumptions above, we prove existence and uniqueness of a weak solution to (4.1)–(4.6), in the class

$$u_{\varepsilon|\Omega^\varepsilon_i} \in L^2(0,T;H^1(\Omega^\varepsilon_i))\,, \qquad i = 1\,,2\,, \tag{4.7}$$

and $u_{\varepsilon|\partial\Omega} = 0$ in the sense of traces (see [6]).

A similar result holds true also for all the other problems which we will encounter in these Notes, which differ just for the nature of the boundary conditions.
In the following, we will show that, if $\gamma^{-1}\varepsilon \le S_\varepsilon(x) \le \gamma\varepsilon$, where S_ε is the initial jump prescribed in (4.5), for a fixed constant $\gamma > 1$, then u_ε becomes stable as $\varepsilon \to 0$ (i.e., it converges to a nonvanishing bounded function). Therefore, let us stipulate that $S_\varepsilon \in H^{1/2}(\Gamma^\varepsilon)$ and

$$S_\varepsilon(x) = \varepsilon S_1\big(x, \frac{x}{\varepsilon}\big) + \varepsilon R_\varepsilon(x)\,, \tag{4.8}$$

where $S_1 : \Omega \times \partial E \to \mathbb{R}$, and

$$\begin{aligned} &\|S_1\|_{L^\infty(\Omega\times\partial E)} < \infty\,, \qquad \|R_\varepsilon\|_{L^\infty(\Omega)} \to 0\,, \text{ as } \varepsilon \to 0;\\ &S_1(x,y) \text{ is continuous in } x, \text{ uniformly over } y \in \partial E,\\ &\text{and periodic in } y, \text{ for each } x \in \Omega. \end{aligned} \tag{4.9}$$

Firstly, we remark that, up to a change of function, we can assume $\beta = 0$; indeed, setting $v_\varepsilon(x,t) = u_\varepsilon(x,t)\mathrm{e}^{\frac{\beta}{\alpha}}$, it follows that v_ε satisfies

$$\begin{aligned} -\operatorname{div}(\sigma_{\mathrm{int}}\nabla v_\varepsilon) &= 0\,, && \text{in } \Omega^\varepsilon_{\mathrm{int}};\\ -\operatorname{div}(\sigma_{\mathrm{out}}\nabla v_\varepsilon) &= 0\,, && \text{in } \Omega^\varepsilon_{\mathrm{out}};\\ \sigma_{\mathrm{int}}\nabla v_\varepsilon^{(\mathrm{int})}\cdot\nu &= \sigma_{\mathrm{out}}\nabla v_\varepsilon^{(\mathrm{out})}\cdot\nu\,, && \text{on } \Gamma^\varepsilon;\\ \frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[v_\varepsilon] &= \sigma_{\mathrm{out}}\nabla v_\varepsilon^{(\mathrm{out})}\cdot\nu\,, && \text{on } \Gamma^\varepsilon;\\ [v_\varepsilon](x,0) &= S_\varepsilon(x)\,, && \text{on } \Gamma^\varepsilon;\\ v_\varepsilon(x,t) &= 0\,, && \text{on } \partial\Omega. \end{aligned}$$

Hence, from now on, we assume $\beta = 0$ in (4.4). The weak formulation of problem (4.1)–(4.6) is

$$\int_0^T\!\!\int_\Omega \sigma\nabla u_\varepsilon\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t - \frac{\alpha}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon}[u_\varepsilon]\frac{\partial}{\partial t}[\psi]\,\mathrm{d}\sigma\,\mathrm{d}t - \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon](0)[\psi](0)\,\mathrm{d}\sigma = 0\,, \tag{4.10}$$

for each $\psi \in L^2(\Omega\times(0,T))$ such that ψ is in the class (4.7), $[\psi] \in H^1(0,T;L^2(\Gamma^\varepsilon))$, and ψ vanishes on $\partial\Omega\times(0,T)$, as well as at $t = T$.
Moreover, multiplying (4.1), (4.2) by u_ε, integrating by parts and using (4.3)–(4.6), for all $0 < t < T$, we obtain the energy estimate

$$\int_0^t\!\!\int_\Omega \sigma|\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau + \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon]^2(x,t)\,\mathrm{d}\sigma = \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon} S_\varepsilon^2(x)\,\mathrm{d}\sigma \le C < +\infty\,, \tag{4.11}$$

where C does not depend on ε and the last inequality is due to (4.8), (4.9), taking into account that $|\Gamma^\varepsilon|_{N-1} \sim 1/\varepsilon$.
Inequality (4.11) together with the following Poincaré type lemma will give the correct estimate, which will be used in Section 4.5 in order to pass to the limit for $\varepsilon \to 0$ in the sequence of the solutions $\{u_\varepsilon\}$ of (4.1)–(4.6).

Lemma 4.1. (Poincaré inequality) *Let $v : \Omega \to \mathbb{R}$ be given by*

$$v_{|\Omega^\varepsilon_{\mathrm{int}}} = v_{1|\Omega^\varepsilon_{\mathrm{int}}}\,, \quad v_{|\Omega^\varepsilon_{\mathrm{out}}} = v_{2|\Omega^\varepsilon_{\mathrm{out}}}\,, \qquad v_1\,,\, v_2 \in H^1_0(\Omega)\,.$$

Then

$$\int_{\Omega} v^2 \,\mathrm{d}x \leq C\Big\{ \int_{\Omega} |\nabla v|^2 \,\mathrm{d}x + \varepsilon^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma \Big\} . \tag{4.12}$$

Here C depends only on Ω and E.

Proof. As v^2 is of class $W^{1,1}$ both in $\Omega^\varepsilon_{\mathrm{int}}$ and in $\Omega^\varepsilon_{\mathrm{out}}$, $v^2 \in BV(\Omega)$; hence, by Theorem 1.41, we obtain

$$\int_{\Omega} v^2 \,\mathrm{d}x \leq \gamma |Dv^2(\Omega)| \leq \gamma \int_{\Omega} |v||\nabla v| \,\mathrm{d}x + \gamma \int_{\Gamma^\varepsilon} |[v^2]| \,\mathrm{d}\sigma , \qquad \gamma = \gamma(\Omega) . \tag{4.13}$$

Indeed the singular part of the variation of v (and therefore of v^2) is concentrated on Γ^ε. Using Theorem 1.26 with δ replaced with $\sqrt{\varepsilon\delta}$, we estimate above last integral by

$$\int_{\Gamma^\varepsilon} |[v]|(|v^{(\mathrm{int})}| + |v^{(\mathrm{out})}|) \,\mathrm{d}\sigma \leq (2\delta\varepsilon)^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma + \frac{\delta\varepsilon}{2} \int_{\Gamma^\varepsilon} (|v^{(\mathrm{int})}|^2 + |v^{(\mathrm{out})}|^2) \,\mathrm{d}\sigma , \tag{4.14}$$

for a $\delta \in (0,1)$ to be chosen presently. Exploiting the periodicity of E, and standard trace inequalities, we check that for each cell $Q_i = \varepsilon(Y + z_i)$, $z_i \in \boldsymbol{Z}^N$,

$$\int_{\Gamma^\varepsilon \cap Q_i} (|v^{(\mathrm{int})}|^2 + |v^{(\mathrm{out})}|^2) \,\mathrm{d}\sigma \leq \gamma\varepsilon^{-1} \int_{\Omega \cap Q_i} \big(v^2 + \varepsilon^2 |\nabla v|^2\big) \,\mathrm{d}x , \tag{4.15}$$

where $\gamma = \gamma(E)$ does not depend on Q_i. Next we add (4.15) over all the cells covering Ω, and use the resulting inequality in (4.14). A further application of Theorem 1.26 to (4.13) yields

$$\int_{\Omega} v^2 \,\mathrm{d}x \leq \gamma\delta^{-1} \int_{\Omega} |\nabla v|^2 \,\mathrm{d}x + \gamma(\delta\varepsilon)^{-1} \int_{\Gamma^\varepsilon} [v]^2 \,\mathrm{d}\sigma + \gamma\delta \int_{\Omega} v^2 \,\mathrm{d}x ,$$

whence (4.12) on selecting a small enough δ. □

Remark 4.2. The factor ε^{-1} in (4.12) is necessary in general, as one can show easily by counterexample. However, if E or E^c is connected, this is not the case: actually, one can prove an estimate similar to (4.12), but with the factor ε^{-1} formally replaced by ε (in this spirit, see Lemma 6 of [31]).

From (4.11) and Lemma 4.1, after an integration in time of (4.12) over the interval $(0,T)$, it follows that

$$\|u_\varepsilon\|^2_{L^2\big(\Omega \times (0,T)\big)} \leq C \left(\int_0^T \int_{\Omega} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{\varepsilon} \int_0^T \int_{\Gamma^\varepsilon} [u_\varepsilon]^2 \,\mathrm{d}\sigma \,\mathrm{d}t \right) \leq C ,$$

where C does not depend on ε. Hence, from Theorem 1.24, we have that there exists a function $u \in L^2\big(0,T; BV(\Omega)\big)$, such that, up to a subsequence, $u_\varepsilon \to u$ weakly in $L^2\big(\Omega \times (0,T)\big)$. As in Section 3, it remains to identify the limit function u, and this will be done in a formal way in Section 4.3 and rigorously in Section 4.5.
We point out that, in this case, the situation is more delicate than in Section 3, since here the convergence of the sequence $\{u_\varepsilon\}$ takes place only weakly in $L^2\big(\Omega \times (0,T)\big)$ (actually also in $L^1_{loc}\big(0,T; L^1(\Omega)\big)$, see (5.9) in [4]). However, no a priori information on the convergence of the sequence $\{\nabla u_\varepsilon\}$ is available.

4.2. **Concentration of the physical problem.** We point out that in the physical setting, the cell membrane has a nonzero thickness, even it is very small with respect to the characteristic length of the cell. Hence, we denote by η the ratio between these two quantities and remark that $\eta << 1$. Moreover, we write Ω as $\Omega = \Omega^{\varepsilon,\eta} \cup \Gamma^{\varepsilon,\eta} \cup \partial\Gamma^{\varepsilon,\eta}$, where $\Omega^{\varepsilon,\eta}$ and $\Gamma^{\varepsilon,\eta}$ are two disjoint open subsets of Ω, $\Gamma^{\varepsilon,\eta}$ is the tubular neighborhood of Γ^{ε} with thickness $\varepsilon\eta$, and $\partial\Gamma^{\varepsilon,\eta}$ is its boundary. In addition, we assume also that $\Omega^{\varepsilon,\eta} = \Omega^{\varepsilon,\eta}_{\text{int}} \cup \Omega^{\varepsilon,\eta}_{\text{out}}$ and $\partial\Gamma^{\varepsilon,\eta} = (\partial\Omega^{\varepsilon,\eta}_{\text{int}} \cup \partial\Omega^{\varepsilon,\eta}_{\text{out}}) \cap \Omega$. Again, $\Omega^{\varepsilon,\eta}_{\text{out}}$, $\Omega^{\varepsilon,\eta}_{\text{int}}$ correspond to the conductive regions, and $\Gamma^{\varepsilon,\eta}$ to the dielectric shell. We assume that, for $\eta \to 0$ and $\varepsilon > 0$ fixed, $|\Gamma^{\varepsilon,\eta}| \sim \varepsilon\eta|\Gamma^{\varepsilon}|_{N-1}$, $\Omega^{\varepsilon,\eta} \to \Omega^{\varepsilon}_{\text{out}} \cup \Omega^{\varepsilon}_{\text{int}}$ and $\partial\Gamma^{\varepsilon,\eta} \to \Gamma^{\varepsilon}$. We employ also the notation $Y = E^{\eta} \cup \Gamma^{\eta} \cup \partial\Gamma^{\eta}$, where E^{η} and Γ^{η} are two disjoint open subsets of Y, Γ^{η} is the tubular neighborhood of Γ with thickness η, and $\partial\Gamma^{\eta}$ is its boundary. Moreover, $E^{\eta} = E^{\eta}_{\text{int}} \cup E^{\eta}_{\text{out}}$ (see Figure 2). For $\eta \to 0$, $E^{\eta} \to E_{\text{int}} \cup E_{\text{out}}$, $|\Gamma^{\eta}| \sim \eta|\Gamma|_{N-1}$ and $\partial\Gamma^{\eta} \to \Gamma$.

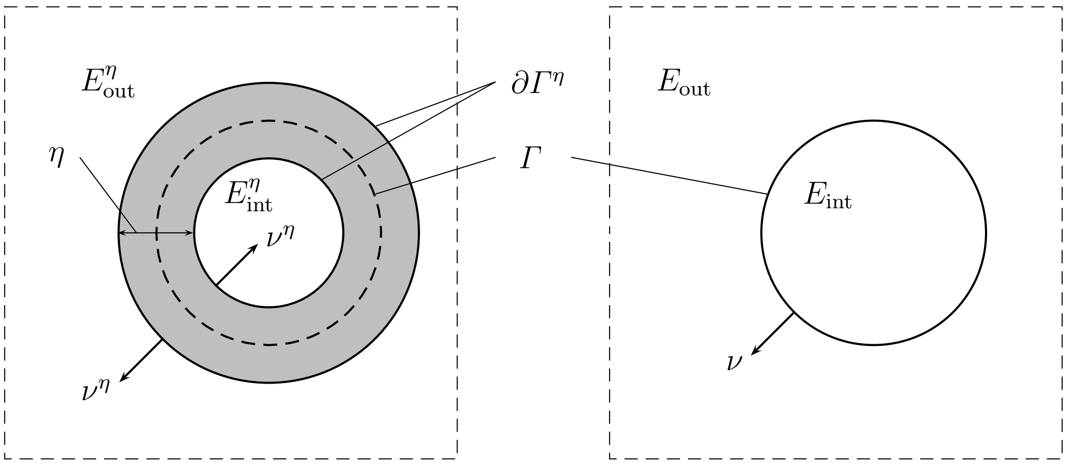

FIGURE 2. The periodic cell Y. *Left*: before concentration; Γ^{η} is the shaded region, and $E^{\eta} = E^{\eta}_{\text{int}} \cup E^{\eta}_{\text{out}}$ is the white region. *Right*: after concentration; Γ^{η} shrinks to Γ as $\eta \to 0$.

The classical governing equation is derived from the Maxwell system in the quasi-static approximation, which gives

$$-\operatorname{div}(A^{\eta}\nabla u^{\eta}_{\varepsilon}) = 0\,, \qquad \text{in } \Omega^{\varepsilon,\eta}; \tag{4.16}$$

$$-\operatorname{div}(B^{\eta}\nabla u^{\eta}_{\varepsilon t}) = 0\,, \qquad \text{in } \Gamma^{\varepsilon,\eta}; \tag{4.17}$$

$$A^{\eta}\nabla u^{\eta}_{\varepsilon} \cdot \nu^{\eta} = B^{\eta}\nabla u^{\eta}_{\varepsilon t} \cdot \nu^{\eta}\,, \qquad \text{on } \partial\Gamma^{\varepsilon,\eta}; \tag{4.18}$$

$$\nabla u^{\eta}_{\varepsilon}(x,0) = \boldsymbol{\mathcal{S}}^{\eta}_{\varepsilon}(x)\,, \qquad \text{in } \Gamma^{\varepsilon,\eta}; \tag{4.19}$$

$$u^{\eta}_{\varepsilon}(x,t) = 0\,, \qquad \text{on } \partial\Omega. \tag{4.20}$$

We assume that the conductivity $A^{\eta} > 0$ is such that $A^{\eta} = \sigma_{\text{int}}$ in $\Omega^{\varepsilon,\eta}_{\text{int}}$, $A^{\eta} = \sigma_{\text{out}}$ in $\Omega^{\varepsilon,\eta}_{\text{out}}$; the permeability $B^{\eta} > 0$ is such that $B^{\eta} = \alpha\,\eta$; and $\boldsymbol{\mathcal{S}}^{\eta}_{\varepsilon} = \nabla\widetilde{S}^{\eta}_{\varepsilon}$, for some $\widetilde{S}^{\eta}_{\varepsilon} \in H^1(\Gamma^{\varepsilon,\eta})$ with $|\boldsymbol{\mathcal{S}}^{\eta}_{\varepsilon}| \sim 1/\eta$.

Remark 4.3. We are interested in preserving, in the limit $\eta \to 0$, the conduction *across* the membrane Γ^{ε} instead of the *tangential* conduction on Γ^{ε}. To this purpose, we need to preserve the flux $B^\eta\nabla u^\eta_{\varepsilon t}\cdot\nu$ and the jump $[u^\eta_{\varepsilon t}]$ across the dielectric shells to be concentrated. This is the reason for which we rescale $B^\eta=\alpha\eta$, instead of scaling $B^\eta=\alpha/\eta$ in $\Gamma^{\varepsilon,\eta}$, as more usual in concentrated-capacity literature.

We are next interested in passing to the limit for $\eta\to 0^+$, keeping $\varepsilon>0$ fixed. In [7] it is proven the following result.

Theorem 4.4. *Under the previous assumptions, when $\eta\to 0^+$, it follows that the concentration of problem (4.16)–(4.20) is given by (4.1)–(4.6). More precisely, as $\eta\to 0^+$ it follows that $u^\eta_\varepsilon\to u_\varepsilon$, weakly in $L^2_{\rm loc}(\Omega\times(0,T))$, where $u_{\varepsilon|\Omega^\varepsilon_{\rm int}}\in L^2_{\rm loc}(0,T;H^1(\Omega^\varepsilon_{\rm int}))$, $u_{\varepsilon|\Omega^\varepsilon_{\rm out}}\in L^2_{\rm loc}(0,T;H^1(\Omega^\varepsilon_{\rm out}))$ and u_ε is the unique solution of (4.1)–(4.6). Moreover, as $\eta\to 0^+$, $\nabla u^\eta_\varepsilon\to\nabla u_\varepsilon$, weakly in $L^2_{\rm loc}(\Omega^\varepsilon_{\rm int}\times(0,T))$ and in $L^2_{\rm loc}(\Omega^\varepsilon_{\rm out}\times(0,T))$.*

We provide here just a formal sketch of the proof, letting $\eta\to 0$, while keeping $\varepsilon>0$ fixed. Let ϕ^η be a smooth testing function in Ω_T, such that $\nabla\phi^\eta(x,T)=0$ in $\Gamma^{\varepsilon,\eta}$, and ϕ^η vanishes in a neighborhood of $\partial\Omega$. Multiplying (4.16), (4.17) by ϕ^η, and integrating by parts, yields after routine calculations,

$$\int_0^T\int_{\Omega^{\varepsilon,\eta}} A^\eta\nabla u^\eta_\varepsilon\nabla\phi^\eta\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_{\Gamma^{\varepsilon,\eta}} B^\eta\nabla u^\eta_\varepsilon\nabla\phi^\eta_t\,\mathrm{d}x\,\mathrm{d}t=\int_{\Gamma^{\varepsilon,\eta}} B^\eta\boldsymbol{S}^\eta_\varepsilon\nabla\phi^\eta(0)\,\mathrm{d}x\,. \tag{4.21}$$

We expect the limit u_ε of u^η_ε to be discontinuous across the interface Γ^ε. Then we may approximate

$$\nabla u^\eta_\varepsilon\cdot\nu\sim\frac{[u^\eta_\varepsilon]}{\varepsilon\eta}\,,\qquad\text{on }\Gamma^\varepsilon, \tag{4.22}$$

where

$$[u^\eta_\varepsilon](P)=u^\eta_\varepsilon(P+\frac{\eta}{2}\nu)-u^\eta_\varepsilon(P-\frac{\eta}{2}\nu)\qquad\forall P\in\Gamma^\varepsilon\,. \tag{4.23}$$

In order to take advantage of (4.22), we select ϕ^η as follows: first, let it equal a smooth function in each of the two components $\Omega^{\varepsilon,\eta}_{\rm int}$ and $\Omega^{\varepsilon,\eta}_{\rm out}$. Then, extend it to $\Gamma^{\varepsilon,\eta}$ in such a way that

$$\nabla\phi^\eta\cdot\nu\sim\frac{[\phi^\eta]}{\varepsilon\eta}\,,\qquad\text{on }\Gamma^\varepsilon.$$

Then (4.21) can be rewritten as

$$\int_0^T\int_{\Omega^{\varepsilon,\eta}} A^\eta\nabla u^\eta_\varepsilon\nabla\phi^\eta\,\mathrm{d}x\,\mathrm{d}t-\varepsilon\eta^2\alpha\int_0^T\int_{\Gamma^\varepsilon}\frac{[u^\eta_\varepsilon]}{\varepsilon\eta}\frac{[\phi^\eta]_t}{\varepsilon\eta}\,\mathrm{d}\sigma\,\mathrm{d}t\simeq\varepsilon\eta^2\alpha\int_{\Gamma^\varepsilon}\frac{[\widetilde{S}^\eta_\varepsilon]}{\varepsilon\eta}\frac{[\phi^\eta(0)]}{\varepsilon\eta}\,\mathrm{d}\sigma\,.$$

We also exploited here the property $|\Gamma^{\varepsilon,\eta}|_N\sim\varepsilon\eta|\Gamma^\varepsilon|_{N-1}$, and we let $\boldsymbol{S}^\eta_\varepsilon\cdot\nu\sim[\widetilde{S}^\eta_\varepsilon]/\varepsilon\eta$, where $[\widetilde{S}^\eta_\varepsilon]$ is defined as in (4.23). Moreover, we assume that $[\widetilde{S}^\eta_\varepsilon]\to S_\varepsilon$ as $\eta\to 0$, for S_ε as in (4.5). Here and in the following we assume that the tangential parts on Γ^ε of the gradients of the involved functions are stable, so that they produce higher order terms in the asymptotic estimate in η and then they are neglected.
Finally taking the limit $\eta\to 0^+$ above, we get

$$\int_0^T\int_{\Omega^\varepsilon_{\rm out}\cup\Omega^\varepsilon_{\rm int}}\sigma\nabla u_\varepsilon\nabla\phi\,\mathrm{d}x\,\mathrm{d}t-\frac{\alpha}{\varepsilon}\int_0^T\int_{\Gamma^\varepsilon}[u_\varepsilon][\phi]_t\,\mathrm{d}\sigma\,\mathrm{d}t=\frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}S_\varepsilon[\phi(0)]\,\mathrm{d}\sigma\,,$$

which is nothing else than the weak formulation of (4.1)–(4.6), as given in (4.10).

4.3. **Formal homogenization of the concentrated problem.** In this section we aim at identifying the form of the homogenized equation, via the two-scale method (see [12], [32], [37]). To this purpose, we introduce the microscopic variables $y \in Y$, $y = x/\varepsilon$ and assume that the following asymptotic expansion holds

$$u_\varepsilon = u_\varepsilon(x, y, t) = u_0(x, y, t) + \varepsilon u_1(x, y, t) + \varepsilon^2 u_2(x, y, t) + \dots \,. \tag{4.24}$$

Note that u_0, u_1, u_2 are periodic in y. Recalling that

$$\operatorname{div} = \frac{1}{\varepsilon} \operatorname{div}_y + \operatorname{div}_x \,, \qquad \nabla = \frac{1}{\varepsilon} \nabla_y + \nabla_x \,, \tag{4.25}$$

we compute

$$\Delta\, u_\varepsilon = \frac{1}{\varepsilon^2} A_0 u_0 + \frac{1}{\varepsilon}(A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + \dots \,, \tag{4.26}$$

Here

$$A_0 = \Delta_y \,, \quad A_1 = \operatorname{div}_y \nabla_x + \operatorname{div}_x \nabla_y \,, \quad A_2 = \Delta_x \,. \tag{4.27}$$

Let us recall explicitly that

$$\nabla u_\varepsilon = \frac{1}{\varepsilon} \nabla_y u_0 + \big(\nabla_x u_0 + \nabla_y u_1\big) + \varepsilon\big(\nabla_y u_2 + \nabla_x u_1\big) + \dots \,, \tag{4.28}$$

and stipulate, in addition to (4.8),

$$S_\varepsilon = S_\varepsilon(x, y) = \varepsilon S_1(x, y) + \varepsilon^2 S_2(x, y) + \dots \,. \tag{4.29}$$

4.3.1. *The term of order* ε^{-2}. Equating the first term on the right hand side of (4.26) to zero, and applying (4.24), (4.28) to (4.1)–(4.5) we find

$$-\sigma\, \Delta_y\, u_0 = 0 \,, \qquad \text{in } E_{\mathrm{int}},\ E_{\mathrm{out}}; \tag{4.30}$$

$$\sigma_{\mathrm{int}} \nabla_y u_0^{(\mathrm{int})} \cdot \nu = \sigma_{\mathrm{out}} \nabla_y u_0^{(\mathrm{out})} \cdot \nu \,, \qquad \text{on } \mathit{\Gamma}; \tag{4.31}$$

$$\alpha \frac{\partial}{\partial t}[u_0] = \sigma_{\mathrm{out}} \nabla_y u_0^{(\mathrm{out})} \cdot \nu \,, \qquad \text{on } \mathit{\Gamma}; \tag{4.32}$$

$$[u_0](x, y, 0) = 0 \,, \qquad \text{on } \mathit{\Gamma}. \tag{4.33}$$

In (4.33) we have also exploited the expansion (4.29). By Theorem 4.6 it follows that

$$u_0 = u_0(x, t) \,. \tag{4.34}$$

4.3.2. *The term of order* ε^{-1}. Proceeding as above, but taking into consideration the second term on the right hand side of (4.26) we obtain

$$-\sigma\, \Delta_y\, u_1 = \sigma A_1 u_0 = 0 \,, \qquad \text{in } E_{\mathrm{int}},\ E_{\mathrm{out}}; \tag{4.35}$$

$$[\sigma \nabla_y u_1 \cdot \nu] = -[\sigma \nabla_x u_0 \cdot \nu] \,, \qquad \text{on } \mathit{\Gamma}; \tag{4.36}$$

$$\alpha \frac{\partial}{\partial t}[u_1] = \sigma_{\mathrm{out}} \nabla_y u_1^{(\mathrm{out})} \cdot \nu + \sigma_{\mathrm{out}} \nabla_x u_0 \cdot \nu \,, \qquad \text{on } \mathit{\Gamma}; \tag{4.37}$$

$$[u_1](x, y, 0) = S_1(x, y) \,, \qquad \text{on } \mathit{\Gamma}. \tag{4.38}$$

In (4.35) and in (4.37) we have made use of (4.34), and of its consequence $[u_0] = 0$.

We recall that in the classical case (see Section 3.1) the analysis of the term of order ε^{-1} leads to the factorization of u_1 in terms of the gradient of u_0 and a suitable cell function. Here the situation is more complicated, due to the presence of the time derivative in the left-hand side of (4.37); nevertheless, we still obtain a sort of factorization of u_1, which will be the crucial point in order to achieve the homogenized equation.

Let $s : \Gamma \to \mathbb{R}$ be a jump function. Consider the problem

$$\begin{aligned} -\sigma\,\Delta_y\, v &= 0\,, && \text{in } E_{\mathrm{int}},\, E_{\mathrm{out}}; && (4.39)\\ [\sigma\nabla_y v\cdot\nu] &= 0\,, && \text{on } \Gamma; && (4.40)\\ \alpha\frac{\partial}{\partial t}[v] &= \sigma_{\mathrm{out}}\nabla_y v^{(\mathrm{out})}\cdot\nu\,, && \text{on } \Gamma; && (4.41)\\ [v](y,0) &= s(y)\,, && \text{on } \Gamma, && (4.42) \end{aligned}$$

where v is a periodic function in Y, such that $\int_Y v = 0$. Define the transform $\mathcal{T}$ by

$$\mathcal{T}(s)(y,t) = v(y,t)\,, \qquad y\in Y\,, t>0\,,$$

and extend the definition of $\mathcal{T}$ to vector (jump) functions, by letting it act componentwise on its argument.
Introduce also the functions $\chi^0 : Y \to \mathbb{R}^N$, $\chi^1 : Y\times(0,T)\to\mathbb{R}^N$ as follows. The components χ^0_h, $h = 1, \dots, N$, satisfy

$$\begin{aligned} -\sigma\,\Delta_y\,\chi^0_h &= 0\,, && \text{in } E_{\mathrm{int}},\, E_{\mathrm{out}}; && (4.43)\\ [\sigma(\nabla_y\chi^0_h - \boldsymbol{e}_h)\cdot\nu] &= 0\,, && \text{on } \Gamma; && (4.44)\\ [\chi^0_h] &= 0\,, && \text{on } \Gamma. && (4.45) \end{aligned}$$

We also require χ^0_h to be a periodic function with vanishing integral average over Y. Moreover χ^1_h is defined from

$$\alpha\chi^1_h = \mathcal{T}\big(\sigma_{\mathrm{out}}(\nabla_y\chi^{0\,(\mathrm{out})}_h - \boldsymbol{e}_h)\cdot\nu\big)\,. \tag{4.46}$$

Let us stipulate that u_1 may be written in the form

$$u_1(x,y,t) = -\chi^0(y)\cdot\nabla_x u_0(x,t) + \mathcal{T}(S_1(x,\cdot))(y,t) - \int_0^t \nabla_x u_0(x,\tau)\cdot\chi^1(y,t-\tau)\,\mathrm{d}\tau\,. \tag{4.47}$$

It is worthwhile making some remarks on the structure of u_1, which is made of three parts: the first one is the standard one (see (3.9)); while the second one keeps into account the effect of the initial datum. The real novelty is the third integral term, which is a non local memory term, due to the capacitive behavior of the membrane; i.e., to the dynamical nature of condition (4.4).

Equations (4.35) are equivalent to (4.43), when we remember that the terms χ^1_h and $\mathcal{T}(S_1(x,\cdot))$ in (4.47) fulfil (4.39). Next, let us impose (4.36) in (4.47). We get, on recalling (4.40)

$$[\sigma\nabla_y u_1\cdot\nu] = -[\sigma\nabla_y\chi^0_h(y)\cdot\nu]u_{0x_h}(x,t) = -[\sigma\nabla_x u_0\cdot\nu] = -[\sigma u_{0x_h}(x,t)\nu_h]\,.$$

In order to satisfy this requirement, we prescribe (4.44). Note that (4.38) is obviously satisfied, owing to the definition of $\mathcal{T}$.

Finally we get to (4.37), which we combine with (4.47) obtaining

$$\alpha\frac{\partial}{\partial t}[u_1] + f'(0)[u_1] = -\alpha[\chi^0(y)\cdot\frac{\partial}{\partial t}\nabla_x u_0(x,t)] + \alpha\frac{\partial}{\partial t}[\mathcal{T}(S_1(x,\cdot))](y,t) \\ - \alpha\nabla_x u_0(x,t)\cdot[\chi^1](y,0) - \alpha\int_0^t \nabla_x u_0(x,\tau)\cdot\frac{\partial}{\partial t}[\chi^1](y,t-\tau)\,\mathrm{d}\tau\,.$$

On the other hand,

$$\sigma_{\mathrm{out}}\nabla_y u_1^{(\mathrm{out})}\cdot\nu + \sigma_{\mathrm{out}}\nabla_x u_0\cdot\nu = -\sigma_{\mathrm{out}}\nabla_y\chi_h^{0\,(\mathrm{out})}(y)\cdot\nu u_{0x_h}(x,t) \\ + \sigma_{\mathrm{out}}\nabla_y\mathcal{T}(S_1(x,\cdot))^{(\mathrm{out})}\cdot\nu - \int_0^t u_{0x_h}(x,\tau)\sigma_{\mathrm{out}}\nabla_y\chi_h^1(y,t-\tau)\cdot\nu\,\mathrm{d}\tau + \sigma_{\mathrm{out}}u_{0x_h}(x,t)\nu_h\,.$$

Hence, (4.45)–(4.46) follow, on equating the quantities above.

Remark 4.5. We note that (4.47) defines u_1 up to an additive function $\widetilde{u}_1(x,t)$ independent of y. However, since for our purpose we will stop the expansion (4.24) to the second order term, $\widetilde{u}_1$ does not play any role, so that we may assume that it is identically zero, without loss of generality.

4.3.3. *The term of order ε^0.* Let us first calculate

$$A_1u_1 = 2\frac{\partial^2 u_1}{\partial x_j\partial y_j}\,,$$

where we employ the summation convention. Therefore, the complete problem involving the third term on the right hand side of (4.26) is

$$-\sigma\,\Delta_y\,u_2 = \sigma\,\Delta_x\,u_0 + 2\sigma\frac{\partial^2 u_1}{\partial x_j\partial y_j}\,, \qquad E_{\mathrm{int}},E_{\mathrm{out}} \tag{4.48}$$

$$[\sigma\nabla_y u_2\cdot\nu] = -[\sigma\nabla_x u_1\cdot\nu]\,, \qquad \text{on } \Gamma; \tag{4.49}$$

$$\alpha\frac{\partial}{\partial t}[u_2] = \sigma_{\mathrm{out}}\nabla_x u_1^{(\mathrm{out})}\cdot\nu + \sigma_{\mathrm{out}}\nabla_y u_2^{(\mathrm{out})}\cdot\nu\,, \qquad \text{on } \Gamma; \tag{4.50}$$

$$[u_2](x,y,0) = S_2(x,y)\,, \qquad \text{on } \Gamma. \tag{4.51}$$

Integrating by parts equation (4.48) both in E_{int} and in E_{out}, and adding the two contributions, we get

$$\Bigg[\int_{E_{\mathrm{int}}} + \int_{E_{\mathrm{out}}}\Bigg]\left\{\sigma\,\Delta_x\,u_0(x,t) + 2\sigma\frac{\partial^2 u_1}{\partial x_j\partial y_j}\right\}\mathrm{d}y \\ = \int_\Gamma\left\{\sigma_{\mathrm{out}}\nabla_y u_2^{(\mathrm{out})}\cdot\nu - \sigma_{\mathrm{int}}\nabla_y u_2^{(\mathrm{int})}\cdot\nu\right\}\mathrm{d}\sigma = \int_\Gamma[\sigma\nabla_y u_2\cdot\nu]\,\mathrm{d}\sigma = -\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma\,.$$

Thus

$$\sigma_0\,\Delta_x\,u_0 = 2\int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma - \int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma = \int_\Gamma[\sigma\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma\,,$$

where we denote

$$\sigma_0 = \sigma_{\mathrm{int}}|E_{\mathrm{int}}| + \sigma_{\mathrm{out}}|E_{\mathrm{out}}|\,. \tag{4.52}$$

We use equality (4.47), where only last two terms on the right hand side have a non zero jump across Γ. Thus we infer from the equality above

$$\sigma_0\,\Delta_x u_0 = \int_\Gamma [\sigma\nabla_x u_1\cdot\nu]\,\mathrm{d}\sigma = -\int_\Gamma [\sigma]\chi_h^0(y)\nu_j\,\mathrm{d}\sigma u_{0x_hx_j}(x,t)$$

$$+\frac{\partial}{\partial x_j}\int_\Gamma [\sigma\mathcal{T}(S_1(x,\cdot))](y,t)\nu_j\,\mathrm{d}\sigma - \int_0^t u_{0x_hx_j}(x,\tau)\int_\Gamma [\sigma\chi_h^1](y,t-\tau)\nu_j\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

We finally write the PDE for u_0 in $\Omega\times(0,T)$ as

$$-\operatorname{div}\Bigg(\sigma_0\nabla_x u_0 + A^0\nabla_x u_0 + \int_0^t A^1(t-\tau)\nabla_x u_0(x,\tau)\,\mathrm{d}\tau - \int_\Gamma [\sigma\mathcal{T}(S_1(x,\cdot))](y,t)\nu\,\mathrm{d}\sigma\Bigg) = 0\,, \tag{4.53}$$

where the two matrices A^i are defined by

$$(A^0)_{jh} = \int_\Gamma [\sigma]\chi_h^0(y)\nu_j\,\mathrm{d}\sigma\,, \qquad (A^1(t))_{jh} = \int_\Gamma [\sigma\chi_h^1](y,t)\nu_j\,\mathrm{d}\sigma\,, \tag{4.54}$$

are symmetric, and $\sigma_0 I + A^0$ is positive definite (see [4], Section 4). Accordingly with (4.6), equation (4.53) must be complemented with the homogeneous boundary datum $u_0(x,t)=0$ on $\partial\Omega\times(0,T)$.
We emphasize again that the decomposition of u_1 stated in (4.47) has been crucial in order to determine the homogenized equation (4.53). That equation is different from the standard elliptic equation which are usually employed in bioimpedenziometric tomography. In fact, the appearance of the memory term in the equation (which, from the mathematical point of view, is a consequence of the structure of the first corrector u_1) seems to be in agreement with the fact that a contribution to current flux is produced not only by the potential applied to the boundary but also by the charge and discharge cycles of the membranes; i.e., to their capacitive behavior.

4.4. **Well-posedness results.** The first result of this section is connected with the well-posedness of all the microscopic problems appearing in our framework.

Theorem 4.6. *Let Ω be an open connected bounded subset of $\mathbb{R}^N$ such that $\Omega=\Omega_1\cup\Omega_2\cup\Gamma$, where Ω_1 and Ω_2 are two disjoint open subset of Ω, $\Gamma=\overline{\partial\Omega_1\cap\Omega}=\overline{\partial\Omega_2\cap\Omega}$ is a compact regular set, and $\Gamma\cap\partial\Omega=\emptyset$. Assume also that Ω, Ω_1 and Ω_2 have Lipschitz boundaries. Let $\alpha>0$ and $\beta\geq 0$. Let $f\in L^2(\Omega\times(0,T))$, $q,h\in L^2(0,T;L^2(\Gamma))$, and $S\in H^{1/2}(\Gamma)$. Therefore, problem*

$$-\sigma\,\Delta v = f(t)\,, \qquad \text{in } \Omega_{\mathrm{int}},\ \Omega_{\mathrm{out}}; \tag{4.55}$$

$$[\sigma\nabla v\cdot\nu] = q(t)\,, \qquad \text{on } \Gamma; \tag{4.56}$$

$$\alpha\frac{\partial}{\partial t}[v] + \beta[v] = \sigma_{\mathrm{out}}\nabla v^{(\mathrm{out})}\cdot\nu + h(t)\,, \qquad \text{on } \Gamma; \tag{4.57}$$

$$[v](x,0) = S\,, \qquad \text{on } \Gamma; \tag{4.58}$$

$$v(x,t) = 0\,, \qquad \text{on } \partial\Omega; \tag{4.59}$$

admits a unique solution $v \in L^2(0,T;\mathcal{H}^1_o(\Omega))$ *with* $[v] \in C(0,T;L^2(\Gamma))$*, where* $\mathcal{H}^1_o(\Omega) = \{u = (u_1, u_2) \mid u_1 := u_{|\Omega_{\rm int}}\,,\ u_2 := u_{|\Omega_{\rm out}} \text{ with } u_1, u_2 \in H^1_o(\Omega)\}$.

The technique employed to prove this theorem relies on a result of existence and uniqueness for abstract parabolic equations (see [39], Chapter 23), to which problem (4.55)–(4.59) can be reduced by means of a suitable identification of the function spaces there involved.

Remark 4.7. Note that the same result as in Theorem 4.6 holds if we assume that $\Omega = Y = (0,1)^N$, $g(\cdot,t)$ is Y-periodic for a.e. $t \in (0,T)$, f and q satisfy the compatibility condition

$$\int_Y f(y,t)\,dy = \int_\Gamma q(y,t)\,dy \qquad \text{for a.e. } t \in (0,T)\,,$$

and we replace (4.59) with the requirement that $v(\cdot,t)$ is Y-periodic.

For equation (4.53), complemented with boundary conditions (e.g. Dirichlet boundary condition), an existence and uniqueness theorem, both for weak and classical solutions, is available (see [5]). Indeed, let us consider the following problem

$$\begin{cases} -\operatorname{div}\left(A(x)\nabla_x u + \displaystyle\int_0^t B(x,t-\tau)\nabla_x u(x,\tau)\,d\tau\right) = f(x,t) & \text{in } \Omega\times(0,T)\,, \\ u = g & \text{in } \partial\Omega\times(0,T)\,, \end{cases} \tag{4.60}$$

where $A(x)$ is a symmetric and positive definite matrix, $B(x,t)$ is a symmetric matrix, $f : \Omega\times(0,T) \to \mathbb{R}$ and $g : \overline{\Omega}\times(0,T) \to \mathbb{R}$ are given functions. Then the two following results hold true.

Theorem 4.8. *Let* $A \in L^\infty(\Omega;\mathbb{R}^{N^2})$ *be a symmetric matrix such that* $\lambda|\xi|^2 \le A(x)\xi\cdot\xi \le \Lambda|\xi|^2$*, for suitable* $0 < \lambda < \Lambda < +\infty$*, for almost every* $x \in \Omega$ *and every* $\xi \in \mathbb{R}^N$*; let* $B \in L^2(0,T;L^\infty(\Omega;\mathbb{R}^{N^2}))$*, and let* $g \in L^2(0,T;H^1(\Omega))$*. Assume that* $f : \Omega\times(0,T) \to \mathbb{R}$ *is a Carathéodory function such that* $f \in L^2(0,T;H^{-1}(\Omega))$*.*
Then, there exists a unique function $u \in L^2(0,T;H^1(\Omega))$ *satisfying in the sense of distributions problem (4.60).*

Theorem 4.9. *Let* $m \ge 0$ *be any fixed integer and let also* $0 < \alpha < 1$*. Let* $A \in C^{1+\alpha}(\overline{\Omega};\mathbb{R}^{N^2})$ *satisfy the assumption of Theorem 4.8 and*

$$B \in C^0([0,T];C^{1+\alpha}(\overline{\Omega};\mathbb{R}^{N^2})) \qquad \textit{be such that} \qquad B' \in L^2(0,T;W^{1,\infty}(\Omega;\mathbb{R}^{N^2}))\,.$$

Assume that $f \in C^0([0,T];C^{m+\alpha}(\overline{\Omega}))$*, and that* $\nabla_x f(x,t)$ *and* $f_t(x,t)$ *exist and are bounded. Let* $g \in C^0([0,T];C^{m+2+\alpha}(\overline{\Omega}))$*, with* $g_t \in L^\infty(0,T;C^{m+2+\alpha}(\overline{\Omega}))$*.*
Then there exists a unique function $u \in C^0([0,T];C^{1+\alpha}(\overline{\Omega})) \cap L^\infty(0,T;C^{m+2+\alpha}(\overline{\Omega}))$ *solving* (4.60) *in the classical sense.*

Both the proofs can be obtained, for example, with a standard delay argument or a fixed point theorem, together with an a-priori estimate in the corresponding function spaces. The a-priori estimates are obtained as in standard elliptic equations, using also the Gronwall's Theorem to deal with the memory term.

4.5. **The homogenization theorem.** The aim of this section is to state the rigorous mathematical proof of the homogenization result.

Theorem 4.10. *Under the assumptions listed in Section 4.1, as $\varepsilon \to 0$, we have that $u_\varepsilon \to u_0$, weakly in $L^2(\Omega \times (0,T))$, and strongly in $L^1_{\mathrm{loc}}(0,T;L^1(\Omega))$, where the limit $u_0 \in L^2(0,T;H^1_0(\Omega))$ solves (4.53).*

Proof. Introduce for $i = 1, \dots, N$, the functions

$$w_i^\varepsilon(x,t) = x_i - \varepsilon\chi_i^0\big(\frac{x}{\varepsilon}\big) - \varepsilon \int_t^T \chi_i^1\big(\frac{x}{\varepsilon}, \tau - t\big)\,\mathrm{d}\tau\,, \tag{4.61}$$

so that explicit calculations reveal

$$-\sigma\,\Delta\, w_i^\varepsilon = 0\,, \qquad \text{in } \Omega^\varepsilon_{\mathrm{int}},\ \Omega^\varepsilon_{\mathrm{out}}; \tag{4.62}$$

$$[\sigma\nabla w_i^\varepsilon\cdot\nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.63}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[w_i^\varepsilon] = -\sigma_{\mathrm{out}}\nabla w_i^{\varepsilon(\mathrm{out})}\cdot\nu\,, \qquad \text{on } \Gamma^\varepsilon. \tag{4.64}$$

Let $\varphi \in C^\infty_o(\Omega)$, and select $w_i^\varepsilon\varphi$ as a testing function in the weak formulation (4.10). We obtain

$$\int_0^T\!\!\int_\Omega \sigma\nabla u_\varepsilon\cdot\nabla w_i^\varepsilon\,\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \sigma\nabla u_\varepsilon\cdot\nabla\varphi\, w_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{\alpha}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon}[u_\varepsilon]\frac{\partial}{\partial t}[w_i^\varepsilon]\varphi\,\mathrm{d}\sigma\,\mathrm{d}t - \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[u_\varepsilon](0)[w_i^\varepsilon](0)\varphi\,\mathrm{d}\sigma = 0\,, \tag{4.65}$$

once we use the obvious relation $[w_i^\varepsilon](x,T) = 0$. Next select $u_\varepsilon\varphi$ as a testing function in the weak formulation of (4.62)–(4.64); in this second step, no integration by parts in t is needed on Γ^ε. We get

$$\int_0^T\!\!\int_\Omega \sigma\nabla w_i^\varepsilon\cdot\nabla u_\varepsilon\,\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \sigma\nabla w_i^\varepsilon\cdot\nabla\varphi\, u_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$$
$$- \frac{\alpha}{\varepsilon}\int_0^T\!\!\int_{\Gamma^\varepsilon}\frac{\partial}{\partial t}[w_i^\varepsilon][u_\varepsilon]\varphi\,\mathrm{d}\sigma\,\mathrm{d}t = 0\,. \tag{4.66}$$

Subtract (4.66) from (4.65) and find, taking (4.8) into account,

$$\int_0^T\!\!\int_\Omega \sigma\nabla u_\varepsilon\cdot\nabla\varphi\, w_i^\varepsilon\,\mathrm{d}x\,\mathrm{d}t = K_{1\varepsilon} + K_{2\varepsilon}\,, \tag{4.67}$$

where we have defined

$$K_{1\varepsilon} = \int_0^T \int_\Omega \sigma \nabla w_i^\varepsilon \cdot \nabla \varphi \, u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t \,,$$

$$K_{2\varepsilon} = -\alpha\varepsilon \int_{\Gamma^\varepsilon} \big(S_1\big(x, \frac{x}{\varepsilon}\big) + R_\varepsilon(x)\big)\varphi(x) \int_0^T [\chi_i^1]\big(\frac{x}{\varepsilon}, \tau\big) \, \mathrm{d}\tau \, \mathrm{d}\sigma \,.$$

We rely here on the energy inequality (4.11) which, together with Lemma 4.1 and Theorem 1.24, imply that, extracting subsequences if needed, we may assume

$$-\sigma\nabla u_\varepsilon \to \xi \,, \quad u_\varepsilon \to u \,, \qquad \text{weakly in } L^2(\Omega \times (0,T)), \tag{4.68}$$

and, by [4, Lemma 7.4]), also

$$u_\varepsilon \to u \,, \qquad \text{strongly in } L^1_{\mathrm{loc}}(0,T; L^1(\Omega)), \tag{4.69}$$

for some $\xi \in L^2(\Omega \times (0,T))^N$, $u \in L^2(\Omega \times (0,T))$. On the other hand, clearly $w_i^\varepsilon \to x_i$ strongly in $L^2(\Omega \times (0,T))$, as $\varepsilon \to 0$. Let us investigate the limiting behavior of $\sigma\nabla w_i^\varepsilon$. Due to the periodicity of the functions χ^i, and to (4.54), one immediately gets

$$\sigma\nabla\left(x_i - \varepsilon\chi_i^0\big(\frac{x}{\varepsilon}\big)\right) \to (\sigma_0 I + A^0)\boldsymbol{e}_i \,, \quad \text{weakly in } L^2(\Omega).$$

By the same token, in the same weak sense,

$$-\sigma\nabla\left(\varepsilon\int_t^T \chi_i^1\big(\frac{x}{\varepsilon}, \tau - t\big)\,\mathrm{d}\tau\right) \to -\int_t^T \int_Y \sigma\nabla_y\chi_i^1(y, \tau - t)\,\mathrm{d}y\,\mathrm{d}\tau = \int_t^T A^1(\tau - t)\boldsymbol{e}_i\,\mathrm{d}\tau \,,$$

where the last equality follows from the definition (4.54) of A^1 and from a trivial integration by parts. Thus, invoking Lemma 7.5 and Remark 7.3 in [4],

$$K_{1\varepsilon} \to \int_0^T\int_\Omega (\sigma_0 I + A^0)\boldsymbol{e}_i \cdot \nabla\varphi u \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega\int_t^T A^1(\tau - t)\boldsymbol{e}_i\,\mathrm{d}\tau \cdot \nabla\varphi u\,\mathrm{d}x\,\mathrm{d}t =: K_{10} \,.$$

Elementary manipulations show that

$$K_{10} = \int_0^T\int_\Omega \Big\{u(x,t)(\sigma_0 I + A^0)\boldsymbol{e}_i + \int_0^t u(x,\tau)A^1(t-\tau)\boldsymbol{e}_i\,\mathrm{d}\tau\Big\} \cdot \nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}t \,.$$

Next we turn to the task of evaluating the limiting behavior of $K_{2\varepsilon}$. Clearly the term involving R_ε vanishes in the limit. Then we appeal to the stipulated regularity of S_1, and apply, with minor changes, the ideas of [38] Lemma 3; we infer

$$K_{2\varepsilon} \to -\alpha\int_\Omega \varphi(x)\int_\Gamma S_1(x,y)\int_0^T [\chi_i^1](y,\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma\,\mathrm{d}x = \int_0^T\int_\Omega \varphi(x)\mathcal{F}_i(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \,,$$

where $\mathcal{F}$ is defined by

$$\mathcal{F}_i(x,t) := -\alpha\int_\Gamma S_1(x,y)[\chi_i^1](y,t)\,d\sigma = \int_\Gamma [\sigma\mathcal{T}(S_1(x,\cdot))](y,t)\nu_i\,d\sigma \,. \tag{4.70}$$

Collecting the results above, let $\varepsilon \to 0$ in (4.67) to arrive at

$$-\int_0^T\!\!\int_\Omega \xi\cdot\nabla\varphi\, x_i\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\!\!\int_\Omega \varphi(x)\mathcal{F}_i(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^T\!\!\int_\Omega \Big\{u(x,t)(\sigma_0 I + A^0)\boldsymbol{e}_i + \int_0^t u(x,\tau)A^1(t-\tau)\boldsymbol{e}_i\,\mathrm{d}\tau\Big\}\cdot\nabla\varphi(x)\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.71}$$

As usual, next we take φx_i as a testing function in (4.10). This test essentially does not detect the boundary Γ^ε, due to (4.3); on letting $\varepsilon \to 0$

$$\int_0^T\!\!\int_\Omega \xi\cdot\nabla\varphi\, x_i\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \xi\cdot\boldsymbol{e}_i\,\varphi\,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{4.72}$$

We substitute (4.72) in (4.71), and differentiate in T the resulting equality; in fact the choice of T is essentially arbitrary in this setting. We obtain (reverting to t as the time variable)

$$\int_\Omega \Big\{u(x,t)(\sigma_0 I + A^0) + \int_0^t u(x,\tau)A^1(t-\tau)\,\mathrm{d}\tau\Big\}\nabla\varphi(x)\,\mathrm{d}x$$
$$= \int_\Omega \xi(x,t)\varphi(x)\,\mathrm{d}x - \int_\Omega \varphi(x)\mathcal{F}(x,t)\,\mathrm{d}x\,,$$

for a.e. $t\in(0,T)$. Using [4, Lemma7.2], it follows that $u\in L^2(0,T;H^1(\Omega))$ and that

$$\xi(x,t) = -(\sigma_0 I + A^0)\nabla u(x,t) - \int_0^t A^1(t-\tau)\nabla u(x,\tau)\,\mathrm{d}\tau + \mathcal{F}(x,t)\,, \tag{4.73}$$

for a.e. $(x,t)\in\Omega\times(0,T)$. Clearly $\operatorname{div}\xi = 0$ in the sense of distributions (see e.g., (4.72) above).

Recalling (4.70), this shows that the limit function u satisfies (4.53). Moreover, as it will be proved in the following Subsection 4.5.1, u satisfies also the homogeneous boundary condition. Therefore, by Theorem 4.8, with $f(x,t) = -\operatorname{div}\mathcal{F}(x,t)$ and $g=0$, the function u coincides with u_0 and hence the whole sequence $\{u_\varepsilon\}$ converges to u_0, so that the result is achieved. □

Remark 4.11. Equality (4.73), which is the constitutive relationship of the homogenized material, expresses the limiting current ξ as a function of the history of the gradient of the potential, ∇u_0.

4.5.1. *u vanishes on $\partial\Omega$.* The trace of u on $\partial\Omega$ exists for a.e. $t\in(0,T)$, because of the already proven regularity of u. It is left to show that this trace is zero.

We understand here u and each u_ε to be defined on $\mathbb{R}^N\times(0,T)$, by extending them as zero outside Ω. Also define,

$$U_\varepsilon(x) = \int_0^T u_\varepsilon(x,t)\,\mathrm{d}t\,,\qquad U(x) = \int_0^T u(x,t)\,\mathrm{d}t\,.$$

Since we already know that the trace on $\partial\Omega$ of each u_ε, and therefore of U_ε, is zero, we infer that for each bounded open set $G\subset\mathbb{R}^N$, the variation $|DU_\varepsilon|(G)$ is given by

$$|DU_\varepsilon|(G)=\int\limits_G\left|\int\limits_0^T\nabla u_\varepsilon\,\mathrm{d}t\right|\,\mathrm{d}x+\int\limits_{\Gamma^\varepsilon\cap G}\left|\int\limits_0^T[u_\varepsilon]\,\mathrm{d}t\right|\,\mathrm{d}\sigma\le\gamma\left(|G|^{1/2}\Big(\int\limits_G\int\limits_0^T|\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/2}\right.$$
$$\left.+(\varepsilon|\Gamma^\varepsilon\cap G|_{N-1})^{1/2}\Big(\frac{1}{\varepsilon}\int\limits_{\Gamma^\varepsilon\cap G}\int\limits_0^T[u_\varepsilon]^2\,\mathrm{d}t\,\mathrm{d}\sigma\Big)^{1/2}\right)\le\gamma\big(|G|^{1/2}+(\varepsilon|\Gamma^\varepsilon\cap G|_{N-1})^{1/2}\big)\,,\quad(4.74)$$

where we have made use of Hölder's inequality and of (4.11). As a first consequence of this estimate, we may invoke classical compactness and semicontinuity results to show that (extracting subsequences if needed)

$$U_\varepsilon\to U\,,\qquad\text{in }L^1(\mathbb{R}^N),\qquad|DU|(G)\le\liminf_{\varepsilon\to0}|DU_\varepsilon|(G)\,,\tag{4.75}$$

for every set $G\subset\mathbb{R}^N$ as above. On the other hand, according to [9, Theorem 3.77] ,

$$|DU|(\partial\Omega)=\int\limits_{\partial\Omega}|U^+-U^-|\,\mathrm{d}\sigma=\int\limits_{\partial\Omega}|U^+|\,\mathrm{d}\sigma\,,\tag{4.76}$$

where the symbol U^+ (respectively, U^-) denotes the trace on $\partial\Omega$ of $U_{0|\Omega}$ (respectively, of $U_{0|\mathbb{R}^N\setminus\overline{\Omega}}\equiv0$).
Define for $0<h<1$ the open set

$$V_h=\{x\in\mathbb{R}^N\mid\operatorname{dist}(x,\partial\Omega)<h\}\,.$$

Combining (4.74)–(4.76), we obtain, as $\partial\Omega\subset V_h$ for all h,

$$\int\limits_{\partial\Omega}|U^+|\,\mathrm{d}\sigma\le|DU(V_h)|\le\gamma\liminf_{\varepsilon\to0}\big(|V_h|^{1/2}+(\varepsilon|\Gamma^\varepsilon\cap V_h|_{N-1})^{1/2}\big)\le\gamma h^{1/2}\,.$$

Indeed, it is readily seen that $|V_h|\le\gamma h$, and that $|\Gamma^\varepsilon\cap V_h|_{N-1}\le\gamma h/\varepsilon$ for all sufficiently small h. Therefore, letting $h\to0$ above we obtain that $U^+=0$ a.e. on $\partial\Omega$. However, U^+ and the trace u^+ of u are related by

$$U^+(x)=\int\limits_0^T u^+(x,t)\,\mathrm{d}t\,,\qquad x\in\partial\Omega\,.$$

Clearly, T stands here for any positive time, so that, differentiating the last equality in time, we obtain $u^+(x,t)=0$ a.e. on $\partial\Omega\times(0,T)$.

4.6. **Comparison with other similar models.** In this regard, different models are obtained corresponding to different scaling with respect to ε (where ε denotes the length of the periodicity cell) of the relevant physical quantity α, entering in the dynamical boundary condition given by

$$\frac{\alpha}{\varepsilon^k}\frac{\partial}{\partial t}[u_\varepsilon]=\sigma\nabla u_\varepsilon^{\mathrm{out}}\cdot\nu\,,\qquad\text{on the membrane interface,}\tag{4.77}$$

with $k\in\boldsymbol{Z}$. In this case the energy estimate becomes

$$\int\limits_0^T\int\limits_\Omega\sigma|\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}\tau+\frac{\alpha}{2\varepsilon^k}\int\limits_{\Gamma^\varepsilon}[u_\varepsilon]^2(x,T)\,\mathrm{d}\sigma=\frac{\alpha}{2\varepsilon^k}\int\limits_{\Gamma^\varepsilon}S_\varepsilon^2(x)\,\mathrm{d}\sigma\,,$$

which, taking into account that $|\Gamma^\varepsilon|_{N-1} \sim 1/\varepsilon$, will be finite if $S_\varepsilon = O(\varepsilon^{(k+1)/2})$. Hence, recalling that $S_\varepsilon = S_\varepsilon(x,y) = S_0(x,y) + \varepsilon S_1(x,y) + \varepsilon^2 S_2(x,y) + \dots$, it follows that $S_0(x,y) = 0$ for $k \geq 0$, $S_1(x,y) = 0$ for $k \geq 2$, and so on. Next, repeating the same arguments as in Section 4.3, we achieve the corresponding homogenized equations. As we saw, the case $k=1$ leads to an elliptic equation with memory; in turn, the case $k=-1$ (see [30] and [35]) leads to a degenerate parabolic system, the well known bidomain model for the cardiac syncithial tissue, where however, in the left hand side of (4.77) an extra term depending on $[u_\varepsilon]$ appears, modeling the nonlinear conductive behavior of the membrane. The case $k=0$ (see [33], [7]) leads to a standard elliptic equation.
In the following we analyze in details the whole family $k \in \boldsymbol{Z}$. Let us start with the homogenized equation in the cases $k \neq -1, 0, 1$:

- for $k \geq 2 \quad \Longrightarrow \quad$ single elliptic equation

$$-\operatorname{div}\Big(((\sigma_{\rm int}|E_{\rm int}| + \sigma_{\rm out}|E_{\rm out}|)I + A^{\mathcal{D}})\nabla_x u_0\Big) = 0 \qquad \text{in } \Omega_T$$

- for $k \leq -2 \quad \Longrightarrow \quad$ system of two independent elliptic equations

$$-\operatorname{div}\Big((\sigma_{\rm int}|E_{\rm int}|I + A^{\mathcal{N}}_{\rm int})\nabla_x u_0^{\rm int}\Big) = 0\,, \qquad \text{in } \Omega_T;$$
$$-\operatorname{div}\Big((\sigma_{\rm out}|E_{\rm out}|I + A^{\mathcal{N}}_{\rm out})\nabla_x u_0^{\rm out}\Big) = 0 \qquad \text{in } \Omega_T;$$

  where $u_0^{\rm int}$ and $u_0^{\rm int}$ are the components of u_0

Remark 4.12. Note that in both the previous cases the dependance on time is only parametric. Moreover, all the matrices $A^{\mathcal{D}}\,, A^{\mathcal{N}}_{\rm int}\,, A^{\mathcal{N}}_{\rm out}$ do not depend on the permeability α.

4.6.1. *Electrical activation of cardiac syncithial tissues (the case $k=-1$).* Let us consider the linearized version of the model proposed by Krassowska and Neu in [30]. In this case we deal with the system of equations

$$\begin{aligned}
-\operatorname{div}(\sigma_{\rm int}\nabla u_\varepsilon^{(\rm out)}) &= 0\,, && \text{in } \Omega^\varepsilon_{\rm int};\\
-\operatorname{div}(\sigma_{\rm out}\nabla u_\varepsilon^{(\rm int)}) &= 0\,, && \text{in } \Omega^\varepsilon_{\rm out};\\
\sigma_{\rm int}\nabla u_\varepsilon^{(\rm int)}\cdot\nu &= \sigma_{\rm out}\nabla u_\varepsilon^{(\rm out)}\cdot\nu\,, && \text{on } \Gamma^\varepsilon;\\
\alpha\varepsilon\frac{\partial}{\partial t}[u_\varepsilon] &= \sigma_{\rm out}\nabla u_\varepsilon^{(\rm out)}\cdot\nu\,, && \text{on } \Gamma^\varepsilon;\\
[u_\varepsilon(x,0) &= S_\varepsilon(x)\,, && \text{on } \Gamma^\varepsilon;\\
u_\varepsilon(x,t) &= 0\,, && \text{on } \partial\Omega.
\end{aligned}$$

Following the notation introduced in Section 3 and assuming that $S_0(x,y) = S_0(x) \neq 0$, it can be found (see [30, 35, 7]) that the macroscopic solution u_0 is actually split into two different functions (this is the reason of the name *bidomain model*), accordingly to

$$u_0(x,y,t) = \begin{cases} u_0^{\rm int}(x,t)\,, & \text{in } E_{\rm int},\\ u_0^{\rm out}(x,t)\,, & \text{in } E_{\rm out}.\end{cases}$$

Moreover, the first corrector u_1 can be factorized as follows

$$u_1(x,y,t) = \begin{cases} -\chi^{\mathcal{N}}(y)\cdot\nabla_x u_0^{\rm int}(x,t) + \widetilde{u}_1^{\rm int}(x,t) & y \in E_{\rm int}\,,\\ -\chi^{\mathcal{N}}(y)\cdot\nabla_x u_0^{\rm out}(x,t) + \widetilde{u}_1^{\rm out}(x,t) & y \in E_{\rm out}\,,\end{cases}$$

where the components $\chi_h^{\mathcal{N}}$ $(h = 1, \dots, N)$ of the cell function $\chi^{\mathcal{N}}$ satisfy

$$-\sigma\,\Delta_y\,\chi_h^{\mathcal{N}} = 0\,, \qquad \text{in } E_{\text{int}},\ E_{\text{out}}; \tag{4.78}$$

$$[\sigma(\nabla_y \chi_h^{\mathcal{N}} - \boldsymbol{e}_h)\cdot\nu] = 0\,, \qquad \text{on } \varGamma; \tag{4.79}$$

$$\sigma_{\text{out}}(\nabla_y \chi_{h,\text{out}}^{\mathcal{N}} - \boldsymbol{e}_h)\cdot\nu = 0\,, \qquad \text{on } \varGamma. \tag{4.80}$$

The components $\chi_h^{\mathcal{N}}$ are also required to be periodic functions in Y, with zero integral average on Y. This implies that only one of the two constants up to which $\chi_{\text{int}}^{\mathcal{N}}$ and $\chi_{\text{out}}^{\mathcal{N}}$ are determined by the previous equations is fixed, but this does not affect the homogenized system below. Note also that, in general, $[\chi_h^{\mathcal{N}}] \neq 0$ on $\varGamma$.
Finally, the functions $\widetilde{u}_1^{\text{int}}$, $\widetilde{u}_1^{\text{out}}$ can be taken equal to zero, and the macroscopic homogenized function u_0 satisfies the degenerate parabolic system given by

$$\begin{cases} |\varGamma|\alpha\dfrac{\partial}{\partial t}(u_0^{\text{out}} - u_0^{\text{int}}) = \operatorname{div}\Big((\sigma_{\text{out}}|E_{\text{out}}|I + A_{\text{out}}^{\mathcal{N}})\nabla_x u_0^{\text{out}}\Big)\,; \\ -\operatorname{div}\Big((\sigma_{\text{int}}|E_{\text{int}}|I + A_{\text{int}}^{\mathcal{N}})\nabla_x u_0^{\text{int}} + (\sigma_{\text{out}}|E_{\text{out}}|I + A_{\text{out}}^{\mathcal{N}})\nabla_x u_0^{\text{out}}\Big) = 0\,; \end{cases}$$

where

$$A_{\text{int}}^{\mathcal{N}} = -\int_{\varGamma} \sigma_{\text{int}}\nu\otimes\chi_{\text{int}}^{\mathcal{N}}\,\mathrm{d}\sigma\,, \qquad A_{\text{out}}^{\mathcal{N}} = \int_{\varGamma} \sigma_{\text{out}}\nu\otimes\chi_{\text{out}}^{\mathcal{N}}\,\mathrm{d}\sigma\,.$$

4.6.2. *Heat transmission with imperfect interfaces (the case $k = 0$).* Let us consider the evolutive version of the model studied by Lipton in [33]. In this case we deal with the system of equations

$$\begin{aligned} -\operatorname{div}(\sigma_{\text{int}}\nabla u_\varepsilon^{(\text{out})}) &= 0\,, && \text{in } \varOmega_{\text{int}}^\varepsilon; \\ -\operatorname{div}(\sigma_{\text{out}}\nabla u_\varepsilon^{(\text{int})}) &= 0\,, && \text{in } \varOmega_{\text{out}}^\varepsilon; \\ \sigma_{\text{int}}\nabla u_\varepsilon^{(\text{int})}\cdot\nu &= \sigma_{\text{out}}\nabla u_\varepsilon^{(\text{out})}\cdot\nu\,, && \text{on } \varGamma^\varepsilon; \\ \alpha\frac{\partial}{\partial t}[u_\varepsilon] &= \sigma_{\text{out}}\nabla u_\varepsilon^{(\text{out})}\cdot\nu\,, && \text{on } \varGamma^\varepsilon; \\ [u_\varepsilon(x,0) &= S_\varepsilon(x)\,, && \text{on } \varGamma^\varepsilon; \\ u_\varepsilon(x,t) &= 0\,, && \text{on } \partial\varOmega. \end{aligned}$$

Following the notation introduced in Section 3 and assuming that $S_0(x,y) = 0$, it can be found (see [33, 7]) that the macroscopic solution u_0, which at a first look can be split into two functions, as follows

$$u_0(x,y,t) = \begin{cases} u_0^{\text{int}}(x,t)\,, & \text{in } E_{\text{int}}, \\ u_0^{\text{out}}(x,t)\,, & \text{in } E_{\text{out}}, \end{cases}$$

actually must satisfy the conditions

$$|\varGamma|\Big(\alpha\frac{\partial}{\partial t}[u_0]\Big) = 0 \qquad + \qquad [u_0]_{|t=0} = 0$$

which imply that $[u_0] = 0$ on $\varGamma$ for all $t \in (0,T)$, so that $u_0^{\text{int}}(x,t) = u_0^{\text{out}}(x,t) = u_0(x,t)$. Moreover, the first corrector can be factorized as follows

$$u_1(x,y,t) = \begin{cases} -\chi^{\mathcal{N}}(y)\cdot\nabla_x u_0(x,t) + \widetilde{u}_1^{\text{int}}(x,t) & y \in E_{\text{int}}\,, \\ -\chi^{\mathcal{N}}(y)\cdot\nabla_x u_0(x,t) + \widetilde{u}_1^{\text{out}}(x,t) & y \in E_{\text{out}}\,, \end{cases}$$

where the components $\chi_h^{\mathcal{N}}$ $(h = 1, \dots, N)$ of the cell function $\chi^{\mathcal{N}}$ satisfies (4.78)–(4.80) as before. Finally, the macroscopic homogenized function u_0 satisfies the standard elliptic equation given by

$$- \operatorname{div} \big((\sigma_0 I + A^{\mathcal{N}}) \nabla_x u_0 \big) = 0 \,,$$

where

$$\begin{aligned} A^{\mathcal{N}} = A^{\mathcal{N}}_{\mathrm{int}} + A^{\mathcal{N}}_{\mathrm{out}} &= - \int_{\Gamma} \sigma_{\mathrm{int}} \nu \otimes \chi^{\mathcal{N}}_{\mathrm{int}} \, \mathrm{d}\sigma + \int_{\Gamma} \sigma_{\mathrm{out}} \nu \otimes \chi^{\mathcal{N}}_{\mathrm{out}} \, \mathrm{d}\sigma \\ &= \int_{\Gamma} [\sigma \nu \otimes \chi^{\mathcal{N}}] \, \mathrm{d}\sigma \end{aligned}$$

and $\sigma_0 = \sigma_{\mathrm{int}} |E_{\mathrm{int}}| + \sigma_{\mathrm{out}} |E_{\mathrm{out}}|$. We note that in this case the homogenized matrix $A^{\mathcal{N}}$ does not depend on the physical properties of the cell membrane; i.e., it does not depend on α, as far as the cell membranes disappear in the homogenization limit.

4.6.3. *Elasticity (the case $k = 1$).* Let us consider the static version of the evolutive model extensively discussed in Sections 4.1–4.5 (see [31]). In this case we deal with the system of equations

$$\begin{aligned} - \operatorname{div}(\sigma_{\mathrm{int}} \nabla u_\varepsilon^{(\mathrm{out})}) &= f(x) \,, && \text{in } \Omega^\varepsilon_{\mathrm{int}}; \\ - \operatorname{div}(\sigma_{\mathrm{out}} \nabla u_\varepsilon^{(\mathrm{int})}) &= f(x) \,, && \text{in } \Omega^\varepsilon_{\mathrm{out}}; \\ \sigma_{\mathrm{int}} \nabla u_\varepsilon^{(\mathrm{int})} \cdot \nu &= \sigma_{\mathrm{out}} \nabla u_\varepsilon^{(\mathrm{out})} \cdot \nu \,, && \text{on } \Gamma^\varepsilon; \\ \frac{\alpha}{\varepsilon} [u_\varepsilon] &= \sigma_{\mathrm{out}} \nabla u_\varepsilon^{(\mathrm{out})} \cdot \nu \,, && \text{on } \Gamma^\varepsilon; \\ u_\varepsilon(x) &= 0 \,, && \text{on } \partial\Omega; \end{aligned}$$

with $f \in L^2(\Omega)$.
Following the notation introduced in Section 3 and assuming that $S_0(x, y) = 0$, it can be found (see [31]) that the macroscopic solution $u_0(x, y) = u_0(x)$ does not depend on y. Moreover, the first corrector can be factorized as $u_1(x, y) = -\chi^{\mathcal{S}}(y) \cdot \nabla_x u_0(x) + \widetilde{u}_1(x)$, where $\widetilde{u}_1$ can be chosen identically equal to zero, and the components $\chi_h^{\mathcal{S}}$ $(h = 1, \dots, N)$ of the cell function $\chi^{\mathcal{S}}$ satisfy

$$\begin{cases} -\sigma \, \Delta_y \, \chi_h^{\mathcal{S}} = 0 \,, & \text{in } E_{\mathrm{int}}, \, E_{\mathrm{out}}; \\ [\sigma (\nabla_y \chi_h^{\mathcal{S}} - \boldsymbol{e}_h) \cdot \nu] = 0 \,, & \text{on } \Gamma; \\ \alpha [\chi_h^{\mathcal{S}}] = \sigma_{\mathrm{out}} (\nabla_y \chi^{\mathcal{S}}_{h,\mathrm{out}} - \boldsymbol{e}_h) \cdot \nu \,, & \text{on } \Gamma. \end{cases}$$

Finally, the macroscopic homogenized function u_0 satisfies the standard elliptic equation given by

$$- \operatorname{div} \left((\sigma_0 I + A^{\mathcal{S}}) \nabla_x u_0 \right) = f(x)$$

where $A^{\mathcal{S}} = \int_\Gamma [\sigma \nu \otimes \chi^{\mathcal{S}}] \, \mathrm{d}\sigma$ and again $\sigma_0 = \sigma_{\mathrm{int}} |E_{\mathrm{int}}| + \sigma_{\mathrm{out}} |E_{\mathrm{out}}|$. We note that in this case, as well as in the corresponding evolutive case, the homogenized matrix $A^{\mathcal{S}}$ depends on the physical properties of the cell membrane, as it can be seen in the system of equations satisfied by the cell function $\chi^{\mathcal{S}}$.

4.6.4. *Final remarks.* We would like to observe that the model considered in details in these notes (i.e., $k = 1$), together with the one corresponding to $k = -1$ in (4.77), preserves memory, in the limit, of the membrane properties (i.e., of the constant α). This is not true for all the other choices of k. Moreover, we expect that, both for cases $k = 1$ and $k = -1$, assigning an alternating potential on the boundary will result in a periodic steady state or a limit cycle as $t \to +\infty$, possibly displaying also a phase delay, as expected in a resistive-capacitive circuit (see the next Section 4.7).

It is not yet clear which one of the two models here presented is more appropriate to describe the physical situation. Indeed, it seems that both of them are valid in their respective frequency ranges. However, the one treated in details in these notes (i.e., model (4.1)–(4.6)) seems to be more suitable to describe the response of a biological tissue to an impulsive potential.

The applicability of this model to real physical situations is connected to the study of an inverse problem, which for the elliptic equation is tipically related to the study of the Neumann-Dirichlet map. This problem has been widely studied. On the contrary (a part from some geometrically simple cases), the inverse problem for equation (4.53) is still open; in this case, the usual Dirichlet-Neumann map should be replaced with a map in which we assign the Dirichlet boundary condition together with the condition:

$$\sigma_0 \frac{\partial u_0}{\partial n} + A^0_{ij} \frac{\partial u_0}{\partial x_i} n_j + \int_0^t A^1_{ij}(t-\tau) \frac{\partial u_0}{\partial x_i}(x,\tau) n_j \, \mathrm{d}\tau = h(x,t)\,,$$

where n is the outward normal to $\partial\Omega$ and h is a given function.

4.7. **Stability.** In this last section we will give a brief description of the asymptotic behavior of $u_\varepsilon(x,t)$ and $u_0(x,t)$ for large times. In the case where a homogeneous Dirichlet boundary condition is satisfied, the following results were proven in [8].

Theorem 4.13. *Let $\Omega^\varepsilon_{\mathrm{int}}, \Omega^\varepsilon_{\mathrm{out}}, \Gamma^\varepsilon, \sigma_{\mathrm{int}}, \sigma_{\mathrm{out}}, \alpha$ be as before. Assume that the initial datum S_ε satisfies* (4.8), (4.9). *Let u_ε be the solution of* (4.1)–(4.6). *Then*

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C(\varepsilon + e^{-\lambda t}) \quad a.e.\ in\ (1,+\infty), \tag{4.81}$$

where C and λ are independent of ε. Moreover, if S_ε has null mean average over each connected component of Γ^ε, it follows that

$$\|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C e^{-\lambda t} \quad a.e.\ in\ (1,+\infty). \tag{4.82}$$

This theorem easily yields the following exponential time-decay estimate for u_0 under homogeneous Dirichlet boundary data.

Corollary 4.14. *Under the assumptions of Theorem 4.13, if $u_\varepsilon \to u_0$ weakly in $L^2(\Omega \times (0,\overline{T}))$ for every $\overline{T} > 0$, then*

$$\|u_0(\cdot,t)\|_{L^2(\Omega)} \le C e^{-\lambda t} \quad a.e.\ in\ (1,+\infty). \tag{4.83}$$

Next we are interested in the case of a nonhomogeneous but time-periodic Dirichlet boundary data for u_ε and u_0. Then we assume

$$u_\varepsilon(x,t) = \Psi(x)\Phi(t) \qquad \text{and} \qquad u_0(x,t) = \Psi(x)\Phi(t)\,, \quad \text{on } \partial\Omega \times (0,+\infty), \tag{4.84}$$

where

$$\Phi(t) \in H^1_\#(\mathbb{R})\,, \qquad \text{and} \qquad \Psi(x) \in H^1(\mathbb{R}^N)\,, \qquad \Delta\Psi = 0 \text{ in } \Omega. \tag{4.85}$$

Here and in the following a subscript $\#$ denotes a space of T-periodic functions, for some fixed $T > 0$.

In order to deal with this case, for every $\varepsilon > 0$ we introduce an auxiliary function $u_\varepsilon^\#$ which solves a time-periodic version of the microscopic differential scheme introduced at the beginning of this chapter

$$-\operatorname{div}(\sigma\nabla u_\varepsilon^\#) = 0\,, \qquad \text{in } (\Omega_{\rm int}^\varepsilon \cup \Omega_{\rm out}^\varepsilon) \times \mathbb{R}; \tag{4.86}$$

$$[\sigma\nabla u_\varepsilon^\# \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon \times \mathbb{R}; \tag{4.87}$$

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon^\#] = \sigma\nabla u_\varepsilon^{\#,\rm out} \cdot \nu\,, \qquad \text{on } \Gamma^\varepsilon \times \mathbb{R}; \tag{4.88}$$

$$u_\varepsilon^\#(x,t) = \Psi(x)\Phi(t)\,, \qquad \text{on } \partial\Omega \times \mathbb{R}; \tag{4.89}$$

$$u_\varepsilon^\#(x,\cdot) \text{ is } T \text{ periodic}, \qquad \forall x \in \Omega\,; \tag{4.90}$$

$$[u_\varepsilon^\#(\cdot,t)] - S_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon\,. \tag{4.91}$$

Indeed, this problem is derived from (4.1)–(4.6), replacing equation (4.5) with (4.90). Equation (4.91) has been added in order to guarantee the uniqueness of the solution, and is suggested by the observation that $[u_\varepsilon(\cdot,t)] - S_\varepsilon(\cdot)$ has null average over each connected component of Γ^ε, as a consequence of (4.1)–(4.4), (4.5).
In [8, Theorem 7] it has been proved that as $\varepsilon \to 0$, the function $u_\varepsilon^\#(x,t)$ approaches a time-periodic function $u_0^\# \in H^1_\#(\mathbb{R}; H^1(\Omega))$ solving

$$-\operatorname{div}\Big(A\nabla u_0^\# + \int_0^{+\infty} B(\tau)\nabla u_0^\#(x,t-\tau)\,\mathrm{d}\tau\Big) = 0\,, \qquad \text{in } \Omega \times \mathbb{R}; \tag{4.92}$$

$$u_0^\# = \Psi(x)\Phi(t)\,, \qquad \text{on } \partial\Omega \times \mathbb{R}. \tag{4.93}$$

Moreover, the following result holds.

Theorem 4.15. *Let $\Omega_{\rm int}^\varepsilon, \Omega_{\rm out}^\varepsilon, \Gamma^\varepsilon, \sigma_{\rm int}, \sigma_{\rm out}, \alpha$ be as before. Assume that the initial datum S_ε satisfies* (4.8), (4.9) *and the boundary datum satisfies* (4.85). *Let $\{u_\varepsilon\}$ and $\{u_\varepsilon^\#\}$ be the sequences of the solutions of* (4.1)–(4.5), (4.84) *and* (4.86)–(4.91), *respectively. Then*

$$\|u_\varepsilon(\cdot,t) - u_\varepsilon^\#(\cdot,t)\|_{L^2(\Omega)} \le C e^{-\lambda t} \quad \text{a.e. in } (1,+\infty), \tag{4.94}$$

where C and λ are positive constants, independent of ε.

This theorem easily yields the following exponential time-decay estimate for $u_0 - u_0^\#$.

Corollary 4.16. *Under the assumption of Theorem 4.15, if $u_\varepsilon \to u_0$ and $u_\varepsilon^\# \to u_0^\#$ weakly in $L^2(\Omega \times (0,\overline{T}))$, for every $\overline{T} > 0$, then the following estimate holds:*

$$\|u_0(\cdot,t) - u_0^\#(\cdot,t)\|_{L^2(\Omega)} \le C e^{-\lambda t} \quad \text{a.e. in } (1,+\infty), \tag{4.95}$$

where C and λ are positive constants, independent of ε.

Finally, expressing the function Φ by means of its Fourier series; i.e.,

$$\Phi(t) = \sum_{k=-\infty}^{+\infty} c_k \mathrm{e}^{i\omega_k t} \tag{4.96}$$

where $\omega_k = 2k\pi/T$ is the k-th circular frequency, and representing the solution $u_\varepsilon^\#(x,t)$ as follows:

$$u_\varepsilon^\#(x,t) = \sum_{k=-\infty}^{+\infty} v_{\varepsilon k}(x)\mathrm{e}^{i\omega_k t}\,, \tag{4.97}$$

we obtain that the complex-valued functions $v_{\varepsilon k}(x) \in L^2(\Omega)$ are such that $v_{\varepsilon k}|_{\Omega_i^\varepsilon} \in H^1(\Omega_i^\varepsilon)$, $i = 1, 2$, and for $k \neq 0$ satisfy the problem

$$-\operatorname{div}(\sigma \nabla v_{\varepsilon k}) = 0\,, \qquad \text{in } \Omega_{\text{int}}^\varepsilon \cup \Omega_{\text{out}}^\varepsilon; \tag{4.98}$$

$$[\sigma \nabla v_{\varepsilon k} \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.99}$$

$$\frac{i\omega_k \alpha}{\varepsilon}[v_{\varepsilon k}] = (\sigma \nabla v_{\varepsilon k} \cdot \nu)^{\text{out}}\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.100}$$

$$v_{\varepsilon k} = c_k \Psi\,, \qquad \text{on } \partial\Omega, \tag{4.101}$$

whereas for $k = 0$ they satisfy the problem

$$-\operatorname{div}(\sigma \nabla v_{\varepsilon 0}) = 0\,, \qquad \text{in } \Omega_{\text{int}}^\varepsilon \cup \Omega_{\text{out}}^\varepsilon; \tag{4.102}$$

$$[\sigma \nabla v_{\varepsilon 0} \cdot \nu] = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.103}$$

$$(\sigma \nabla v_{\varepsilon 0} \cdot \nu)^{\text{out}} = 0\,, \qquad \text{on } \Gamma^\varepsilon; \tag{4.104}$$

$$v_{\varepsilon 0} = c_0 \Psi\,, \qquad \text{on } \partial\Omega; \tag{4.105}$$

$$[v_{\varepsilon 0}] - S_\varepsilon(\cdot) \quad \text{has null average over each connected component of } \Gamma^\varepsilon\,. \tag{4.106}$$

Note that any solution $v_{\varepsilon k}$ of Problem (4.98)–(4.101) is such that $[v_{\varepsilon k}]$ has null average over each connected component of Γ^ε.
Finally, in [8] the following homogenization result is proven:

Theorem 4.17. *Let $\Omega_{\text{int}}^\varepsilon, \Omega_{\text{out}}^\varepsilon, \Gamma^\varepsilon$, $\sigma_{\text{int}}, \sigma_{\text{out}}, \alpha$ be as before. Assume that the boundary datum satisfies* (4.85). *Then, for $k \in \boldsymbol{Z} \setminus \{0\}$ [respectively, $k = 0$, under the further assumptions* (4.8), (4.9)*], the solution $v_{\varepsilon k}$ of Problem* (4.98)–(4.101) *[respectively, Problem* (4.102)–(4.106)*] strongly converges in $L^2(\Omega)$ to a function $v_{0k} \in H^1(\Omega)$ which is the unique solution of the problem*

$$-\operatorname{div}(A^{\omega_k} \nabla v_{0k}) = 0\,, \qquad \textit{in } \Omega; \tag{4.107}$$

$$v_{0k} = c_k \Psi\,, \qquad \textit{on } \partial\Omega; \tag{4.108}$$

where

$$A^{\omega_k} = A^0 + \int_0^{+\infty} A^1(t)\, e^{-i\omega_k t}\, \mathrm{d}t\,, \tag{4.109}$$

with A^0 and $A^1(t)$ defined in (4.54).

Remark 4.18. Experimental measurements in clinical applications are currently performed by assigning time-harmonic boundary data and assuming that the resulting electric potential is time-harmonic, too. This assumption, which is often referred to as the limiting amplitude principle, leads to the commonly accepted mathematical model based on the complex elliptic Problem (4.107)–(4.108) for the electric potential [14, 25]. In [8], in view of the preceding theorem, this phenomenological equations have been mathematically justified and, moreover, a quasi-explicit relation between the circular frequency ω and the coefficient A^{ω_k} has been found.

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