On the Laplacean transfer across fractal mixtures

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Abstract
Laplacean transport across and towards irregular interfaces have been used to model many phenomena in nature and technology. The peculiar aspect is that these phenomena take place in domains with small bulk and large interfaces in order to produce rapid and efficient transport. In this paper we perform the asymptotic homogenization analysis of Robin problems in domains with a fractal boundary.

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1 Introduction
Robin problems for the Laplace equation in irregular domains have been used to model many phenomena in physics, physiology, electrochemistry and chemical engineering. The peculiar aspect is that these phenomena take place in domains with small bulk and large interfaces in order to produce rapid and efficient transport (see, for a detailed discussion and other motivations, [18], [19], and [37]). From this point of view, domains with a fractal boundary provide a natural setting for studying these phenomena.

Robin problems in fractal domains have been studied by R.F. Bass, K. Burdzy, Z. Chen [5] using a probabilistic approach. We are interested in an analytical approach.
Here we propose to face the fractal problem via the homogenization theory. We think that this approach provides a sound physical basis to the model.

The homogenization approach for reinforcement problems, in the classical setting of regular domains, was widely used from the 1970s in conjunction with numerous applications. The works include those by P. H. Hung, E. Sánchez-Palencia [20], J. R. Cannon, G. H. Meyer [8], H. Brézis, L. Caffarelli, A. Friedman [6], E. Acerbi, G. Buttazzo [1], and G. Buttazzo, G. Dal Maso, U. Mosco [7], as well as the book by H. Attouch [3], whose bibliography contains an exhaustive list.

The use of the homogenization approach for reinforcement problems in domains with a fractal (or prefractal) boundary is recent, the first papers being, to our knowledge, [33], [25], [34], and [35]. These works actually concern singular homogenization results. More precisely, the fractal layer is approximated by a two dimensional highly conductive thin layer with vanishing thickness and increasing conductivity. The scaling assumptions can be roughly expressed by the condition that the thickness of the fiber tends to zero and the conductivity diverges to $+\infty$, while the product tends to a positive number, as $n \to +\infty$. From the PDE’s point of view, in the limit problem, highly conductive layers bring out a second order transmission condition on the fractal layer.

In this paper instead we deal with insulating layers: both thickness and conductivity of the fiber tend to zero. A technical aspect, which is peculiar to insulating layers, is that there is a loss of coerciveness as $n \to +\infty$, and uniform $H^1$-estimate fails. We overcome this difficulty by establishing Poincaré type inequalities (see Theorem 6.1 and Theorem 7.3) and by using some delicate tools such as the extension theorem for $(\varepsilon, \infty)$ domains due to Jones (Theorem 1 in [21]). As in the classical case, the limit problem depends on the asymptotic behavior of the ratio between the conductivity and the thickness. If the ratio tends to a positive number, as $n \to +\infty$, then a first order condition on the fractal boundary - the Robin condition - appears in the limit problem. Here we consider also the cases in which the previous condition is not satisfied. More precisely, if the ratio between the conductivity and the thickness tends to zero, then a different first order condition on the fractal boundary - the Neumann condition - appears in the limit problem. If the ratio diverges, then the Dirichlet condition appears in the limit problem.

Now we describe the layout and the main results of this paper.

In the second section, we fix notation and preliminary results. We consider a simple-connected domain $\Omega^{(\xi)}$ whose boundary is a fractal curve and reinforced (approximating) domains $\Omega^{(\xi)}_{\varepsilon,n}$ which are constructed by adding $\varepsilon$–thin, polygonal, 2–dimensional fibers $\Sigma^{(\xi)}_{\varepsilon,n}$ to the prefractal approximating domains $\Omega^{(\xi)}_{n}$. In these domains, we consider suitable energy forms
\( a_n(\cdot, \cdot) \) (defined in (2.6)) that are the sum of a “weighted” Dirichlet integral on \( \Omega^{(\xi),n} \) and a volume term of zero order with a positive parameter \( \delta_n \). The coefficients \( a^n \) are equal to one on \( \Omega^{(\xi),n} \), and equal to the product of the weight \( w^n \) and two positive constants \( c_n \) and \( \sigma_n \) on the fiber \( \Sigma^{(\xi),n} \). The weight \( w^n \) is equal to one on the prefractional domains \( \Omega^{(\xi),n} \) and behaves like the thickness of the fiber on each fiber \( \Sigma^{(\xi),n} \), while the constants \( c_n \) describe the conductivity of the material and \( \sigma_n \) are normalizing factors that depend on the intrinsic parameters of the limit mixture. We state the existence and the uniqueness of the functions \( u_n \) that realize the minimum of the complete energy functionals that are the sum of the energy forms \( a_n(\cdot, \cdot) \), a volume term and a layer contribution on the boundary \( \partial \Omega^{(\xi),n} \) (see Theorem 2.1). Moreover we study the limit problems and we state the existence and the uniqueness of the relative variational solutions (see Theorems 2.2 and 2.3).

In Section 3, we state the main results. We consider the Dirichlet problems for elliptic operators introduced in the previous section and we show that the asymptotic behavior of the solutions \( u_n \) depends on the conductivity parameters \( c_n \). More precisely, we prove that the sequence \( u_n \) converges to the variational solution of Robin problem, Neumann problem, or Dirichlet problem on \( \Omega^{(\xi)} \) when the sequence \( c_n \) tends to a positive number, to zero, or it diverges (see Theorems 3.1 and 3.3). Moreover, we state the convergence of the spectral structures associated with the above operators (see Theorems 3.2 and 3.4).

In Section 4 we recall the definition and main properties of M-convergence, and we prove M-convergence of the functionals corresponding to the reinforcement problems (see Theorems 4.1 and 4.2).

Section 5 concerns the convergence of spectral structures. As in our framework the operators act in different Hilbert spaces, we introduce the definition and main properties of M-K-S-convergence (that is a generalization of M-convergence to variable Hilbert spaces) and we prove Theorems 3.2 and 3.4.

In Section 6 we give the proof of Theorems 3.1 and 3.3 where sharp estimates in the trace theorems on polygonal boundaries, extension results and a suitable Poincaré inequality (Theorem 6.1) play an important role. In particular, given that the constant \( C_p \) appearing in Poincaré inequality does not depend on the increasing number of sides we are able to prove that suitable extensions \( u^*_n \) of the minimizers \( u_n \) are equi-bounded in the \( H^1 \)–norm with respect to \( n \) and \( \varepsilon \).

Section 7 concerns Laplacean transfer across fractal mixtures. The mathematical model of the so-called Laplacean transport and of some other phenomena involves mixed Dirichlet-Robin conditions on the boundary. In fact, species characterized by their concentration diffuses in the bulk from a source (Dirichlet condition) towards a semi-permeable interface (Robin condition) on which it disappears at a given rate (either by transferring across
the interface, or by reacting in the case of a catalytic cell) (see [19]). Ir-
regular semi-permeable interfaces give rise to rapid and efficient flows. In
the present setting the geometry of the bulk is represented by the domain
\( \tilde{\Omega}^{(\xi)} = \Omega^{(\xi)} \setminus B(P_0, \frac{1}{8}) \), where \( B(P_0, \frac{1}{8}) \)
is the ball with center \( P_0 = (\frac{1}{2}, -\frac{1}{2}) \) and radius \( \frac{1}{8} \). In a similar way, we construct the prefractal domains \( \tilde{\Omega}^{(\xi),n} = \Omega^{(\xi),n} \setminus B(P_0, \frac{1}{8}) \), and the reinforced domains \( \tilde{\Omega}\varepsilon^{(\xi),n} = \Omega\varepsilon^{(\xi),n} \setminus B(P_0, \frac{1}{8}) \).
Existence and uniqueness results for the variational solution of the mixed
Dirichlet-Robin problem modeling Laplacean transport (on \( \tilde{\Omega}^{(\xi)} \)) can be
proved as in [9] and [10] by making some natural changes due to the different
geometry of the domain. In the above mentioned papers, prefractal mixed
Dirichlet-Robin problems (where the fractal curve is replaced by polygonal
curves approximating the fractal) have been also considered and existence,
uniqueness and regularity results were established. The convergence of the
solutions of approximating problems to the solution of the fractal problem
has been proved in [11] and [12], where a suitable choice of the coefficients
in the Robin condition on the polygonal curves is a crucial condition.

In the framework of the homogenization approach, the results proved in
the previous sections can be adapted to the geometry of \( \tilde{\Omega}^{(\xi)} \). Actually the
presence of the Dirichlet condition (in a part of the boundary) allows us to
improve the previous results (see Theorems 7.1, 7.2 and 7.4). In particular,
we establish a Poincaré inequality where the constant \( C_P \) does not depend
on the increasing number of sides (see Theorems 7.3) and we prove that suit-
able extensions \( u^*_n \) of the minimizers \( u_n \) are equi-bounded in the \( H^1 \)-norm
with respect to \( n \) and \( \varepsilon \). In a previous paper ([13]) we have already addressed
mixed Dirichlet-Robin problems with the homogenization approach. How-
ever Theorem 4.2 in [13] concerns only the case in which the sequence \( c_n \)
is constant and positive. In particular the case in which the sequence \( c_n \)
diverges as well as the results concerning the spectral structures are new.
Moreover the variational problems considered in Theorems 7.2 and 7.4 differ
from those in [13] because of the presence of the layer terms in the data.
We remark that these terms appear in the models of Laplacean transport
being due to the source terms (see [9]). From the mathematical point of
view these terms can be managed by means of trace results (see Theorem
8.1).

The last section includes definitions and properties of the scale irregular
Koch curves and some trace and extension results stated in the specific
geometry of the domains.

2 Notation and preliminary results

In this section we introduce the notation and some preliminary results.

Let \( \Omega_0 \) be the square \( \{(x, y) : 0 < x < 1, -1 < y < 0\} \) with vertices
\( A = (0, 0), B = (1, 0), C = (1, -1), \) and \( D = (0, -1) \). On each of the 4 sides
we construct a scale irregular Koch curve (for definition and main properties, see Section 8). More precisely, we consider the set $\Omega^{(\xi)}$ bounded by 4 scale irregular Koch curves $K^{(\xi)}_j$, $j = 1, 2, 3, 4$ with endpoints, respectively, $A$ and $B$, $B$ and $C$, $C$ and $D$, $D$ and $A$.

We consider the sets $\Omega^{(\xi)}_n$ bounded by 4 approximating prefractal curves $K^{(\xi)}_{j,n}$ starting from the segments $K_j$ with endpoints $A$, $B$, $B$, and $C$, $C$, $D$, and $D$ respectively, and the reinforced domain $\Omega^{(\xi)}_{\varepsilon,n}$ constructed by an iteration procedure and starting from a suitable initial fiber $\Sigma_{\varepsilon}$. More precisely, let $\Omega_0$ be the square introduced before and let $K_1$ be the interval with end-points $A$ and $B$. For every $0 < \varepsilon \leq \varepsilon_0 < \frac{1}{2}$, we define the fiber $\Sigma_{1,\varepsilon}$ $\varepsilon$-neighborhood of $K_1$, to be the (open) polygon whose vertices are the points $A$, $B$, $P_2$, $P_1$, where

$$P_1 = (\varepsilon, \varepsilon), P_2 = (1 - \varepsilon, \varepsilon).$$

We then subdivide $\Sigma_{1,\varepsilon}$ as the union of the rectangle $R_{1,\varepsilon}$ and the two triangles $T_{1,h,\varepsilon}$, $h = 1, 2$. Here, $R_{1,\varepsilon}$ is the rectangle with vertices $P_1$, $P_2$, $P_3$, $P_4$; $T_{1,1,\varepsilon}$ is the triangle with vertices $A$, $P_1$, $P_4$ and $T_{1,2,\varepsilon}$ is the triangle with vertices $B$, $P_3$, $P_2$ where

$$P_4 = (\varepsilon, 0), P_3 = (1 - \varepsilon, 0).$$

For every $n$ and $\varepsilon$ as above, we define the fiber $\Sigma^{(\xi)}_{1,\varepsilon}$, $\varepsilon$-neighborhood of $K^{(\xi)}_{1,n}$, to be the (open) set

$$\Sigma^{(\xi)}_{1,\varepsilon} = \bigcup_{i|n} \Sigma^{(\xi)}_{i|n},$$

where

$$\Sigma^{(\xi)}_{i|n} = \psi_{i|n}(\Sigma_{1,\varepsilon})$$

(see Figures 2.1, 2.2, 2.3, 2.4).

We proceed in a similar way in order to construct the fiber $\Sigma^{(\xi)}_{j,\varepsilon}$, $\varepsilon$-neighborhood of $K^{(\xi)}_{j,n}$ ($j = 2, 3, 4$), and we define the fiber $\Sigma^{(\xi)}_{\varepsilon}$, $\varepsilon$-neighborhood of $\partial \Omega^{(\xi)}_n$, $\Sigma^{(\xi)}_{\varepsilon} = \bigcup_{j=1}^4 \Sigma^{(\xi)}_{j,\varepsilon}$ and

$$\Omega^{(\xi)}_\varepsilon = \Omega_n \cup \Sigma^{(\xi)}_{\varepsilon}$$

(see Figure 2.5).

We define a weight $w^{(\xi)}_{n}$ as follows.
Figure 2.1: The "\(\varepsilon\)-neighborhood" \(\Sigma_{1,\varepsilon}\).

Figure 2.2: The "\(\varepsilon\)-neighborhood" \(\Sigma_{j,\varepsilon}^{(\xi),1}\).

Figure 2.3: The "\(\varepsilon\)-neighborhood" \(\Sigma_{j,\varepsilon}^{(\xi),2}\).

Figure 2.4: The "\(\varepsilon\)-neighborhood" \(\Sigma_{j,\varepsilon}^{(\xi),3}\).
Let $P$ – for some $i|n$ – belong to the boundary $\partial(\Sigma^{(\xi),i|n})$ of $\Sigma^{(\xi),i|n}$ and let $P^\perp$ be the orthogonal projection of $P$ on $K^{(\xi),i|n}$. If $(x,y)$ belongs to the segment with end-points $P$ and $P^\perp$, we set, in our current notation, 
\[ w_{\xi,n}^n(x,y) = |P - P^\perp|, \]
where $|P - P^\perp|$ is the (Euclidean) distance between $P$ and $P^\perp$ in $\mathbb{R}^2$.

We proceed in a similar way in order to construct the weights $w_{\xi,\varepsilon}^n$ on $\Sigma_{j,\varepsilon}^{(\xi),n}$ ($j = 2, 3, 4$) and we define $w_{\xi,\varepsilon}^n$ on $\Omega_{\varepsilon}^{(\xi),n}$
\[ w_{\xi,\varepsilon}^n(x,y) = \begin{cases} w_{\xi,\varepsilon}^n(x,y) & \text{if } (x,y) \in \Sigma_{j,\varepsilon}^{(\xi),n} \\ 1 & \text{if } (x,y) \in \Omega_{\varepsilon}^{(\xi),n}. \end{cases} \tag{2.1} \]

Associated with the weight $w_{\xi,\varepsilon}^n$, we consider the Sobolev spaces $H^1(\Omega_{\varepsilon}^{(\xi),n}; w_{\xi,\varepsilon}^n)$ and $H^1_0(\Omega_{\varepsilon}^{(\xi),n}; w_{\xi,\varepsilon}^n)$, defined as the completion of $C^\infty(\Omega_{\varepsilon}^{(\xi),n})$ and $C^\infty_0(\Omega_{\varepsilon}^{(\xi),n})$, respectively, in the norm
\[ \|u\|_{H^1(\Omega_{\varepsilon}^{(\xi),n}; w_{\xi,\varepsilon}^n)} = \left\{ \int_{\Omega_{\varepsilon}^{(\xi),n}} u^2 dx dy + \int_{\Omega_{\varepsilon}^{(\xi),n}} |\nabla u|^2 w_{\xi,\varepsilon}^n dx dy \right\}^{\frac{1}{2}}. \tag{2.2} \]

We stress the fact that the gradient is uniquely defined as the weight $w_{\xi,\varepsilon}^n$ and its reciprocal $\frac{1}{w_{\xi,\varepsilon}^n}$ belong to $L^1(\Omega_{\varepsilon}^{(\xi),n})$ (see, e.g., [16]).

We define the coefficients
\[ a_{\xi,\varepsilon}^n(x,y) = \begin{cases} c_n \sigma_n w_{\xi,\varepsilon}^n(x,y) & \text{if } (x,y) \in \Sigma_{\xi,\varepsilon}^{(\xi),n} \\ 1 & \text{if } (x,y) \in \Omega_{\varepsilon}^{(\xi),n}, \end{cases} \tag{2.3} \]
where
\[ c_n > 0 \tag{2.4} \]
and
\[ \sigma_n = \ell^{(\xi)}(n) = \prod_{i=1}^n \ell_{\xi_i}, \tag{2.5} \]

Here $\ell_{\xi_i}$ denotes the reciprocal of the contraction factor of the family $\Psi^{(\xi_i)}$ (see Section 8).

From now on, when it does not give rise to misunderstanding, we denote by $C$ positive, possibly different constants that do not depend on $n$ and on $\varepsilon$.

The following theorem states the existence and the uniqueness of the variational solution of the reinforcement problem. We consider the bilinear form associated with the reinforcement problem
\[ a_n(u,v) := \int_{\Omega_{\varepsilon}^{(\xi),n}} a_{\xi,\varepsilon}^n \nabla u \nabla v dx dy + \delta_n \int_{\Omega_{\varepsilon}^{(\xi),n}} u v dx dy \tag{2.6} \]
where $a_{\xi,\varepsilon}^n$ is defined in (2.3), (2.5), (2.4), and $\delta_n > 0$.  

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Theorem 2.1 Let \( \sigma_n \) be as in (2.5) and \( d_n \in \mathbb{R} \). Then, for any \( f_n \in L^2(\Omega_n^{(\xi)}; w^{n}_n) \), there exists one and only one solution \( u_n \) of the following problem

\[
\begin{aligned}
\text{find } u_n \in H^1_0(\Omega_n^{(\xi)}; w^{n}_n) & \text{ such that } \\
\ell_n(u_n, v) &= \int_{\Omega_n^{(\xi)}} f_n v \, dx + \sigma_n d_n \int_{\partial \Omega_n^{(\xi)}} v \, ds \quad \forall v \in H^1_0(\Omega_n^{(\xi)}; w^{n}_n),
\end{aligned}
\]

where \( \ell_n(\cdot, \cdot) \) is defined in (2.6). Moreover, \( u_n \) is the only function that realizes the minimum of the energy functional

\[
\min_{v \in H^1_0(\Omega_n^{(\xi)}; w^{n}_n)} \left\{ \ell_n(v, v) - 2 \int_{\Omega_n^{(\xi)}} f_n v \, dx \right\}.
\]

**Proof.** We use Lax-Milgram Theorem. By Theorems 8.1 and 8.3, we have

\[
|\sigma_n d_n \int_{\partial \Omega_n^{(\xi)}} v \, ds| \leq \sigma_n |d_n| ||v||_{L^2(\partial \Omega_n^{(\xi)}; w^{n})} \sqrt{|\partial \Omega_n^{(\xi)}; w^{n}|} \leq 2|d_n| \sqrt{C_1 C_J} ||v||_{H^1(\Omega_n^{(\xi)}; w^{n})}
\]

(\text{where here and in the following } |\partial \Omega_n^{(\xi)}; w^{n}| \text{ denotes the arc length measure of the polygonal curve } \partial \Omega_n^{(\xi)}; w^{n}) \text{ and}

\[
|\int_{\Omega_n^{(\xi)}} f_n v \, dx| \leq ||f_n||_{L^2(\Omega_n^{(\xi)}; w^{n})} ||v||_{L^2(\Omega_n^{(\xi)}; w^{n})}.
\]

Moreover the bilinear form \( \ell_n(u, v) \) satisfies

\[
\ell_n(v, v) \geq \min(\delta_n, c_n \sigma_n, 1) ||v||^2_{H^1_0(\Omega_n^{(\xi)}; w^{n}_n)}
\]

and

\[
|\ell_n(u, v)| \leq \max(\delta_n, c_n \sigma_n, 1) ||u||_{H^1_0(\Omega_n^{(\xi)}; w^{n}_n)} ||v||_{H^1_0(\Omega_n^{(\xi)}; w^{n}_n)}.
\]
In the following theorems, we state the existence and uniqueness of the variational solution of the Robin, Neumann, and Dirichlet problems on the domain $\Omega^{(\xi)}$.

We consider the bilinear form associated with the Robin problem

$$a_{c_0}(u, v) := \int_{\Omega^{(\xi)}} \nabla u \nabla v \, dx \, dy + \delta_0 \int_{\Omega^{(\xi)}} u \, v \, dx \, dy + c_0 \int_{\partial\Omega^{(\xi)}} \gamma_0 u \, \gamma_0 v \, d\mu^{(\xi)}$$

(2.10)

where $\mu^{(\xi)}$ is the measure on $\partial\Omega^{(\xi)}$ that coincides, on each $K_j^{(\xi)}$, with the Radon measure defined in (8.4) and $\gamma_0 u$ denotes the trace of the function $u$ on the boundary of $\Omega^{(\xi)}$ (see (8.5)). From now on, we suppress $\gamma_0$ in the notation, when it does not give rise to misunderstanding, by writing simply $v$ instead of $\gamma_0 v$ and similar expressions.

We assume that

$$c_0 \geq 0, \quad \delta_0 \geq 0, \quad \text{and} \quad \max(c_0, \delta_0) > 0.$$  

(2.11)

**Theorem 2.2** Let us assume (2.11) and $d \in \mathbb{R}$. Then, for any $f \in L^2(\Omega^{(\xi)})$, there exists one and only one solution $u$ of the following problem

$$\begin{cases}
\text{find } u \in H^1(\Omega^{(\xi)}) \quad \text{such that} \\
a_{c_0}(u, v) = \int_{\Omega^{(\xi)}} f \, v \, dx \, dy + d \int_{\partial\Omega^{(\xi)}} v \, d\mu^{(\xi)} \quad \forall \ v \in H^1(\Omega^{(\xi)})
\end{cases}$$

(2.12)

where $a_{c_0}(\cdot, \cdot)$ is defined in (2.10). Moreover, $u$ is the only function that realizes the minimum of the energy functional

$$\min_{v \in H^1(\Omega^{(\xi)})} \left\{ a_{c_0}(v, v) - 2 \int_{\Omega^{(\xi)}} f \, v \, dx \, dy - 2d \int_{\partial\Omega^{(\xi)}} v \, d\mu^{(\xi)} \right\}.$$  

(2.13)

**Proof.** By Theorems 8.2 and 8.4, we have that

$$|d \int_{\partial\Omega^{(\xi)}} v \, ds| \leq 2d ||v||_{L^2(\partial\Omega^{(\xi)})} \leq 2 \sqrt{C_1^* C_R} |d||v||_{H^1(\Omega^{(\xi)})}$$

and

$$| \int_{\Omega^{(\xi)}} f \, v \, dx \, dy | \leq ||f||_{L^2(\Omega^{(\xi)})} ||v||_{L^2(\Omega^{(\xi)})}.$$  

The bilinear form $a_{c_0}(u, v)$ is continuous. Indeed, by using Theorem 8.2 again,

$$|a_{c_0}(u, v)| \leq (\max(\delta_0, 1) + c_0 C_1^* C_R) ||u||_{H^1(\Omega^{(\xi)})} ||v||_{H^1(\Omega^{(\xi)})}.$$  

Moreover, the form is coercive. In fact, if $\delta_0 > 0$ we obtain trivially

$$a_{c_0}(v, v) \geq \min(\delta_0, 1)||v||_{H^1(\Omega^{(\xi)})}^2.$$  

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Instead, if $\delta_0 = 0$ and $c_0 > 0$ by using generalized Poincaré inequality (see Lemma 3.1.1 in [28]), we obtain that
\[ a_{c_0}(v, v) \geq C \min(c_0, 1) \|v\|_{H^1(\Omega)}^2. \]

In a similar way, we prove the following result.

We consider the bilinear form associated with the Dirichlet problem
\[ a_\infty(u, v) := \int_{\Omega(\xi)} \nabla u \nabla v \, dx \, dy + \delta_0 \int_{\Omega(\xi)} u \, v \, dx \, dy. \]  
(2.14)

We assume that
\[ \delta_0 \geq 0. \]  
(2.15)

**Theorem 2.3** Let us assume (2.15). Then, for any \( f \in L^2(\Omega(\xi)) \), there exists one and only one solution \( u \) of the following problem
\[ \begin{align*}
&\text{find } u \in H^1_0(\Omega(\xi)) \text{ such that } \\
&\quad a_\infty(u, v) = \int_{\Omega(\xi)} f \, v \, dx \, dy \quad \forall \, v \in H^1_0(\Omega(\xi))
\end{align*} \]  
(2.16)

where \( a_\infty(\cdot, \cdot) \) is defined in (2.14). Moreover, \( u \) is the only function that realizes the minimum of the energy functional
\[ \min_{v \in H^1_0(\Omega(\xi))} \left\{ a_\infty(v, v) - 2 \int_{\Omega(\xi)} f \, v \, dx \, dy \right\}. \]  
(2.17)

3 Main results

In this section we state the main results.

Let \( \Omega^* \) be an open regular domain such that \( \Omega^* \supset \Omega_{n}^{(\xi), n} \), for \( n \) : in order to fix notation we choose as \( \Omega^* \) the ball with the center in the point \( P_0 = (\frac{1}{2}, -\frac{1}{2}) \) and radius 1.

We use an extension operator \( Ext_J \) from \( H^1(\Omega(\xi), n) \) to the space \( H^1(\mathbb{R}^2) \) whose norm is independent of the (increasing) number of sides (see Theorem 8.3 in Section 8).

We set
\[ u^*_n = (Ext_J u_n|_{\Omega(\xi), n})|_{\Omega^*}. \]  
(3.1)

where \( u_n \) is the solution of the problem (2.7).

In order to study the asymptotic behaviour of the functions \( u_n \), we fix the further assumptions
\[ f_n, f \in L^2(\Omega^*), \text{ and } f_n \to f \in L^2(\Omega^*), \text{ as } n \to +\infty, \]  
(3.2)
\( \delta_n > 0 \) and \( \delta_n \to \delta_0 \) as \( n \to +\infty \), \quad (3.3) \\
c_n > 0 \) and \( c_n \to c_0 \) as \( n \to +\infty \), \quad (3.4) \\
d_n, d \in \mathbb{R}, \) and \( d_n \to d \) as \( n \to +\infty \). \quad (3.5)

**Theorem 3.1** Let us assume conditions (3.4), (3.5), (3.3), (2.11), and (3.2). Let \( \varepsilon = \varepsilon(n) \) be an arbitrary sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Then the sequence of the restrictions to \( \Omega(\xi) \) of the functions \( u_n^\varepsilon \) (defined in (3.1)) converges to the function \( u \) (defined in (2.12)) weakly in \( H^1(\Omega(\xi)) \).

For a fixed \( n \), the second order operator associated with the form (2.6) \( A_n(u) = -\text{div}(a_n^\varepsilon(x,y)\nabla u) + \delta_n u \) (with homogeneous Dirichlet condition) has a compact resolvent in \( L^2(\Omega(\xi)^{n}) \) and the spectrum consists of a increasing sequence of real eigenvalues \( \lambda_{n,k}^\varepsilon \) such that \( \lambda_{n,k}^\varepsilon \to +\infty \) when \( k \to +\infty \). We denote the associated spectral resolution of \( A_n \) with \( P_n(\lambda) \). If \( c_0 > 0 \) the second order operator associated with the form (2.10) \( A_{c_0}(u) = -\text{div}(\nabla u) + \delta_0 u \) (with Robin condition) has a compact resolvent in \( L^2(\Omega(\xi)) \) and the spectrum consists of a increasing sequence of real eigenvalues \( \lambda_k \) such that \( \lambda_k \to +\infty \) when \( k \to +\infty \) (see e.g. [15]). If \( c_0 = 0 \) the second order operator associated with the form (2.10) \( A_{c_0}(u) = -\text{div}(\nabla u) + \delta_0 u \) (with Neumann condition) has a compact resolvent in \( L^2(\Omega(\xi)) \) and the spectrum consists of a increasing sequence of real eigenvalues \( \lambda_k \) such that \( \lambda_k \to +\infty \) when \( k \to +\infty \) (see e.g. [27]). In both cases, we denote by \( P_{c_0}(\lambda) \) the spectral resolution associated with \( A_{c_0} \).

We recall that for every interval \( [\lambda, \mu] \) of \( \mathbb{R} \), \( P_n((\mu - \lambda]) = P_n(\mu) - P_n(\lambda) \) is a projector operator in the Hilbert space \( L^2(\Omega(\xi)^{n}) \) and \( P_{c_0}((\mu - \lambda]) = P_{c_0}(\mu) - P_{c_0}(\lambda) \) is a projector operator in the Hilbert space \( L^2(\Omega(\xi)) \).

**Theorem 3.2** Let \( \varepsilon = \varepsilon(n) \) be an arbitrary sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Let us assume conditions (3.4) and (3.3). Then \( \forall \ 0 < \mu < \lambda \), points of continuity for the spectral resolution \( P_{c_0} \) of the operator \( A_{c_0} \), we have that the projector operators \( P_n((\mu - \lambda]) \) of the spectral resolutions \( P_n \) of the operators \( A_n \) in \( L^2(\Omega(\xi)^{n}) \) converge to the projector \( P_{c_0}((\mu - \lambda]) \) of the spectral resolution \( P_{c_0} \) strongly as \( n \to +\infty \), in the sense of Definition 5.5 of the Section 5.

Theorems 3.1 and 3.2 concern the case in which the conductivity of the thin fibers vanishes at the same rate of the thickness of the fiber and the
case in which the conductivity of the thin fibers vanishes faster than the thickness of the fiber.

Now we consider the case when the conductivity of the thin fibers vanishes slower than the thickness of the fiber: more precisely, we suppose

\[ c_n w_n \to 0, \ c_n \to +\infty, \ \varepsilon(n) \geq \frac{1}{\ell(n)}. \]  

(3.6)

Theorem 3.3 Let \( \varepsilon = \varepsilon(n) \) be a sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Let us assume (3.5), (3.3), (3.6), and (3.2). Then the sequence of the restrictions to \( \Omega^{(c)} \) of the functions \( u^*_n \) (defined in (3.1)) converges to the function \( u \) (defined in (2.16)) weakly in \( H^1(\Omega^{(c)}) \).

The second order operator associated with the form (2.14) \( A_D(u) = -\text{div}(\nabla u) + \delta_0 u \) (with Dirichlet condition) has a compact resolvent in \( L^2(\Omega^{(c)}) \) and the spectrum consists of an increasing sequence of real eigenvalues \( \lambda_k \) such that \( \lambda_k \to +\infty \) when \( k \to +\infty \) (see e.g. [27]). We denote by \( P_D(\lambda) \) the spectral resolution associated with \( A_D \).

Theorem 3.4 Let \( \varepsilon = \varepsilon(n) \) be a sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Let us assume conditions (3.3) and (3.6). Then \( \forall 0 < \mu < \lambda \), points of continuity for the spectral resolution \( P_D \) of the operator \( A_D \), we have that the projector operators \( P^n((\mu - \lambda]) \) of the spectral resolutions \( P^n \) of the operators \( A^n \) in \( L^2(\Omega^{(c)},\varepsilon) \) converge to the projector \( P_D((\mu - \lambda]) \) of the spectral resolution \( P_D \) strongly as \( n \to +\infty \), in the sense of Definition 5.5 of the Section 5.

The proofs of Theorems 3.1, 3.3, 3.2, 3.4 are postponed in Sections 6 and 5. More precisely, in Section 4 we recall the definition and the main properties of M-convergence of forms in a fixed Hilbert space and we prove the convergence of the functionals corresponding to the forms (2.6) and (2.10). In Section 5 we recall the definition and the main properties of M-K-S-convergence for the case of variable Hilbert spaces according to K. Kuwae and T. Shioya and we prove Theorems 3.2 and 3.4. Theorems 3.1 and 3.3 will be proved in Section 6.

4 Mosco convergence

We recall the notion of M-convergence of functionals, introduced in [29], (see also [30]).
Definition 4.1 A sequence of functionals \( F^n : H \to (-\infty, +\infty] \) is said to \( M \)–converge to a functional \( F : H \to (-\infty, +\infty] \) in a Hilbert space \( H \), if

(a) For every \( u \in H \) there exists \( u_n \) converging strongly to \( u \) in \( H \) such that
\[
\limsup \ F^n[u_n] \leq F[u], \quad \text{as} \quad n \to +\infty. \tag{4.1}
\]

(b) For every \( v_n \) converging weakly to \( u \) in \( H \)
\[
\liminf \ F^n[v_n] \geq F[u], \quad \text{as} \quad n \to +\infty. \tag{4.2}
\]

We consider the sequence of weighted energy functionals in \( L^2(\Omega^*) \)
\[
F^n \varepsilon[u] = \begin{cases} 
\int_{\Omega^{(\varepsilon)}} a^n_{\varepsilon}(x,y)|\nabla u|^2 dx\,dy + \delta_n \int_{\Omega^{(\varepsilon)},n} u^2 dx\,dy & \text{if } u|_{\Omega^{(\varepsilon)},n} \in H^1_0(\Omega^{(\varepsilon)},n ; w^n_{\varepsilon}) \\
+\infty & \text{otherwise in } L^2(\Omega^*)
\end{cases}
\tag{4.3}
\]
(The coefficients \( a^n_{\varepsilon} \) are defined in (2.3), (2.5), (2.4), \( \delta_n > 0 \)) and
\[
F_{c_0}[u] = \begin{cases} 
\int_{\Omega^{(\varepsilon)}} |\nabla u|^2 dx\,dy + \delta_0 \int_{\Omega^{(\varepsilon)}} u^2 dx\,dy + c_0 \int_{\partial\Omega^{(\varepsilon)}} u^2 d\mu(\varepsilon) & \text{if } u|_{\Omega^{(\varepsilon)}} \in H^1(\Omega^{(\varepsilon)}) \\
+\infty & \text{otherwise in } L^2(\Omega^*)
\end{cases}
\tag{4.4}
\]

Theorem 4.1 Let us assume (3.4) and (3.3). Let \( \varepsilon = \varepsilon(n) \) be an arbitrary sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Then, the sequence of functionals \( F^n_{\varepsilon(n)} \), defined in (4.3), \( M \)–converges in \( L^2(\Omega^*) \) to the functional \( F_{c_0} \) defined in (4.4) as \( n \to +\infty \).

In order to prove Theorem 4.1 we state the following convergence result.

Proposition 4.1 Let \( \sigma_n \) be as in (2.5). Then, for every sequence \( g_n \in H^1(\Omega^{(\varepsilon)}) \) weakly converging towards \( g^* \) in \( H^1(\Omega^{(\varepsilon)}) \), we have
\[
\sigma_n \int_{\partial\Omega^{(\varepsilon)},n} g_n \, ds \to \int_{\partial\Omega^{(\varepsilon)}} g^* d\mu(\varepsilon), \quad \text{as} \quad n \to +\infty. \tag{4.5}
\]

Proof. From Theorems 8.1 and 8.4,
\[
\sqrt{\sigma_n} ||g||_{L^2(\partial\Omega^{(\varepsilon)},n)} \leq C_T ||g||_{H^\alpha(\Omega^{(\varepsilon)})} \forall g \in H^\alpha(\Omega^{(\varepsilon)}), \tag{4.6}
\]
where \( \alpha \) is a fixed number in the interval \((1/2, 1)\) and the constant \( C_T = \sqrt{C_R C_\alpha} \) does not depend on \( n \) (not even on \( g \)). On the other hand, by the compactness of the embedding of the space \( H^\alpha(\Omega^{(\varepsilon)}) \) with \( \alpha \in (1/2, 1) \) in the space \( H^1(\Omega^{(\varepsilon)}) \) (see, for instance, Theorem 1.4.6.2 in [27] and Theorem
16.2 in [26]), the sequence \( g_n \) strongly converges towards \( g^* \) in \( H^\alpha(\Omega^{(\xi)}) \). Then

\[
\sqrt{\sigma_n}||g_n - g^*||_{L^2(\partial\Omega^{(\xi)},n)} \leq C_T||g_n - g^*||_{H^\alpha(\Omega^{(\xi)})} \to 0 , \text{ as } n \to +\infty \quad (4.7)
\]

and also

\[
|\sigma_n \int_{\partial\Omega^{(\xi)},n} (g_n - g^*)ds| \leq \sqrt{\sigma_n}||g_n - g^*||_{L^2(\partial\Omega^{(\xi)},n)} \sqrt{\sigma_n} \sqrt{|\partial\Omega^{(\xi)}|}, \quad (4.8)
\]

hence

\[
\sigma_n \int_{\partial\Omega^{(\xi)},n} (g_n - g^*)ds \to 0 , \text{ as } n \to +\infty. \quad (4.9)
\]

By an approximation result for the measure \( \mu^{(\xi)} \) (see Theorem 2.1 in [12] and also Lemma 8.4 in [17]), we have that

\[
\sigma_n \int_{\partial\Omega^{(\xi)},n} \varphi ds - \int_{\partial\Omega^{(\xi)}} \varphi \mu^{(\xi)} \to 0 , \text{ as } n \to +\infty \quad (4.10)
\]

\( \forall \varphi \in C^1(\Omega^{(\xi)}) \). Lastly, we use the fact that the space \( C^1(\Omega^{(\xi)}) \) is a dense subspace of the space in \( H^1(\Omega^{(\xi)}) \) and Theorem 8.2 to complete the proof of Proposition 4.1. \( \square \)

From now on, we simply denote \( \varepsilon(n) \) by \( \varepsilon \) and in order to further simplify notation, when it does not give rise to misunderstanding, we write \( F^n \) in place of \( F^{\varepsilon(n)} \), \( w^n \) in place of \( w^{\varepsilon(n)} \), as well as \( a^n \) in place of \( a^{\varepsilon(n)} \). Moreover we suppress the index \( (\xi) \) in the notation of the fibers.

**Proof of Theorem 4.1** First we proceed with the proof of condition (a) in Definition 4.1. We consider a given function \( u \) as in condition (a) and we observe that, without loss of generality, we can assume that \( u|_{\Omega^{(\xi)}} \in H^1(\Omega^{(\xi)}) \), otherwise the inequality (4.1) becomes trivial.

We assume, in addition, that \( u|_{\Omega^{(\xi)}} \in C^1(\Omega^{(\xi)}) \). For every \( n \) and \( \varepsilon \) as above, we define \( u_{1,n} \) on \( \Sigma_{1,\varepsilon} \) by

\[
u_{1,n} = \bigcup_{i,n} \Sigma_{1,\varepsilon}^{i,n} \]

\[
\begin{align*}
u_{1,n}(x,y) = G_{1,\varepsilon}(u \circ \psi_{i,n}^{-1}) \circ \psi_{i,n}^{-1}(x,y) & \quad \text{if } (x,y) \in \Sigma_{1,\varepsilon}^{i,n} \quad (4.11)
\end{align*}
\]

where \( G_{1,\varepsilon} \) is the operator from \( C^1(\Sigma_{1,\varepsilon}) \) to \( Lip(\Sigma_{1,\varepsilon}) \) defined in the following way. For every \( \zeta \in (0,1) \), we define \( P_+ = P_+(\zeta) = (\zeta, \hat{\eta}_+ (\zeta)) \in \partial \Sigma_{1,\varepsilon} \) to be the intersection of \( \partial \Sigma_{1,\varepsilon} \) with the vertical line through the point \( (\zeta,0) \in K_1 \). Then, for a given \( g \in C^1(\Sigma_{1,\varepsilon}) \) we put

\[
G_{1,\varepsilon}(g)(\zeta, \eta) = \begin{cases} g(0,0) & \text{if } (\zeta, \eta) = (0,0) \\
g(0,0) - \frac{\eta - \hat{\eta}_+}{\eta_+ - \hat{\eta}_+} & \text{if } (\zeta, \eta) \in \Sigma_{1,\varepsilon} \setminus \{A,B\}, \quad (4.12) \\
g(1,0) & \text{if } (\zeta, \eta) = (1,0).
\end{cases}
\]
We construct, in a similar way, \( G_{j,\varepsilon} \) on \( \Sigma_{j,\varepsilon} \) and \( u_{j,n} \) on \( \Sigma_{j,\varepsilon}^n \) for \( j = 2, 3, 4 \).

We set
\[
\overline{u} = \begin{cases} 
  u & \text{on } \Omega^{(\xi)} \\
  0 & \text{on } \Omega^* \setminus \Omega^{(\xi)}
\end{cases}
\]
and for every \( n \) and \( \varepsilon \) as above, we define
\[
u_n(x, y) = \begin{cases} 
  \overline{u}(x, y) & \text{if } (x, y) \in \Omega^{(\xi)} \setminus \Sigma_{j,\varepsilon}^n \\
  u_{j,n}(x, y) & \text{if } (x, y) \in \Sigma_{j,\varepsilon}^n.
\end{cases}
\] (4.13)

We note that \( \nu_n \) tend to \( \overline{u} \) in \( L^2(\Omega^*) \), \( \sup_{\Sigma_{j,\varepsilon}^n} |u_n| \leq \sup_{\Omega^{(\xi)}} |u| \) and the restrictions on \( \Omega^{(\xi),n} \) of the functions \( u_n \) defined in (4.13) belong to \( H^1_{0}(\Omega^{(\xi),n}) \).

For each \( n \), we split the integral \( F^n[u_n] \) in three terms, taking into account the definitions of \( a_n \) and \( u_n \),
\[
F^n[u_n] = \int_{\Omega^{(\xi),n}} |\nabla u|^2 \, dx \, dy + \delta_n \int_{\Omega^{(\xi),n}} u_n^2 \, dx \, dy + \sigma_n c_n \int_{\Sigma_{j,\varepsilon}^n} |\nabla u_n|^2 w_n \, dx \, dy
\]

Since the sets \( \Omega^{(\xi),n} \) tend to the set \( \Omega^{(\xi)} \) as \( n \to +\infty \), we get
\[
\lim_{n \to +\infty} \int_{\Omega^{(\xi),n}} |\nabla u|^2 \, dx \, dy = \int_{\Omega^{(\xi)}} |\nabla u|^2 \, dx \, dy,
\] (4.14)
\[
\lim_{n \to +\infty} \delta_n \int_{\Omega^{(\xi),n}} u_n^2 \, dx \, dy = \delta_0 \int_{\Omega^{(\xi)}} u^2 \, dx \, dy,
\] (4.15)
\[
\lim_{n \to +\infty} \delta_n \int_{\Sigma_{j,\varepsilon}^n} u_n^2 \, dx \, dy = 0.
\] (4.16)

Now we prove
\[
\lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_{j,\varepsilon}^n} |\nabla u_n|^2 w_n \, dx \, dy = c_0 \int_{\partial \Omega^{(\xi)}} u^2 \, d\mu^{(\xi)}.
\] (4.17)

We split the integral on \( \Sigma_{j,\varepsilon}^n \) as the sum of the 4 integrals on the sets \( \Sigma_{j,\varepsilon}^n \), \( j = 1, 2, 3, 4 \). We now consider \( \Sigma_{j,\varepsilon}^n \) : in all the other cases we proceed in a similar manner.

We decompose each \( \Sigma_{j,\varepsilon}^n \) as the union of one rectangle and two triangles and we evaluate the corresponding integrals by making use of the coordinates change provided by the map \( \psi_{i,n} \). Thus, we write
\[
c_n \sigma_n \int_{\Sigma_{j,\varepsilon}^n} |\nabla u_n|^2 w_n \, dx \, dy \equiv R_{1,0} + \sum_{h=1}^2 X_{1,h}
\]
where
\[
R_{1,0} = c_n \sigma_n \int_{\psi_{i,n}(R_{1,\varepsilon})} |\nabla u_n|^2 w_n \, dx \, dy, \quad X_{1,h} = c_n \sigma_n \int_{\psi_{i,n}(T_{1,h,\varepsilon})} |\nabla u_n|^2 w_n \, dx \, dy, \quad h = 1, 2.
\]
We note that for \((x,y) \in \Sigma_1^{1,n}\), \((x,y) = \psi_{i|n}(\zeta, \eta)\) and \(w_n(x,y) = \frac{\ell_\varepsilon(\zeta)}{l(n)}\), where

\[
\ell_\varepsilon(\zeta) = \begin{cases} 
\zeta & 0 < \zeta < \varepsilon \\
\varepsilon & \varepsilon < \zeta < 1 - \varepsilon \\
1 - \varepsilon & 1 - \varepsilon < \zeta < 1 .
\end{cases}
\]

Put \(g(\zeta, \eta) := (u_n \circ \psi_{i|n})(\zeta, \eta)\) and set \(M = ||u||^2_{C^1(\mathbb{R}^n)}\).

We have

\[
R_{1,0} = c_n \frac{\varepsilon}{4^n} \int_0^\varepsilon dy \int_\varepsilon^{1-\varepsilon} |g_\zeta(\zeta, 0)(1 - \frac{\eta}{\varepsilon})|^2 d\zeta + c_n \frac{\varepsilon}{4^n} \int_0^\varepsilon d\eta \int_\varepsilon^{1-\varepsilon} |g(\zeta, 0)\frac{1}{\varepsilon}|^2 d\zeta \equiv A_n + B_n
\]

Now

\[
A_n = c_n \frac{\varepsilon^2}{4^n l(n)} \int_{\psi_{i|n}(\varepsilon, 1-\varepsilon)} |\nabla u_n|^2 ds \leq M \frac{c_n \varepsilon^2}{4^n l(n)^2} . \tag{4.18}
\]

Moreover,

\[
B_n = \frac{c_n l(n)}{4^n} \int_{\psi_{i|n}(\varepsilon, 1-\varepsilon)} |u_n(x,y)|^2 ds. \tag{4.19}
\]

Now we evaluate the terms

\[
X_{1,h} = c_n \sigma_n \int_{\psi_{i|n}(T_1,h,\varepsilon)} |\nabla u_n|^2 w^n dx dy, \quad h = 1, 2:
\]

we have

\[
X_{1,1} = c_n \sigma_n \int_{\psi_{i|n}(T_1,1,\varepsilon)} |\nabla u_n|^2 w^n dx dy =
\]

\[
= \frac{c_n}{4^n} \int_0^\varepsilon \int_0^\zeta |g_\zeta(\zeta, 0)(1 - \frac{\eta}{\zeta}) + g(\zeta, 0) \frac{\eta}{\zeta}|^2 d\zeta d\eta + \frac{c_n}{4^n} \int_0^\varepsilon \int_0^\zeta |g(\zeta, 0)\frac{1}{\zeta}|^2 d\zeta d\eta \equiv C_n + D_n.
\]

The first term can be estimated as follows

\[
C_n \leq C \frac{c_n M \varepsilon}{4^n} . \tag{4.20}
\]

The second term can be estimated as before

\[
D_n = \frac{c_n l(n)}{4^n} \int_{\psi_{i|n}(0,\varepsilon)} u_n^2 ds. \tag{4.21}
\]

As \(X_{1,2}\) is analogous, we obtain

\[
c_n \sigma_n \int_{\Sigma_1^{1,n}} |\nabla u_n|^2 w^n dx dy = \sum_{i|n} \left( c_n \frac{l(n)}{4^n} \int_{\psi_{i|n}(K_i)} u_n^2 ds + C \frac{c_n M \varepsilon}{4^n} \right) \quad (4.22)
\]

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and
\[ \lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_1} |\nabla u_n|^2 w^n dxdy = \lim_{n \to +\infty} c_n \frac{l(n)}{4^n} \int_{K_1^n} u_n^2 ds. \] (4.23)

In a similar way, we can estimate the other terms for \( j = 2, 3, 4 \), hence
\[ \lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_2} |\nabla u_n|^2 w^n dxdy = \lim_{n \to +\infty} c_n \frac{l(n)}{4^n} \int_{\partial \Omega^2(n)} u_n^2 ds. \] (4.24)

As \( u_n = u \) on \( \partial \Omega^2(n) \), by Proposition 4.1 (for \( g_n = u \)) we obtain
\[ \lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_1} |\nabla u_n|^2 w^n dxdy = c_0 \int_{\partial \Omega^2} u^2 d\mu(n). \] (4.25)

We complete the proof of part (a) of the Theorem, by making use of the diagonal formula of Corollary 1.16 of [3].

Now we prove condition (b) of Definition 4.1. Let \( v_n \) be a sequence as in (b) that is,
\[ v_n \rightharpoonup u \quad \text{in} \quad L^2(\Omega^s). \] (4.26)
In order to prove the inequality (4.2), it is not restrictive to assume that
\[ \liminf_{n \to +\infty} F^n[v_n] \leq C^* < +\infty. \] (4.27)
Then, from (4.26) and (4.27), up to passing to a subsequence, we deduce that
\[ ||v_n||_{H^1(\Omega^s)} \leq C \]
where \( C \) is a constant independent of \( n \). By Theorem 8.3, there exists a bounded linear extension operator \( \text{Ext}_J : H^1(\Omega^s) \to H^1(\mathbb{R}^2) \), whose norm is independent of \( n \), that is,
\[ ||\text{Ext}_J v_n||_{H^1(\mathbb{R}^2)} \leq C_J ||v_n||_{H^1(\Omega^s)} \] (4.28)
with \( C_J \) independent of \( n \). We put
\[ \hat{v}_n = (\text{Ext}_J v_n|_{\Omega^s})|_{\Omega^s}, \] (4.29)
then there exist \( \hat{v} \in H^1(\Omega^s) \) and a subsequence of \( \hat{v}_n \), denoted by \( \hat{v}_n \) again, weakly converging to \( \hat{v} \) in \( H^1(\Omega^s) \). By a direct calculation, we can prove that the sequence \( \hat{v}_n \) weakly converges to \( u \) in \( L^2(\Omega^s) \) hence
\[ \hat{v}_n \rightharpoonup u |_{\Omega^s} \quad \text{in} \quad H^1(\Omega^s), \] (4.30)
\[ \liminf_{n \to +\infty} \int_{\Omega^s} |\nabla v_n|^2 dxdy \geq \int_{\Omega^s} |\nabla u|^2 dxdy \] (4.31)
and
\[
\lim_{n \to +\infty} \delta_n \int_{\Omega_n^{(\xi)}} v_n^2 \, dx\,dy = \delta_0 \int_{\Omega^{(\xi)}} u^2 \, dx\,dy. \tag{4.32}
\]

In order to conclude the proof, we only have to show that (if \(c_0 > 0\))
\[
\liminf_{n \to +\infty} c_n \sigma_n \int_{\Sigma_0^\varepsilon} |\nabla v_n|^2 w^n \, dx\,dy \geq c_0 \int_{\partial \Omega^{(\xi)}} u_m^2 \, d\mu^{(\xi)}. \tag{4.33}
\]

Let us consider a sequence of functions \(u_m \in C^1(\Omega^\ast)\) such that
\[
\lim_{m \to +\infty} \|u_m - u\|_{L^2(\Omega^\ast)} = 0
\]
and
\[
\lim_{m \to +\infty} \|u_m - u|_{\Omega^{(\xi)}}\|_{H^1(\Omega^{(\xi)})} = 0. \tag{4.34}
\]

As in the proof of (a) (\textit{lim sup condition}), for any (fixed) \(m\), we construct the functions \(u_{m,n} = (u_m)_n\) – starting with \(u_m\) – as in (4.13) and we obtain that
\[
\lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_0^\varepsilon} |\nabla u_{m,n}|^2 w^n \, dx\,dy = c_0 \int_{\partial \Omega^{(\xi)}} u_m^2 \, d\mu^{(\xi)}. \tag{4.35}
\]

Now we prove that
\[
\lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_0^\varepsilon} (\nabla u_{m,n}, \nabla v_n) w^n \, dx\,dy = c_0 \int_{\partial \Omega^{(\xi)}} u_m u \, d\mu^{(\xi)}. \tag{4.36}
\]

We proceed as in the proof of condition (a) and we obtain
\[
\sum_{i\in[n]} c_n \sigma_n \int_{\Sigma_i^{(\psi_{i,n})}} (\nabla u_{m,n}, \nabla v_n) w^n \, dx\,dy = \sum_{i\in[n]} c_n \sigma_n \int_{\psi_{i,n}([0,1])} u_{m,n} v_n \, ds + C \sqrt{C^* c_n M \varepsilon} \tag{4.37}
\]
where we have used (4.27).

As the terms on the other sides can be evaluated in the same way, we obtain
\[
\lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma_0^\varepsilon} (\nabla u_{m,n}, \nabla v_n) w^n \, dx\,dy = \lim_{n \to +\infty} c_n \sigma_n \int_{\partial \Omega^{(\xi)},n} u_{m,n} v_n \, ds. \tag{4.38}
\]

From Proposition 4.1 (with \(g_n = \hat{v}_n u_m\), see (4.30) and (4.29))
\[
\lim_{n \to +\infty} \sigma_n \int_{\partial \Omega^{(\xi),n}} u_{m,n} v_n \, ds = \int_{\partial \Omega^{(\xi)}} u_m u \, d\mu^{(\xi)} \tag{4.39}
\]
and so (4.36) is proved.
Combining (4.36) and (4.35),

$$\liminf_{n \to +\infty} c_n \sigma_n \int_{\Sigma^n_\varepsilon} |\nabla v_n|^2 w^n \, dx \, dy \geq$$

$$\lim_{n \to +\infty} c_n \sigma_n \left( 2 \int_{\Sigma^n_\varepsilon} (\nabla u_{m,n}, \nabla v_n) w^n \, dx \, dy - \int_{\Sigma^n_\varepsilon} |\nabla u_{m,n}|^2 w^n \, dx \, dy \right) =$$

$$= 2c_0 \int_{\partial\Omega^{(\xi)}} u_m \, d\mu^{(\xi)} - c_0 \int_{\partial\Omega^{(\xi)}} u_m^2 \, d\mu^{(\xi)} ;$$

letting \( m \to +\infty \) we conclude the proof. \( \Box \)

Now we consider the case where the layer is weakly insulating (see (3.6)) and we introduce the following functional (4.40) in \( L^2(\Omega^*) \)

$$F_{\infty}[u] = \left\{ \begin{array}{ll}
\int_{\Omega^{(\xi)}} |\nabla u|^2 \, dx \, dy + \delta_0 \int_{\Omega^{(\xi)}} u^2 \, dx \, dy & \text{if } u|_{\Omega^{(\xi)}} \in H^1_0(\Omega^{(\xi)}) \\
+\infty & \text{otherwise in } L^2(\Omega^*)
\end{array} \right. \quad (4.40)$$

**Theorem 4.2** Let \( \varepsilon = \varepsilon(n) \) be a sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Let us assume (3.6) and (3.3). Then the sequence of functionals \( F_{\varepsilon(n)}^{\infty} \) defined in (4.3), \( M \)–converges in \( L^2(\Omega^*) \) as \( n \to +\infty \) to the energy functional \( F_{\infty}[u] \) defined in (4.40).

**Proof.** We now proceed with the proof of condition (a) in Definition 4.1. We consider a given function \( u \) as in condition (a) and we observe that, without loss of generality, we can assume that \( u|_{\Omega^{(\xi)}} \in H^1_0(\Omega^{(\xi)}) \), otherwise the inequality (4.1) becomes trivial. We assume, in addition, that \( u|_{\Omega^{(\xi)}} \in C^1_0(\Omega^{(\xi)}) \). For every \( n \) and \( \varepsilon \) as above, we define \( u_{1,n} \) on \( \Sigma^n_{1,\varepsilon} \)

$$u_{1,n}(x,y) = G_{1,\varepsilon}(u \circ \psi_{1|n}^{-1}(x,y)) \quad \text{if } (x,y) \in \Sigma^n_{1,\varepsilon} \quad (4.41)$$

where \( G_{1,\varepsilon} \) is the operator from \( C^1(\Sigma_{1,\varepsilon}) \) to \( Lip(\Sigma_{1,\varepsilon}) \) defined in the following way. For every \( \zeta \in (0,1) \), we define \( P_+ = P_+ (\zeta) = (\zeta, \hat{\eta}_+(\zeta)) \in \partial \Sigma_{1,\varepsilon} \) to be the intersection of \( \partial \Sigma_{1,\varepsilon} \) with the vertical line through the point \( (\zeta,0) \in K_1 \). Then, for a given \( g \in C^1(\Sigma_{1,\varepsilon}) \) we put

$$G_{1,\varepsilon}(g)(\zeta, \eta) = \begin{cases}
0 & \text{if } (\zeta, \eta) = (0,0) \\
g(\zeta,0) \frac{\eta - \eta_+}{\eta_+} & \text{if } (\zeta, \eta) \in \Sigma_{1,\varepsilon} \setminus \{A,B\} \quad (4.42) \\
0 & \text{if } (\zeta, \eta) = (1,0).
\end{cases}$$

$$\tilde{u} = \begin{cases} u \text{ on } \Omega^{(\xi)} \\
0 \text{ on } \Omega^* \setminus \Omega^{(\xi)} \end{cases}$$
and for every \( n \) and \( \varepsilon \) as above, we define
\[
  u_n(x, y) = \begin{cases} 
    \pi(x, y) & \text{if } (x, y) \in \Omega^* \setminus \overline{\Sigma^n_{\varepsilon}} \\
    u_j, n(x, y) & \text{if } (x, y) \in \Sigma^n_{j, \varepsilon}. 
  \end{cases}
\]  

(4.44)

We note that \( u_n \) tend to \( \pi \) in \( L^2(\Omega^*) \), \( \sup_{\Sigma^n_{\varepsilon}} |u_n| \leq \sup_{\Omega^*} |u| \) and the restriction on \( \Omega_{\varepsilon}^{(\kappa), n} \) of the functions \( u_n \) defined in (4.44) belong to \( H^1_0(\Omega_{\varepsilon}^{(\kappa), n}, w^n) \).

We proceed as in the proof of previous Theorem 4.1 and we note that the only difference is that we have to prove
\[
  \lim_{n \to +\infty} c_n \sigma_n \int_{\Sigma^n_{\varepsilon}} |\nabla u_n|^2 w^n dxdy = 0 \tag{4.45}
\]

hence we need to evaluate the terms like (4.19) and (4.21) by taking into account the assumptions of Theorem 4.2. By using the definition (4.42), we obtain
\[
  \sup_{\psi_{i_n}([0, 1])} |u_n| \leq \sqrt{M} l(n) \tag{4.46}
\]

where \( M = ||u||^2_{C^1(\Pi^{(\kappa)})} \). Starting from formula (4.23), we evaluate the term on the right end side by taking into account (4.46): more precisely
\[
  c_n \frac{l(n)}{4^n} \int_{K^n_1} u_n^2 ds \leq M \frac{c_n}{l(n)^2}. \tag{4.47}
\]

By using (3.6) we conclude
\[
  \lim_{n \to +\infty} c_n \frac{l(n)}{4^n} \int_{K^n_1} u_n^2 ds \leq \lim_{n \to +\infty} M \frac{c_n}{l(n)^2} = 0. \tag{4.48}
\]

In a similar way, we can evaluate the terms on \( \Sigma^n_{j, \varepsilon} \), \( j = 2, 3, 4 \), and the proof (4.45) is concluded. We complete the proof of part (a) of the Theorem, by making use of the diagonal formula of Corollary 1.16 of [3].

Now we prove condition (b) of Definition 4.1. Let \( v_n \) be a sequence as in (b) of Definition 4.1, that is,
\[
  v_n \rightharpoonup u \quad \text{in} \quad L^2(\Omega^*). \tag{4.49}
\]

In order to prove the inequality (4.2), it is not restrictive assume that the condition (4.27) holds and (up to passing to a subsequence) \( v_n \in H^1_0(\Omega_{\varepsilon}^{(\kappa), n}, w^n) \).

As the sequences \( c_n \) in the definition of the coefficients \( a_{\varepsilon}^n \) (see (2.3)) tend to \(+\infty\), for any \( k > 0 \), there exists \( \overline{\pi} \) such that for any \( n > \overline{\pi} \), \( c_n > k \). Then,
\[
  F^n[v_n] = \int_{\Omega_{\varepsilon}^{(\kappa), n}} |\nabla v_n|^2 dxdy + c_n \sigma_n \int_{\Sigma^n_{\varepsilon}} |\nabla v_n|^2 w^n dxdy + \delta_n \int_{\Omega_{\varepsilon}^{(\kappa), n}} v_n^2 dxdy >
\]
\[
\int_{\Omega(\xi)} |\nabla v_n|^2 dxdy + k\sigma_n \int_{\Sigma} |\nabla v_n|^2 w^2 dxdy + \delta_n \int_{\Omega(\xi)} v_n^2 dxdy
\]
for any \( n > n_0 \). By (4.31) and (4.33) in the proof of Theorem 4.1, we obtain

\[
\liminf_{n \to +\infty} F^n[v_n] \geq \int_{\Omega(\xi)} |\nabla u|^2 dxdy + k \int_{\partial\Omega(\xi)} u^2 d\mu(\xi) + \delta_0 \int_{\Omega(\xi)} u^2 dxdy,
\]
for any \( k > 0 \). Then we have

\[
\liminf_{n \to +\infty} F^n[v_n] \geq \int_{\Omega(\xi)} |\nabla u|^2 dxdy + \delta_0 \int_{\Omega(\xi)} u^2 dxdy,
\]
and, for any \( k > 0 \),

\[
\frac{1}{k} \liminf_{n \to +\infty} F^n[v_n] \geq \int_{\partial\Omega(\xi)} u^2 d\mu(\xi),
\]
that is, \( u = 0 \) on \( \partial\Omega(\xi) \). \( \square \)

5 M-K-S-convergence

We recall that M-convergence of forms implies the convergence of semigroups, resolvents and spectral structures relative to the associated operators (see [30]). As in the framework of the present paper the operators act in different Hilbert spaces we need to introduce the definition and the main properties of M-K-S-convergence that is a generalization of M-convergence to the case of variable Hilbert spaces.

**Definition 5.1** A sequence of Hilbert spaces \( H^n \) is said to converge to the Hilbert space \( H \) (\( H^n \to H \)) if there exists a dense subspace \( C \subset H \) and a sequence of linear maps \( \Phi_n : C \to H^n \) such that

\[
||\Phi_n u||_{H^n} \to ||u||_H,
\]

as \( n \to +\infty \), for every \( u \in C \).

**Lemma 5.1** The Hilbert spaces \( H^n = L^2(\Omega^{(\xi),n}) \) converge to the Hilbert space \( H = L^2(\Omega^{(\xi)}) \) in the sense of Definition 5.1 as \( n \to +\infty \).

**Proof.** We choose \( C = C(\Omega^{(\xi)}) \) and, for every \( n \), we define the map \( \Phi_n \)

\[
\Phi_n u = \begin{cases} u & \text{in } \Omega^{(\xi)} \cap \Omega^{(\xi),n} \\ 0 & \text{in } \Omega^{(\xi),n} \setminus \Omega^{(\xi)} \end{cases}
\]
The convergence of the spaces in the sense of Definition 5.1 follows from the convergence of the sets \( \Omega^{(\xi)}_n \) to the set \( \Omega^{(\xi)} \) as \( n \to +\infty \). □

We recall the notion of convergence for functionals and operators given in [29], see also [30], as adapted by K. Kuwae - T. Shioya and A. Kolesnikov in the papers [24], [23] to variable Hilbert spaces as in Definition 5.1.

We start by giving the definitions of strong convergence and of weak convergence of vectors in variable Hilbert spaces according to K. Kuwae - T. Shioya (see [24]).

**Definition 5.2** Let \( H^n \to H \) according to Definition 5.1. A sequence of vectors \( u_n \in H^n \) converges K-S-strongly to a vector \( u \in H \) (\( u_n \to u \) K-S-strongly) if there exists a sequence \( \tilde{u}_m \in \mathcal{C} \) such that both conditions below hold:

\[
||\tilde{u}_m - u||_H \to 0 \quad (5.2)
\]

as \( m \to +\infty \),

\[
\lim_m \limsup_n ||\Phi_n \tilde{u}_m - u_n||_{H^n} = 0. \quad (5.3)
\]

as \( n \to +\infty \) and \( m \to +\infty \).

**Definition 5.3** Let \( H^n \to H \) according to Definition 5.1. A sequence of vectors \( u_n \in H^n \) converges K-S-weakly to a vector \( u \in H \) (\( u_n \to u \) K-S-weakly), if

\[
(u_n, v)_{H^n} \to (u, v)_H \quad (5.4)
\]

for every \( v_n \to v \) K-S-strongly, as \( n \to +\infty \).

**Definition 5.4** Let \( H^n \to H \) according to Definition 5.1. Let \( F^n : H^n \to (-\infty, +\infty] \) for every \( n \) and let \( F : H \to (-\infty, +\infty] \). The functionals \( F^n \) M-K-S-converge to \( F \) as \( n \to +\infty \) if both conditions below hold:

(a) For every \( u \in H \) there exists \( u_n \in H^n \) converging K-S-strongly to \( u \) such that

\[
\limsup_n F^n[u_n] \leq F[u], \quad \text{as } n \to +\infty. \quad (5.5)
\]

(b) For every \( v_n \in H^n \) converging K-S-weakly to \( u \in H \)

\[
\liminf_n F^n[v_n] \geq F[u], \quad \text{as } n \to +\infty. \quad (5.6)
\]

Convergence of operators is defined as follows.

**Definition 5.5** Let \( H^n \to H \) according to Definition 5.1. A sequence of bounded linear operators \( B^n \in \mathcal{L}(H^n) \) converges strongly to the linear bounded operator \( B \in \mathcal{L}(H) \) if for every \( u_n \in H^n \) converging K-S-strongly to \( u \in H \), the sequence \( B^n(u_n) \in H^n \) converges K-S-strongly to \( B(u) \in H \).
The following Proposition 5.1 collects some convergence properties from [23] (see also [24]).

**Proposition 5.1** Let $H^n \to H$ according to Definition 5.1. Then the following properties hold:

(i) For every $u \in H$ there exist vectors $u_n \in H^n$ that converge K-S-strongly to $u$.

(ii) The vectors $u_n \in H^n$ converge K-S-strongly to $u \in H$ if and only if

$$||u_n||_{H^n} \to ||u||_H, \text{ and } (u_n, \Phi_n \varphi)_{H^n} \to (u, \varphi)_H$$

for every $\varphi \in C$.

We are now in position to prove Theorem 3.2 and Theorem 3.4.

**Proof of Theorem 3.2.** We note that Theorem 3.2 and Theorem 4.1 have the same hypotheses. Then the functionals $F_{\varepsilon}^n$ M-converge in $L^2(\Omega^*)$ to the functional $F_{c_0}$. We show that M-convergence of the functionals $F_{\varepsilon}^n$ implies M-K-S-convergence of the functionals $E_{\varepsilon}^n$ to the functional $E_{c_0}$ where

$$E_{\varepsilon}^n[u] = \begin{cases} \int_{\Omega^*} |\nabla u| dx + \delta_n \int_{\Omega^*} u^2 dx + \int_{\partial \Omega} \varphi \cdot \nabla u & \text{if } u \in H^1(\Omega^*,\varepsilon) \\ +\infty & \text{if } u \notin H^1(\Omega^*,\varepsilon) \end{cases}$$

(5.7)

(the coefficients $a^\varepsilon_n$ are defined in (2.3), (2.5), (2.4), $\delta_n > 0$) and

$$E_{c_0}[u] = \begin{cases} \int_{\Omega} |\nabla u| dx + \delta \int_{\Omega} u^2 dx + c \int_{\partial \Omega} u^2 d\mu & \text{if } u \in H^1(\Omega) \\ +\infty & \text{if } u \notin H^1(\Omega) \end{cases}$$

(5.8)

If $u \in L^2(\Omega^*) \setminus H^1(\Omega)$ then condition (a) in Definition 5.4 follows from property (i) of Proposition 5.1. If instead $u \in H^1(\Omega^*)$, we put

$$\overline{\mu} = \begin{cases} u & \text{in } \Omega^* \\ 0 & \text{in } \Omega^* \setminus \Omega^* \end{cases}$$

(5.9)

and by condition (a) in Definition 4.1 there exists a sequence $u^*_n \in L^2(\Omega^*)$ strongly convergent to $\overline{\mu}$ (in $L^2(\Omega^*)$) such that $u_n := u^*_n |_{\Omega^*} \in H^1(\Omega^*,\varepsilon)$ and

$$\lim \sup E_{\varepsilon}^n[u_n] = \lim \sup F_{\varepsilon}^n[u_n] \leq F_{c_0}[\overline{\mu}] = E_{c_0}[u] < +\infty.$$
\[
\int_{\Omega^{(n)}} u_n^2 dx dy \to \int_{\Omega^{(n)}} u^2 dx dy, \quad \text{as } n \to +\infty.
\]

As the increasing sequence of the sets \(\Omega^{(n)}\) converges to the set \(\Omega^{(\xi)}\) in the Hausdorff metric we deduce, passing to the limit on \(n\), that

\[
\liminf_{n \to +\infty} \int_{\Omega^{(n)}} u_n^2 dx dy \geq \lim_{n \to +\infty} \lim_{n \to +\infty} \int_{\Omega^{(n)}} u_n^2 dx dy = \int_{\Omega^{(\xi)}} u^2 dx dy.
\]

On the other hand,

\[
\limsup_{n \to +\infty} \int_{\Omega^{(n)}} u_n^2 dx dy \leq \lim_{n \to +\infty} \int_{\Omega^{(n)}} (u_n^*)^2 dx dy = \int_{\Omega^*} \tilde{u}^2 dx dy = \int_{\Omega^{(\xi)}} u^2 dx dy.
\]

hence the first condition in item (ii) of Proposition 5.1 is proved. Consider now \(\varphi \in \mathcal{C}_0(\Omega^{(\xi)})\) and set

\[
\varphi = \begin{cases} 
\varphi & \text{in } \Omega^{(\xi)} \\
0 & \text{in } \Omega^* \setminus \Omega^{(\xi)} \end{cases} \quad (5.10)
\]

and

\[
\varphi_n = \begin{cases} 
\varphi & \text{in } \Omega^{(\xi),n} \\
0 & \text{in } \Omega^* \setminus \Omega^{(\xi),n} \end{cases} \quad (5.11)
\]

It is easy to note that the sequence of the functions \(\varphi_n\) is weakly convergent to the function \(\varphi\) in \(L^2(\Omega^*)\).

Then we have

\[
\int_{\Omega^{(\xi),n}} u_n \Phi_n \varphi dx dy = \int_{\Omega^{(n)}) \cap \Omega^{(\xi)}} u_n \varphi dx dy = \int_{\Omega^{(\xi)}} u_n \varphi_n dx dy \to \int_{\Omega^{(\xi)}} u \varphi dx dy
\]
as \(n \to +\infty\) and also the second condition in item (ii) of Proposition 5.1 is proved.

Let us consider condition (b) in Definition 5.4. We assume that the sequence \(v_n\) converges K-S-weakly to \(u\) and \(\liminf E^{(n)}_v [v_n] \leq C^*\) then, up to passing to a subsequence still denoted by \(v_n\), the function \(v_n\) belongs to the space \(H^1_0(\Omega^{(n)}; w^n_\varepsilon)\) and \(\lim E^{(n)}_v [v_n] = \lim F^{(n)}_v [v_n] \leq C^*\).

We set

\[
\overline{v}_n = \begin{cases} 
v_n & \text{in } \Omega^{(n)} \\
0 & \text{in } \Omega^* \setminus \Omega^{(n)} \end{cases} \quad (5.12)
\]

and we prove that the sequence of the functions \(\overline{v}_n\) is weakly convergent (in \(L^2(\Omega^*)\)) to the function \(\overline{v}\) defined in (5.9). In fact for any \(g \in \mathcal{C}_0(\Omega^*)\) we set

\[
g_n = \begin{cases} 
g & \text{in } \Omega^{(n)} \\
0 & \text{in } \Omega^* \setminus \Omega^{(n)} \end{cases} \quad (5.13)
\]
and by using Proposition 5.1 we prove that the sequence $g_n$ converges K-S-
strongly to $g$.

Hence

$$
\int_{\Omega^*} (\overline{v}_n - \overline{u}) g dx dy = \int_{\Omega^*} v_n g dx dy - \int_{\Omega^*} u g dx dy \to 0
$$

and so condition (b) in Definition 5.4 follows from condition (b) in Definition 4.1.

From Theorem 2.4 of [24], we deduce that $\forall \ 0 < \mu < \lambda$, points of continuity for the spectral resolution $P_{c_0}$ of the operator $A_{c_0}$, the projector operators $P^n((\mu - \lambda))$ of the spectral resolutions $P^n$ of the operators $A^n$ in $L^2(\Omega^*_{\varepsilon,n})$ converge to the projector $P_{c_0}((\mu - \lambda))$ of the spectral resolution $P_{c_0}$ strongly as $n \to +\infty$, in the sense of Definition 5.5. $\square$.

Proof of Theorem 3.4. In the same way we obtain that from Theorem 4.2 it follows the convergence of the functionals $E_n^\varepsilon$ defined in (5.7) to the functional $E^\infty$ according the Definition (5.4) where

$$
E^\infty[u] = \begin{cases} 
\int_{\Omega^*} |\nabla u|^2 dx dy + \delta_0 \int_{\Omega^*} |u|^2 dx dy & \text{if } u \in H^1_0(\Omega^*(\varepsilon)) \\
\infty & \text{if } u \in L^2(\Omega^*(\varepsilon)) \setminus H^1_0(\Omega^*(\varepsilon)).
\end{cases}
$$

From Theorem 2.4 of [24], we deduce that $\forall \ 0 < \mu < \lambda$, points of continuity for the spectral resolution $P_D$ of the operator $A_D$, the projector operators $P^n((\mu - \lambda))$ of the spectral resolutions $P^n$ of the operators $A^n$ in $L^2(\Omega^*_{\varepsilon,n})$ converge to the projector $P_D((\mu - \lambda))$ of the spectral resolution $P_D$ strongly as $n \to +\infty$, in the sense of Definition 5.5. $\square$.

6 Asymptotics

In this section we prove Theorems 3.1 and 3.3. We note that the coerciveness estimate (2.9) is not sharp enough for our aim and we need to improve it: therefore, before proving Theorem 3.1, we establish a Poincaré type inequality where the relevant fact is that the constant $C^*_P$ is independent of $n$.

Theorem 6.1 There exists a constant $C^*_P$ independent of $n$, such that,

$$
||u||_{L^2(\Omega^*_{\varepsilon,n})} \leq C^*_P(||\nabla u||^2_{L^2(\Omega^*(\varepsilon,n))} + \sigma_n \int_{\Sigma^{\varepsilon,n}_n} |\nabla u|^2 w^*_n dx dy)^{1/2} \quad (6.1)
$$

for all $u \in H^1_0(\Omega^*_{\varepsilon,n}; w^*_n)$.
Proof. Suppose the statement to be proved is false: for every \( n \in \mathbb{N} \), there exists \( v_n \in H^1_0(\Omega^{(\xi)\cdot n}; w^{n}_\epsilon) \), such that

\[
\frac{||v_n||^2_{L^2(\Omega^{(\xi)\cdot n})}}{n^2} > n^2 (||\nabla v_n||^2_{L^2(\Omega^{(\xi)\cdot n})} + \sigma_n \int_{\Omega^{(\xi)\cdot n}} |\nabla v_n|^2 w^{n}_\epsilon dxdy). \tag{6.2}
\]

Set

\[ u_n = \frac{v_n}{||v_n||_{L^2(\Omega^{(\xi)\cdot n})}}. \]

We extend \( u_n \) from the set \( \Omega^{(\xi)\cdot n} \) to \( \Omega^* \)

\[ \tilde{u}_n = \begin{cases} u_n & \text{in } \Omega^{(\xi)\cdot n} \\ 0 & \text{in } \Omega^* \setminus \Omega^{(\xi)\cdot n}. \end{cases} \tag{6.3} \]

Therefore, there exists a function \( u^* \in L^2(\Omega^*) \) and a subsequence (still denoted by \( \tilde{u}_n \)) that converges to \( u^* \) weakly in \( L^2(\Omega^*) \).

From (6.2), we obtain

\[
||\nabla u_n||^2_{L^2(\Omega^{(\xi)\cdot n})} + \sigma_n \int_{\Omega^{(\xi)\cdot n}} |\nabla u_n|^2 w^{n}_\epsilon dxdy < \frac{1}{n^2} \tag{6.4}
\]

and, in particular,

\[
||u_n||^2_{H^1(\Omega^{(\xi)\cdot n})} < 1 + \frac{1}{n^2}.
\]

We extend \( u_n \) from the set \( \Omega^{(\xi)\cdot n} \) to \( \mathbb{R}^2 \) (see Theorem 8.3): we set

\[ u^{**}_n = (\text{Ext}_{\Omega^{(\xi)\cdot n}} u_n)|_{\Omega^{(\xi)\cdot n}} \tag{6.5} \]

and we have

\[
||u^{**}_n||_{H^1(\Omega^{(\xi)\cdot n})} \leq \sqrt{C_J} ||u_n||_{H^1(\Omega^{(\xi)\cdot n})} \leq C \sqrt{C_J}. \tag{6.6}
\]

Therefore, there exists a function \( u^{**} \) in \( H^1(\Omega^{(\xi)}) \) and a subsequence still denoted by \( u^{**}_n \) that converges to \( u^{**} \) weakly in \( H^1(\Omega^{(\xi)}) \) and strongly in \( L^2(\Omega^{(\xi)}) \).

We fix \( n_0 \) : then, as \( u^{**}_n = u_n = \tilde{u}_n \) on \( \Omega^{(\xi)\cdot n_0} \), we deduce \( u^{**} = u^* \) in \( \Omega^{(\xi)\cdot n_0} \).

As the increasing sequence of the sets \( \Omega^{(\xi)\cdot n} \) converges to the set \( \Omega^{(\xi)} \) in the Hausdorff metric we deduce, passing to the limit on \( n_0 \), that \( u^{**} = u^* \) on \( \Omega^{(\xi)} \). Hence the sequence \( u_n \) converges to \( u^* \) weakly in \( H^1(\Omega^{(\xi)\cdot n_0}) \) : then

\[
||\nabla u^*||_{L^2(\Omega^{(\xi)\cdot n_0})} \leq \liminf_n ||\nabla u_n||_{L^2(\Omega^{(\xi)\cdot n_0})} \leq 0, \tag{6.7}
\]

i.e. the function \( u^* \) is constant on \( \Omega^{(\xi)\cdot n_0} \) and, passing to the limit on \( n_0 \) again, \( u^* \) is constant on \( \Omega^{(\xi)} \).
Since the functions $\tilde{u}_n$ converge to $u^*$ weakly in $L^2(\Omega^*)$ and $\tilde{u}_n|_{\Omega^*(\xi),n} = u_n \in H^1_0(\Omega^*(\xi),n; w^\alpha_n)$, we can proceed as in the proof of Theorem 4.1 (see (4.33)) and we obtain

$$\int_{\partial\Omega^*}(u^*)^2d\mu^{(\xi)} \leq \liminf_{n \to +\infty} \sigma_n \int_{\Omega^*(\xi),n} |\nabla u_n|^2 w^\alpha_n dxdy. \tag{6.8}$$

As $u^*$ is constant on $\Omega^*(\xi)$, from (6.4) and (6.8) we deduce that $u^* = 0$ on $\Omega^*(\xi)$.

By a direct calculation, we obtain

$$||u_n||^{2}_{L^2(\Sigma^*(\xi),n)} \leq \sigma_n \int_{\Sigma^*(\xi),n} |\nabla u_n|^2 w^\alpha_n dxdy \leq \frac{1}{n^2}, \tag{6.9}$$

therefore, as

$$1 = ||u_n||^{2}_{L^2(\Omega^*(\xi),n)} = ||u_n||^{2}_{L^2(\Omega^*(\xi),n)} + ||u_n - u_n^*||^{2}_{L^2(\Omega^*(\xi),n)} \leq ||u_n^*||^{2}_{L^2(\Omega^*(\xi),n)} + \frac{1}{n^2},$$

passing to the limit, we obtain a contradiction. \hfill \Box

We now prove Theorem 3.1.

**Proof.** Let $u_n$ be the solution of the problem (2.7). Then

$$a_n(u_n, u_n) \leq ||f_n||^{2}_{L^2(\Omega^*(\xi),n)}||u_n||^{2}_{L^2(\Omega^*(\xi),n)} + C|d_n||u_n||^{2}_{H^1(\Omega^*(\xi),n)} \tag{6.10}$$

where we have used Theorems 8.3 and 8.1.

Suppose first that $\delta_0 > 0$. Then, from (6.10), by the definition of the form $a_n(\cdot, \cdot)$ (see (2.6)) we deduce

$$\delta_n||u_n||^{2}_{L^2(\Omega^*(\xi),n)} + ||\nabla u_n||^{2}_{L^2(\Omega^*(\xi),n)} \leq ||f_n||^{2}_{L^2(\Omega^*(\xi),n)}||u_n||^{2}_{L^2(\Omega^*(\xi),n)} + C|d_n||u_n||^{2}_{H^1(\Omega^*(\xi),n)} \tag{6.11}$$

and hence

$$||u_n||^{2}_{H^1(\Omega^*(\xi),n)} \leq C^*, \tag{6.12}$$

where the constant $C^*$ does not depend on $n$.

By assumption (2.11), if $\delta_0 = 0$, then $c_0 > 0$. From inequality (6.1) we derive

$$||u_n||^{2}_{L^2(\Omega^*(\xi),n)} \leq (C^*_p)^2(||\nabla u_n||^{2}_{L^2(\Omega^*(\xi),n)} + \sigma_n \int_{\Omega^*(\xi),n} |\nabla u_n|^2 w^\alpha_n dxdy). \tag{6.13}$$

From (6.10) and by the definition of the form $a_n(\cdot, \cdot)$, we deduce that

$$||\nabla u_n||^{2}_{L^2(\Omega^*(\xi),n)} + \sigma_n c_n \int_{\Omega^*(\xi),n} |\nabla u_n|^2 w^\alpha_n dxdy \leq \tag{6.14}$$
\[ \|f_n\|_{L^2(\Omega^{(\xi),n})} + \|u_n\|_{L^2(\Omega^{(\xi),n})} + C|d_n|\|u_n\|_{H^1(\Omega^{(\xi),n})} \]

and, as a consequence of (6.13) and (6.14),

\[ \|u_n\|_{H^1(\Omega^{(\xi),n})}^2 \leq C^{**}, \quad (6.15) \]

where the constant \( C^{**} \) does not depend on \( n \).

In any case, we shall consider the function \( u_n^* \), which, as before, is the extension of the function \( u_n \) from the set \( \Omega^{(\xi),n} \) to the set \( \Omega^* \), and in any case for every \( n \), from either (6.12) or (6.15) we derive

\[ \|u_n^*\|_{H^1(\Omega^*)} \leq \sqrt{C_J}\|u_n\|_{H^1(\Omega^{(\xi),n})} \leq C. \quad (6.16) \]

Therefore, there exists a subsequence still denoted by \( u_n^* \) that weakly converges to a function \( u^* \) in \( H^1(\Omega^*) \). Now we prove that \( u^*|_{\Omega^{(\xi)}} = u \). By using condition (b) of M-convergence we obtain that

\[ F[u^*] - 2 \int_{\Omega^{(\xi)}} f u^* \, dxdy - 2d \int_{\partial\Omega^{(\xi)}} u^* \, d\mu^{(\xi)} \leq \liminf \left( F_{\varepsilon}[u_n^*] - 2 \int_{\Omega^{(\xi),n}} f_n u_n^* \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} u_n^* \, ds \right). \quad (6.17) \]

By using condition (a) of M-convergence there exists \( v_n \in L^2(\Omega^*) \), \( v_n|_{\Omega^{(\xi),n}} \in H^1_0(\Omega^{(\xi),n}; \epsilon_n^2) \) converging strongly in \( L^2(\Omega^*) \) to \( \tilde{u} \) (defined in (4.43)) such that

\[ \lim F_{\varepsilon}^n[v_n] = F[\tilde{u}] = F[u], \]

as \( n \to +\infty \). Then by Proposition 4.1 (where \( g_n \) is a suitable extension of \( v_n \) we obtain

\[ \lim \left( F_{\varepsilon}[v_n] - 2 \int_{\Omega^{(\xi),n}} f_n v_n \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} v_n \, ds \right) = F[u] - 2 \int_{\Omega^{(\xi)}} f u \, dxdy - 2d \int_{\partial\Omega^{(\xi)}} u \, d\mu^{(\xi)}. \quad (6.18) \]

As, by direct calculations, we obtain

\[ \liminf \left( F_{\varepsilon}[u_n^*] - 2 \int_{\Omega^{(\xi),n}} f_n u_n^* \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} u_n^* \, ds \right) \leq \liminf \left( F_{\varepsilon}[u_n^*] - 2 \int_{\Omega^{(\xi),n}} f_n u_n^* \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} u_n \, ds \right) \leq \liminf \left( F_{\varepsilon}[u_n] - 2 \int_{\Omega^{(\xi),n}} f_n u_n \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} u_n \, ds \right) \]

\[ \leq \liminf \left( F_{\varepsilon}[u_n] - 2 \int_{\Omega^{(\xi),n}} f_n u_n \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} u_n \, ds \right) \]

and

\[ F_{\varepsilon}[u_n] - 2 \int_{\Omega^{(\xi),n}} f_n u_n \, dxdy - 2\sigma_n d_n \int_{\partial\Omega^{(\xi),n}} u_n \, ds = \quad (6.20) \]
\[ v \in H_0^1(\Omega_n; w(\xi), n \in \mathbb{N}) \]

\[ F_n[v_n] - 2 \int_{\partial \Omega(\xi)} v_n \, d\lambda_n \leq F_n[u_n] - 2 \int_{\partial \Omega(\xi)} u_n \, d\lambda_n, \]

combining (6.17), (6.19), (6.20), and (6.18) we obtain that

\[ F[u^*] - 2 \int_{\Omega(\xi)} f \, u^* \, dx \, dy - 2d \int_{\partial \Omega(\xi)} u^* \, d\mu(\xi) \leq F[\tilde{u}] - 2 \int_{\Omega(\xi)} f \, u \, dx \, dy - 2d \int_{\partial \Omega(\xi)} u \, d\mu(\xi). \]

By the uniqueness of the solution (2.12), we conclude that \( u^* \big|_{\Omega(\xi)} = u \), and \( u^*_n \big|_{\Omega(\xi)} \) converges to \( u \) weakly in \( H^1(\Omega(\xi)) \).

\section{Laplacean transport}

The mathematical model of the so-called Laplacean transport and of some other phenomena involve mixed Dirichelet-Robin conditions on the boundary.

In the present setting the geometry of the bulk could be represented by the domain \( \tilde{\Omega}(\xi) = \Omega(\xi) \setminus B(P_0, \frac{1}{8}) \), where \( B(P_0, \frac{1}{8}) \) is the ball with center \( P_0 = \left( \frac{1}{2}, -\frac{1}{2} \right) \) and radius \( \frac{1}{8} \) where \( \Omega(\xi) \) is defined in Section 2.

In a similar way, we consider the prefractal domains \( \tilde{\Omega}(\xi)_n = \Omega(\xi)_n \setminus B(P_0, \frac{1}{8}) \), and the reinforced domains \( \tilde{\Omega}(\xi)_n = \Omega(\xi)_n \setminus B(P_0, \frac{1}{8}) \).

We consider the bilinear form associated with the Dirichelet Robin problem

\[ \tilde{a}(u, v) := \int_{\tilde{\Omega}(\xi)} \nabla u \nabla v \, dx \, dy + c_0 \int_{\partial \Omega(\xi)} u \, v \, d\mu(\xi), \quad (7.1) \]

where \( \mu(\xi) \) is the measure on \( \partial \Omega(\xi) \) that coincides, on each \( K_j(\xi) \), \( j = 1, 2, 3, 4 \), with the Radon measure defined in (8.4).

\textbf{Theorem 7.1} For any \( d \in \mathbb{R}, c_0 \geq 0 \), there exists one and only one solution \( u \) of the following problem

\[ \begin{cases} \text{find } u \in V(\tilde{\Omega}(\xi)) : & u \in H^1(\tilde{\Omega}(\xi)) \text{ such that} \\ \tilde{a}(u, v) = d \int_{\partial \Omega(\xi)} v \, d\mu(\xi) & \forall v \in V(\tilde{\Omega}(\xi)) \end{cases} \]

(7.2)

where \( \tilde{a}(\cdot, \cdot) \) is defined in (7.1). Moreover, \( u \) is the only function that realizes the minimum of the energy functional

\[ \min_{v \in V(\tilde{\Omega}(\xi))} \left\{ \tilde{a}(v, v) - 2d \int_{\partial \Omega(\xi)} v \, d\mu(\xi) \right\}. \]
Theorem 7.1 can be proved via the Lax-Milgram Theorem, the principal tools being a trace result (Theorem 8.2) and a generalized Poincaré inequality (Lemma 3.1.1 in [28]). Since the proof is similar to the proof of Theorem 2.2, but easier, we skip it.

The following theorem concerns the Dirichlet problems in the reinforced domains.

**Theorem 7.2** Let \( d_n \in \mathbb{R} \) and \( \sigma_n \) be as in (2.5). Then, there exists one and only one solution \( u_n \) of the problem

\[
\begin{cases}
  \text{find } u_n \in H^1_0(\tilde{\Omega}_n^{(\xi)}; w^n) & \text{such that } \\
  \tilde{a}_n(u, v) = \sigma_n d_n \int_{\partial \Omega(\xi), \varepsilon} v \, ds & \forall \ v \in H^1_0(\tilde{\Omega}_n^{(\xi)}, w^n),
\end{cases}
\]

where the bilinear form \( \tilde{a}_n(\cdot, \cdot) \) is

\[
\tilde{a}_n(u, v) := \int_{\tilde{\Omega}_n^{(\xi)}} a_n^n \nabla u \nabla v \, dxdy
\]

and the coefficients \( a_n^n \) are defined in (2.3) (see also (2.5) and (2.4)).

Moreover, \( u_n \) is the only function that realizes the minimum of the functional

\[
\min_{v \in H^1_0(\tilde{\Omega}_n^{(\xi)}, w^n)} \left\{ \tilde{a}_n(v, v) - 2\sigma_n d_n \int_{\partial \Omega(\xi), \varepsilon} v \, ds \right\}.
\]

In order to prove Theorem 7.2, we state a generalized Poincaré inequality for functions belonging to \( H^1(\tilde{\Omega}_n^{(\xi)}, w^n) \) where the relevant fact is that the constant \( C_P \) is independent of \( n \).
Theorem 7.3 There exists a constant $C_P$ independent of $n$, such that,

$$
\|u\|_{L^2(\Omega^{(\xi),n})} \leq C_P (\|\nabla u\|_{L^2(\Omega^{(\xi),n})} + \|u\|_{L^2(\partial B(P_0,\frac{1}{8}))}) \tag{7.6}
$$

for all $u \in H^1(\Omega^{(\xi),n})$.

**Proof.** Set, for $u \in H^1(\Omega^{(\xi),n})$,

$$
G(u) = \|u\|_{L^2(\partial B(P_0,\frac{1}{8}))}. \tag{7.7}
$$

Suppose the statement to be proved is false: for every $n \in \mathbb{N}$, there exists $v_n \in H^1(\Omega^{(\xi),n})$, such that

$$
\|v_n\|_{L^2(\Omega^{(\xi),n})} > n (\|\nabla v_n\|_{L^2(\Omega^{(\xi),n})} + G(v_n)). \tag{7.8}
$$

Set

$$
u_n = \frac{v_n}{\|v_n\|_{H^1(\Omega^{(\xi),n})}}. \tag{7.9}
$$

We extend $u_n$ from $\Omega^{(\xi),n}$ to $\mathbb{R}^2$ by using an extension operator whose norm is independent of the (increasing) number of sides (see Theorem 8.3) and for $u^*_n = Ext_J u_n|_{\Omega^{(\xi)}}$ we have

$$
\|u^*_n\|_{H^1(\Omega^{(\xi)})} \leq \sqrt{C_J} \|u_n\|_{H^1(\Omega^{(\xi),n})} = \sqrt{C_J}. \tag{7.10}
$$

Then, there exists a function $u^*$ in $H^1(\Omega^{(\xi)})$ and a subsequence (still denoted by $u^*_n$) that converges to $u^*$ weakly in $H^1(\Omega^{(\xi)})$ and strongly in $H^s(\Omega^{(\xi)})$ for $0 \leq s < 1$ (see Theorem 1.4.6.2 in [27] and Theorem 16.2 in [26]). From (7.8), we obtain

$$
\|\nabla u_n\|_{L^2(\Omega^{(\xi),n})} < \frac{1}{n}. \tag{7.11}
$$

We fix $n_0$ : for all $n \geq n_0$ as $\Omega^{(\xi),n_0} \subseteq \Omega^{(\xi),n}$

$$
\|\nabla u_n\|_{L^2(\Omega^{(\xi),n_0})} < \frac{1}{n} \tag{7.12}
$$

and hence

$$
\|\nabla u^*_n\|_{L^2(\Omega^{(\xi),n_0})} \leq \liminf_n \|\nabla u^*_n|_{\Omega^{(\xi),n_0}}\|_{L^2(\Omega^{(\xi),n_0})} = 0, \tag{7.13}
$$

i.e. the function $u^*$ is constant on $\Omega^{(\xi),n_0}$ and, passing to the limit on $n_0$, we obtain that $u^*$ is constant on $\Omega^{(\xi)}$.

On the other hand, from (7.8), we obtain

$$
G(u_n) < \frac{1}{n} : \tag{7.14}
$$
so

\[ G(u^*) = ||u^*||_{L^2(\partial B(P_0, \frac{1}{k}))} = \lim_{n \to \infty} ||u^*_n||_{\tilde{\Omega}^n} ||u^*_n||_{L^2(\partial B(P_0, \frac{1}{k}))} = 0 \]

and then \( u^* = 0 \) on \( \tilde{\Omega}^\xi \). Therefore, as

\[ 1 = ||u_n||^2_{H^1(\tilde{\Omega}^\xi, n)} = ||u_n||^2_{L^2(\tilde{\Omega}^\xi, n)} + ||\nabla u_n||^2_{L^2(\tilde{\Omega}^\xi, n)} \leq ||u^*_n||^2_{L^2(\tilde{\Omega}^\xi)} + \frac{1}{n^2}, \]

passing to the limit, we obtain a contradiction. \( \square \)

Now we prove Theorem 7.2.

**Proof.** We use the Lax-Milgram theorem. Indeed, by Theorems 8.1 and 8.3, we have that

\[ |\sigma_n| d_n \int_{\partial \Omega(\xi, n)} v \, ds \leq |\sigma_n| d_n ||v||_{L^2(\partial \Omega(\xi, n))} \sqrt{|\partial \Omega(\xi, n)|} \leq 2 \sqrt{C_1 C_J} d_n ||v||_{H^1(\tilde{\Omega}^\xi, n)} \leq 2 \sqrt{C_1 C_J} d_n ||v||_{H^1_0(\tilde{\Omega}^\xi, n; w^n_\xi)}. \]

Obviously, the bilinear form \( \tilde{\alpha}_n(u, v) \) is continuous, \( i.e. \)

\[ |\tilde{\alpha}_n(u, v)| \leq \max(1, \sigma_n c_n) ||u||_{H^1_0(\tilde{\Omega}^\xi, n; w^n_\xi)} ||v||_{H^1_0(\tilde{\Omega}^\xi, n; w^n_\xi)}. \]

In order to prove the coerciveness, we first obtain by means of a direct calculation that

\[ ||u||^2_{L^2(\Sigma(\xi, n))} \leq \sigma_n \int_{\Sigma(\xi, n)} |\nabla u|^2 w^n_\xi \, dx dy, \quad (7.15) \]

for all \( u \in H^1_0(\tilde{\Omega}^\xi, n; w^n_\xi) \).

From Poincaré inequality (7.6), we deduce that

\[ ||u||_{L^2(\tilde{\Omega}^\xi, n)} \leq C_P ||\nabla u||_{L^2(\tilde{\Omega}^\xi, n)}, \quad (7.16) \]

for all \( u \in H^1_0(\tilde{\Omega}^\xi, n; w^n_\xi) \).

Then, from (7.15) and (7.16), we obtain that the bilinear form is coercive, \( i.e. \)

\[ \tilde{\alpha}_n(u, u) \geq \frac{1}{2} \min(1, \sigma_n c_n, c_n, \frac{1}{C_P}) ||u||^2_{H^1_0(\tilde{\Omega}^\xi, n; w^n_\xi)}, \quad (7.17) \]

\( \square \)

For any \( n \) we denote by \( \tilde{u}_n \) the trivial extension on \( B(P_0, \frac{1}{k}) \), \( i.e. \) \( (\tilde{u}_n)|_{B(P_0, \frac{1}{k})} = 0 \), we then extend the functions \( \tilde{u}_n \) from \( \Omega(\xi, n) \) to \( \mathbb{R}^2 \) by using an extension operator whose norm is independent of the (increasing) number of sides (see Theorem 8.3) and we set

\[ u^*_n = (Ext \tilde{u}_n|_{\Omega(\xi, n)})|_{\tilde{\Omega}^\xi}. \quad (7.18) \]

The following theorem, that can be proved as Theorem 3.1, establishes the asymptotic behavior of the functions \( u^*_n \) (defined in (7.18)).
Theorem 7.4 Let us assume (3.4) and (3.5). Let \( \varepsilon = \varepsilon(n) \) be an arbitrary sequence such that \( \varepsilon(n) \to 0 \) as \( n \to +\infty \). Then the sequence of the restrictions to \( \bar{\Omega}(\xi) \) of the functions \( u^\varepsilon_n \) (defined in (7.18)) converges to the function \( u \) (defined in (7.2)) weakly in \( H^1(\bar{\Omega}(\xi)) \).

8 APPENDIX

8.1. Scale irregular Koch curves.

We recall the definition of scale irregular Koch curves built on two families of contractive similitudes (for general irregular scale fractals and their main properties, see [4], [31], and [32]).

Let \( A = \{1, 2\} \): let \( 2 < \ell_1 < \ell_2 < 4 \), and, for each \( a \in A \), let

\[
\Psi^{(a)} = \{ \psi_1^{(a)}, \ldots, \psi_4^{(a)} \}
\]

be the family of contractive similitudes \( \psi_i^{(a)} : \mathbb{C} \to \mathbb{C} \), \( i = 1, \ldots, 4 \), with contraction factor \( \ell_a^{-1} \):

\[
\psi_1^{(a)}(z) = \frac{z}{\ell_a}, \quad \psi_2^{(a)}(z) = \frac{z}{\ell_a} e^{i \theta(\ell_a)} + \frac{1}{\ell_a},
\]

\[
\psi_3^{(a)}(z) = \frac{z}{\ell_a} e^{-i \theta(\ell_a)} + \frac{1}{\ell_a} + i \sqrt{\frac{1}{\ell_a} - \frac{1}{4}}, \quad \psi_4^{(a)}(z) = \frac{z - 1}{\ell_a} + 1,
\]

where

\[
\theta(\ell_a) = \arcsin \left( \frac{\sqrt{\ell_a(4 - \ell_a)}}{2} \right).
\]

Let \( \Xi = A^\mathbb{N} \): we call \( \xi \in \Xi \) an environment. We define a left shift \( S \) on \( \Xi \) such that if \( \xi = (\xi_1, \xi_2, \xi_3, \ldots) \), then \( S\xi = (\xi_2, \xi_3, \ldots) \). For \( \mathcal{O} \subset \mathbb{R}^2 \), we set

\[
\Phi^{(a)}(\mathcal{O}) = \bigcup_{i=1}^{4} \psi_i^{(a)}(\mathcal{O})
\]

and

\[
\Phi^{(\xi)}(\mathcal{O}) = \Phi(\xi_1) \circ \cdots \circ \Phi(\xi_n)(\mathcal{O}).
\]

Let \( K \) be the line segment of unit length with \( A = (0, 0) \) and \( B = (1, 0) \) as endpoints. We set, for each \( n \) in \( \mathbb{N} \), \( K^{(\xi)} = \Phi^{(\xi)}(K) \): \( K^{(\xi)} \) is the so-called \( n \)-th prefractal curve.

The fractal \( K^{(\xi)} \) associated with the environment sequence \( \xi \) is defined by
\[ K^{(\xi)} = \bigcup_{n=1}^{+\infty} \Phi_n^{(\xi)}(\Gamma) \]

where \( \Gamma = \{A, B\} \) with \( A = (0,0) \) and \( B = (1,0) \). We remark that these fractals do not have any exact self-similarity, i.e. there is no scaling factor which leaves the set invariant; the family \( \{ K^{(\xi)}, \xi \in \Xi \} \), however, satisfies the following relation
\[ K^{(\xi)} = \Phi^{(\xi)}(K^{(S\xi)}). \quad (8.3) \]

Moreover, the spatial symmetry is preserved and the set \( K^{(\xi)} \) is locally spatially homogeneous, i.e. the volume measure \( \mu^{(\xi)} \) on \( K^{(\xi)} \) satisfies the following locally spatially homogeneous condition \( (8.4) \). Before describing this measure, we introduce some notations. For \( \xi \in \Xi \), we set \( i|n = (i_1, \ldots, i_n) \) and \( \psi_i|n = \psi_{i_1}^{(\xi_1)} \circ \cdots \circ \psi_{i_n}^{(\xi_n)} \) and, for any set \( O \subset \mathbb{R}^2 \), \( \psi_i|n(O) = O_{i|n} \).

The volume measure \( \mu^{(\xi)} \) is the only Radon measure on \( K^{(\xi)} \) such that
\[ \mu^{(\xi)}(\psi_{i|n}(K^{(S\xi)})) = \frac{1}{4^n} \quad (8.4) \]
(see Section 2 in [4]) as, for each \( a \in \mathcal{A} \), the family \( \Psi^{(a)} \) has 4 contractive similitudes.

### 8.2. Trace and Extension Theorems.

We collect some trace and extension results used in the previous sections. We recall that, for \( v \) in \( L^1_{\text{loc}}(D) \), where \( D \) is an arbitrary open set of \( \mathbb{R}^2 \), the trace operator \( \gamma_0 \) is defined as
\[ \gamma_0 v(P) := \lim_{r \to 0} \frac{1}{m_2(B(P,r) \cap D)} \int_{B(P,r) \cap D} v(x,y) \, dx \, dy \quad (8.5) \]
\( (m_2 \) denotes the 2-dimensional Lebesgue measure) at every point \( P \in \overline{D} \) where the limit exists (see, for example, page 15 in [22]). We suppress \( \gamma_0 \) in the notation, when it does not give rise to misunderstanding, by writing simply \( v \) instead of \( \gamma_0 v \).

The following theorem characterizes the trace to the set \( \partial \Omega^{(\xi)}_{n} \) of a function belonging to Sobolev spaces \( H^\alpha(\mathbb{R}^2) \) (for the definitions and the main properties of Sobolev spaces, see [2]). We recall that
\[ \sigma_n = \frac{\ell^{(\xi)}(n)}{4^n} \quad \text{with} \quad \ell^{(\xi)}(n) = \prod_{i=1}^{n} \ell_{\xi_i} \quad (8.6) \]

**Theorem 8.1** Let \( u \in H^\alpha(\mathbb{R}^2) \) and \( \sigma_n \) as in (8.6). Then, for \( \frac{1}{2} < \alpha \leq 1 \),
\[ ||u||_{L^2(\partial \Omega^{(\xi)}_{n})} \leq \frac{C_\alpha}{\sigma_n} ||u||_{H^\alpha(\mathbb{R}^2)}, \quad (8.7) \]
where \( C_\alpha \) is independent of \( n \).
For the proof, see Theorem 3.5 in [14].

The following theorem that characterizes the trace to the set $\partial \Omega^{(\xi)}$ of a function belonging Sobolev spaces $H^\alpha(\mathbb{R}^2)$ is a consequence of Theorem 3.9 in [14] (see also Chapter V in [22]). We remark that the fractal is not necessarily a d-set.

**Theorem 8.2** Let $u \in H^\alpha(\mathbb{R}^2)$. Then, for $\frac{1}{2} < \alpha \leq 1$,

$$||u||_{L^2(\partial \Omega^{(\xi)})}^2 \leq C^*_\alpha ||u||_{H^\alpha(\mathbb{R}^2)}^2.$$  \hspace{1cm} (8.8)

Following Theorem 8.3 provides an extension operator from $H^1(\Omega^{(\xi)},n)$ to the space $H^1(\mathbb{R}^2)$ whose norm is independent of the (increasing) number of sides. This result is a consequence of the extension theorem due to Jones (Theorem 1 in [21]) as the domains $\Omega^{(\xi),n}$ are $(\varepsilon,\infty)$ domains with $\varepsilon$ independent of $n$ (for the proof, see Lemma 3.3 in [12]).

**Theorem 8.3** There exists a bounded linear extension operator $Ext_J : H^1(\Omega^{(\xi)},n) \rightarrow H^1(\mathbb{R}^2)$, such that

$$||Ext_J v||_{H^1(\mathbb{R}^2)}^2 \leq C_J ||v||_{H^1(\Omega^{(\xi)},n)}^2$$ \hspace{1cm} (8.9)

with $C_J$ independent of $n$.

We conclude this Section with an extension theorem for fractional Sobolev spaces $H^\alpha(\Omega^{(\xi)})$ with $\frac{1}{2} < \alpha \leq 1$, (see Theorem 5.8 in [12] and Theorem 8 in [36]).

**Theorem 8.4** There exists a linear extension operator $Ext_R$ such that, for any $\alpha \in (\frac{1}{2},1]$, $Ext_R : H^\alpha(\Omega^{(\xi)}) \rightarrow H^\alpha(\mathbb{R}^2)$,

$$||Ext_R v||_{H^\alpha(\mathbb{R}^2)}^2 \leq C_R ||v||_{H^\alpha(\Omega^{(\xi)})}^2$$ \hspace{1cm} (8.10)

with $C_R$ depending on $\alpha$.

**Remark.** We note that Theorem 8 in [36] is an improvement of the extension theorem by Jones as it provides a degree-independent Sobolev extension operator for $(\varepsilon,\delta)$ domains acting on all ordinary Sobolev spaces. We observe, finally, that if $\frac{1}{2} < \alpha < 1$, estimate (8.10) can be also deduced from Theorem 1 on page 103 in [22] (see also Theorem 3 on page 155 in [22]) since the domain $\Omega^{(\xi)}$ can be seen as a 2-set.
References


