Approximate controllability for linear degenerate parabolic problems with bilinear control

Piermarco Cannarsa,
Dipartimento di Matematica ,
Università di Roma "Tor Vergata",
I-00161 Roma, Italy

Giuseppe Floridia,
Dipartimento di Matematica e Informatica ,
Università di Catania,
I-95125 Catania, Italy

Abstract

In this work we study the global approximate multiplicative controllability for the linear degenerate parabolic Cauchy-Neumann problem

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in} \quad Q_T = (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm 1} = 0 & t \in (0,T) \\ v(0,x) = v_0(x) & x \in (-1,1)\ , \end{cases}$$

with the bilinear control $\alpha(t,x) \in L^\infty(Q_T)$. The problem is strongly degenerate in the sense that $a \in C^1([-1,1])$, positive on $(-1,1)$, is allowed to vanish at ± 1 provided that a certain integrability condition is fulfilled. We will show that the above system can be steered in $L^2(\Omega)$ from any nonzero, nonnegative initial state into any neighborhood of any desirable nonnegative target-state by bilinear static controls. Moreover, we extend the above result relaxing the sign constraint on v_0.

Key words: approximate controllability, degenerate parabolic equations, bilinear control

AMS subject classifications: 35K65, 93B05, 34B24

1 Introduction

Motivation

Climate depends on various parameters such as temperature, humidity, wind intensity, the effect of greenhouse gases, and so on. It is also affected by a complex set of interactions in the atmosphere, oceans and continents, that involve physical, chemical, geological and biological processes.

One of the first attempts to model the effects of interaction between large ice masses and solar radiation on climate is the one due, independently, by Budyko

[5, 6] and Sellers [25] (see also [12, 13, 17] and the references therein). Such a model studies how extensive the climate response is to an event such as a sharp increase in greenhouse gases; in this case we talk about climate sensitivity. A process that changes climate sensitivity is called *feedback*. If the process increases the intensity of response we say that it has *positive feedback*, whereas it has *negative feedback* if it reduces the intensity of response.

The Budyko-Sellers model studies the role played by continental and oceanic areas of ice on climate change. In such a model, the sea level mean zonally averaged temperature $u(t, x)$ on the Earth, where t denotes time and x the sine of latitude, satisfies the following degenerate *Cauchy-Neumann* problem (1.1) in the bounded domain $(-1, 1)$.

The effect of solar radiation on climate can be summarized in the following figure

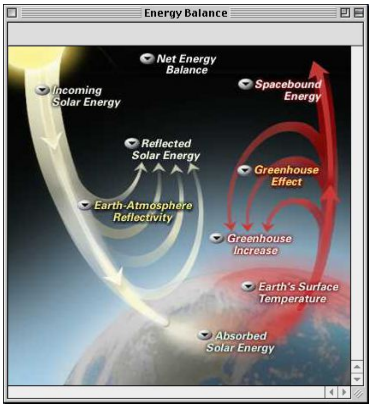

Figure 1: www.edu-design-principles.org (copyrighted by DPD)

We have the following *energy balance* :

Heat variation $= R_a - R_e + D$

- R_a = absorbed energy
- R_e = emitted energy
- D = diffusion

The general formulation of the Budyko-Sellers model on a compact surface $\mathcal{M}$ without boundary is as follows

$$u_t - \Delta_{\mathcal{M}} u = R_a(t, x, u) - R_e(t, x, u)$$

where $u(t, x)$ is the distribution of temperature and $\Delta_{\mathcal{M}}$ is the classical Laplace-Beltrami operator. Moreover,

- $R_a(t, x, u) = Q(t, x)\beta(x, u)$
- $R_e(t, x, u) = A(t, x) + B(t, x)u$

In the above, Q is the *insolation* function, and β is the *coalbedo* function (that is, 1-*albedo* function).
Albedo is the reflecting power of a surface. It is defined as the ratio of reflected radiation from the surface to incident radiation upon it. It may also be expressed as a percentage, and is measured on a scale from zero for no reflecting power of a perfectly black surface, to 1 for perfect reflection of a white surface.

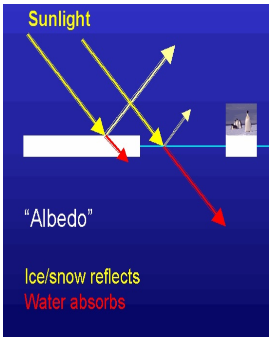

Figure 2: www.esr.org (copyrighted by ESR)

The main difference between Budyko's model and the one by Sellers, is that in the former the coalbedo function is discontinuous, while in the latter it is a continuous function. In fact we have

- Budyko

$$\beta(u) = \begin{cases} \beta_0 & u < -10 \\ [\beta_0, \beta_1] & u = -10 \\ \beta_1 & u > -10 \ , \end{cases}$$

- Sellers

$$\beta(u) = \begin{cases} \beta_0 & u < u_- \\ \text{line} & u_- \leq u \leq u_+ \\ \beta_1 & u > u_+ \ , \end{cases}$$

where $u_\pm = -10 \pm \delta, \delta > 0.$

On $\mathcal{M} = \Sigma^2$ the Laplace-Beltrami operator is

$$\Delta_{\mathcal{M}} = \frac{1}{\sin\phi}\left\{\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin\phi}\frac{\partial^2 u}{\partial\lambda^2}\right\}$$

where ϕ is the *colatitude* and λ is the *longitude*.

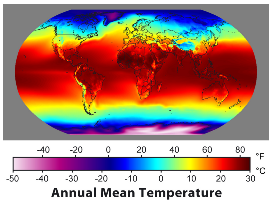

Figure 3: www.globalwarmingart.com (copyrighted by Global Warming Art)

In the one-dimensional Budyko-Sellers we take the average of the temperature at $x = \cos\phi$ and the Budyko-Sellers model reduces to

$$\begin{cases} u_t - \left((1-x^2)u_x\right)_x = g(t,x)\,h(x,u) + f(t,x), & x \in (-1,1) \\ (1-x^2)u_{x|_{x=\pm 1}} = 0 \ . \end{cases} \tag{1.1}$$

Problem formulation

Let us consider the following *Cauchy-Neumann* strongly degenerate boundary linear problem in divergence form, governed in the bounded domain $(-1,1)$ by means of the *bilinear control* $\alpha(t,x)$

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \text{in} \qquad Q_T \;=\; (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm 1} = 0 & t\in(0,T) \\ v(0,x) = v_0(x) & x\in(-1,1)\;. \end{cases} \tag{1.2}$$

We assume that

1. $v_0 \in L^2(-1,1)$
2. $\alpha \in L^\infty(Q_T)$
3. $a \in C^1([-1,1])$ satisfies
   (a) $a(x) > 0 \;\forall\, x\in(-1,1), \quad a(-1) = a(1) = 0$
   (b) $A \in L^1(-1,1)$, where $A(x) = \int_0^x \frac{ds}{a(s)}$.

Remark We observe that

1. $\frac{1}{a} \notin L^1(-1,1)$, so $a(x)$ is strongly degenerate
2. the principal part of the operator in (1.2) coincides with that of the Budyko-Sellers model for $a(x) = 1 - x^2$. In this case $A(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \in L^1(-1,1)$
3. a sufficient condition for 3.b) is that $a'(\pm 1) \neq 0$ (if $a \in C^2([-1,1])$ the above condition is also necessary).

□

We are interested in studying the multiplicative controllability of problem (1.2) by the *bilinear control* $\alpha(t,x)$. In particular, for the above linear problem, we will discuss results guaranteeing global nonnegative approximate controllability in large time (for multiplicative controllability see [20, 23, 8]).
Now we recall one definition from control theory.

Definition 1.2

We say that the system (1.2) is nonnegatively globally approximately controllable in $L^2(-1,1)$, if for every $\varepsilon > 0$ and for every nonnegative $v_0(x)$, $v_d(x) \in L^2(-1,1)$ with $v_0 \not\equiv 0$ there are a $T = T(\varepsilon, v_0, v_d)$ and a bilinear control $\alpha(t,x) \in L^\infty(Q_T)$ such that for the corresponding solution $v(t,x)$ of (1.2) we obtain

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \le \varepsilon\,.$$

In the following we will sometimes use $\|\cdot\|$ instead of $\|\cdot\|_{L^2(-1,1)}$.

Main results

In this work at first the *nonnegative global approximate controllability* result is obtained for the linear system (1.2) in the following theorem.

Theorem 1.3

The linear system (1.2) is nonnegatively approximately controllable in $L^2(-1,1)$ by means of static controls in $L^\infty(-1,1)$. Moreover, the corresponding solution to (1.2) remains nonnegative at all times.

Then the results present in Theorem 1.3 can be extended to a larger class of initial states.

Theorem 1.4

For any $v_d \in L^2(-1,1), v_d \geq 0$ and any $v_0 \in L^2(-1,1)$ such that

$$\int_{-1}^{1} v_0 v_d dx > 0, \tag{1.3}$$

for every $\varepsilon > 0$, there are $T = T(\varepsilon, v_0, v_d) \geq 0$ and a static bilinear control, $\alpha = \alpha(x)$, $\alpha \in L^\infty(-1,1)$ such that

$$\|v(T,\cdot) - v_d\|_{L^2(-1,1)} \leq \varepsilon\,.$$

Remark The solution $v(t,x)$ of the problem (1.2) in the assumptions of Theorem 1.4 does not remain nonnegative in Q_T, like in Theorem 1.3, but it can also assume negative values. $\square$

Mathematical motivation

This note is inspired by [20, 8]. In [20] A.Y. Khapalov studied the global nonnegative approximate controllability of the one dimensional *non-degenerate* semilinear convection-diffusion-reaction equation governed in a bounded domain via the bilinear control $\alpha \in L^\infty(Q_T)$. In [8], the same approximate controllability property is derived in suitable classes of functions that change sign.
In this note we extend some of the results of [20] to degenerate linear equations.

General references for *multiplicative controllability* are, e.g., [18, 19, 21, 22, 23, 3]. In control theory, boundary and interior locally distributed controls are usually employed (see, e.g., [9, 10, 11, 14, 15, 16]). These controls are additive terms in the equation and have localized support. However, such models are unfit to study several interesting applied problems such as chemical reactions controlled by catalysts, and also smart materials, which are able to change their principal parameters under certain conditions. This explains the growing interest in *multiplicative controllability*.

2 Preliminaries

Positive and negative part

Given $\Omega \subseteq \mathbb{R}^n$, $v : \Omega \longrightarrow \mathbb{R}$ we consider the positive-part function

$$v^+(x) = \max(v(x), 0), \qquad \forall x \in \Omega,$$

and the negative-part function

$$v^-(x) = \max(0, -v(x)), \qquad \forall x \in \Omega.$$

Then we have the following equality

$$v = v^+ - v^- \qquad \text{in } \Omega$$

For the functions v^+ and v^- the following result of regularity in Sobolev's spaces will be useful (see [24], Appendix A).

Theorem 2.1

Let $\Omega \subset \mathbb{R}^n$, $u : \Omega \longrightarrow \mathbb{R}$, $u \in H^{1,s}(\Omega)$, $1 \le s \le \infty$. Then

$$u^+,\, u^- \in H^{1,s}(\Omega)$$

and for $1 \le i \le n$

$$(u^+)_{x_i} = \begin{cases} u_{x_i} & \text{in } \{x \in \Omega : u(x) > 0\} \\ 0 & \text{in } \{x \in \Omega : u(x) \le 0\}\,, \end{cases} \tag{2.4}$$

and

$$(u^-)_{x_i} = \begin{cases} -u_{x_i} & \text{in } \{x \in \Omega : u(x) < 0\} \\ 0 & \text{in } \{x \in \Omega : u(x) \ge 0\}\,. \end{cases} \tag{2.5}$$

Gronwall's Lemma

Lemma 2.2 *Gronwall's inequality (differential form).*

Let $\eta(t)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. $t \in [0,T]$ the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \tag{2.6}$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$.
Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds\right] \tag{2.7}$$

for all $0 \leq t \leq T$.
In particular, if $\psi(t) \equiv 0$ in (2.6), i.e. $\eta' \leq \phi\,\eta$ for a.e. $t \in [0,T]$, and $\eta(0) = 0$, then

$$\eta \equiv 0 \qquad \textit{in } [0,T].$$

Well-posedness in weighted Sobolev spaces

In order to deal with the well-posedness of problem (1.2), it is necessary to introduce the following Sobolev weighted spaces

$$H^1_a(-1,1) :=$$
$$:= \{u \in L^2(-1,1) : u \text{ locally absolutely continuous in } (-1,1), \sqrt{a}u_x \in L^2(-1,1)\}$$

and

$$H^2_a(-1,1) := \{u \in H^1_a(-1,1) |\, au_x \in H^1(-1,1)\} =$$
$$= \{u \in L^2(-1,1) | u \text{ locally absolutely continuous in } (-1,1),$$
$$au \in H^1_0(-1,1),\ au_x \in H^1(-1,1) \text{ and } (a\,u_x)(\pm 1) = 0\}$$

respectively with the following norms

$$\|u\|^2_{H^1_a} := \|u\|^2_{L^2(-1,1)} + |u|^2_{1,a} \text{ and } \|u\|^2_{H^2_a} := \|u\|^2_{H^1_a} + \|(au_x)\|^2_{L^2(0,1)};$$

where $|u|_{1,a} = \|\sqrt{a}u_x\|_{L^2(-1,1)}$ is a seminorm.

In this note we obtain the following result.

Lemma 2.3

$$H^1_a(-1,1) \hookrightarrow L^2(-1,1) \qquad \textit{with compact embedding.} \tag{2.8}$$

Proof:
Given $u \in H^1_a(-1,1)$, let

$$\bar{u}(x) = \begin{cases} u & \text{if } x \in [-1,1] \\ 0 & \text{elsewere} \quad . \end{cases}$$

It is sufficient to prove that, for every $R > 0$,

$$\sup_{\|u\|_{1,a} \le R} \int_{\mathbb{R}} |\bar{u}(x+h) - \bar{u}(x)|^2\, dx \longrightarrow 0, \qquad \text{as } h \to 0 \tag{2.9}$$

Let $h > 0$([1]) and let $u \in H^1_a(-1,1)$ be such that $\|u\|_{1,a} \le R$, we have the following equality

$$\int_{\mathbb{R}} |\bar{u}(x+h) - \bar{u}(x)|^2\, dx =$$

$$= \int_{-1-h}^{-1} |u(x+h)|^2\, dx + \int_{-1}^{1-h} |u(x+h) - u(x)|^2\, dx + \int_{1-h}^{1} |u(x)|^2\, dx =$$

$$= \int_{-1}^{-1+h} |u(x)|^2\, dx + \int_{-1}^{1-h} |u(x+h) - u(x)|^2\, dx + \int_{1-h}^{1} |u(x)|^2\, dx$$

First, let us prove that

$$\sup_{\|u\|_{1,a} \le R} \int_{-1}^{1-h} |u(x+h) - u(x)|^2\, dx \longrightarrow 0, \qquad \text{as } h \to 0^+. \tag{2.10}$$

Recalling that $A(x) = \int_0^x \frac{ds}{a(s)}$, we have

$$|u(x+h) - u(x)| \le \int_x^{x+h} \sqrt{a(s)}|u'(s)| \frac{1}{\sqrt{a(s)}}\, ds \le$$

$$\le \left(\int_{-1}^{1} a(s)|u'(s)|^2\, ds \right)^{\frac{1}{2}} \left(\int_x^{x+h} \frac{ds}{a(s)} \right)^{\frac{1}{2}} = |u|_{1,a} \left[A(x+h) - A(x) \right]^{\frac{1}{2}} .$$

By integrating on $[-1, 1-h]$, since $A \in L^1(-1,1)$ (by assumption 3.b)), we obtain

$$\int_{-1}^{1-h} |u(x+h) - u(x)|^2\, dx \le |u|^2_{1,a} \int_{-1}^{1-h} \left(A(x+h) - A(x) \right)\, dx \le$$

$$\le R^2 \left[\int_{-1+h}^{1} A(x)\, dx - \int_{-1}^{1-h} A(x)\, dx \right] =$$

[1] In the case $h < 0$ we proceed similarly.

$$= R^2 \left[\int_{1-h}^{1} A(x)\, dx \; - \; \int_{-1}^{-1+h} A(x)\, dx \right] \longrightarrow 0, \qquad \text{as } h \to 0^+.$$

Now, let us prove that

$$\sup_{\|u\|_{1,a} \le R} \int_{1-h}^{1} |u(x)|^2\, dx \longrightarrow 0, \qquad\qquad \text{as } h \to 0^+. \tag{2.11}$$

We have

$$|u(0)| \le |u(x)| + \int_0^x \sqrt{a(s)}|u'(s)| \frac{1}{\sqrt{a(s)}}\, ds \le$$

$$\le |u(x)| + \left(\int_{-1}^{1} a(s)|u'(s)|^2\, ds \right)^{\frac{1}{2}} \left(\int_0^x \frac{ds}{a(s)} \right)^{\frac{1}{2}} \le |u(x)| + |u|_{1,a} \sqrt{A(x)}\,.$$

By integrating on $[0,1]$, we obtain

$$|u(0)| \le \int_0^1 |u(x)|\, dx + |u|_{1,a} \int_0^1 \sqrt{A(x)}\, dx \le$$

$$\le \|u\|_{L^2(-1,1)} + |u|_{1,a} \int_0^1 \sqrt{A(x)}\, dx \le C \|u\|_{1,a}\,.$$

Then,

$$|u(0)| \le C\, R\,. \tag{2.12}$$

Now, it follows that

$$|u(x)|^2 \le 2|u(0)|^2 + 2A(x)|u|_{1,a}^2 \le C\, R^2 + 2A(x) R^2\,.$$

Finally, since $A \in L^1(-1,1)$, by integrating on $[1-h, 1]$ we obtain

$$\int_{1-h}^{1} |u(x)|^2\, dx \le C\, hR^2 + 2R^2 \int_{1-h}^{1} A(x)\, dx \longrightarrow 0, \qquad\qquad \text{as } h \to 0^+.$$

Similarly, we can prove that

$$\sup_{\|u\|_{1,a} \le R} \int_{-1}^{-1+h} |u(x)|^2\, dx \longrightarrow 0, \qquad\qquad \text{as } h \to 0^+. \tag{2.13}$$

By (2.10), (2.11) and (2.13) we obtain (2.9). □

We now recall the existence and uniqueness result for system (1.2) obtained in [7] (see also [1]). Let us consider, first, the operator $(A_0, D(A_0))$ defined by

$$\begin{cases} D(A_0) = H^2_a(-1,1) \\ \\ A_0 u = (a u_x)_x\,, \;\; \forall\, u \in D(A_0)\,. \end{cases} \tag{2.14}$$

Observe that A_0 is a closed, self-adjoint, dissipative operator with dense domain in $L^2(-1,1)$. Therefore, A_0 is the infinitesimal generator of a C_0 − semigroup of contractions in $L^2(-1,1)$.

Next, given $\alpha \in L^\infty(-1,1)$, let us introduce the operator

$$\begin{cases} D(A) = D(A_0) \\ \\ A = A_0 + \alpha I \ . \end{cases} \tag{2.15}$$

For such an operator we have the following proposition.

Proposition 2.4

- *$D(A)$ is compactly embedded and dense in $L^2(-1,1)$.*
- *$A : D(A) \longrightarrow L^2(-1,1)$ is the infinitesimal generator of a strongly continuous semigroup, e^{tA}, of bounded linear operator on $L^2(-1,1)$.*

Observe that problem (1.2) can be recast in the Hilbert space $L^2(-1,1)$ as

$$\begin{cases} u'(t) = A\,u(t)\,, \qquad t > 0 \\ \\ u(0) = u_0 \ . \end{cases} \tag{2.16}$$

where A is the operator in (2.15).

We recall that a *weak solution* of (2.16) is a function $u \in C^0([0,T]; L^2(-1,1))$ such that for every $v \in D(A^*)$ the function $\langle u(t), v\rangle$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^* v\rangle\,,$$

for almost $t \in [0,T]$ (see [2]).

Theorem 2.5

For every $\alpha \in L^\infty((0,T)\times(-1,1))$ and every $u_0 \in L^2(-1,1)$, there exists a unique

$$u \in C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1))$$

weak solution to (1.2), which coincides with $e^{tA}u_0$.

In the space

$$\mathcal{B}(0,T) = C^0([0,T]; L^2(-1,1)) \cap L^2(0,T; H^1_a(-1,1))$$

let us define the following norm

$$\|u\|^2_{\mathcal{B}(0,T)} = \sup_{t\in[0,T]} \|u(t,\cdot)\|^2_{L^2(-1,1)} + 2\int_0^T \int_{-1}^1 a(x)u_x^2 dx\,,\ \forall u \in \mathcal{B}(0,T)\,. \quad (2.17)$$

3 Some auxiliary lemmas and the proofs of main results

Let $A = A_0 + \alpha I$, where the operator A_0 is defined in (2.14) and $\alpha \in L^\infty(-1,1)$. Since A is self-adjoint and $D(A) \hookrightarrow L^2(-1,1)$ is compact (see Proposition 2.4), we have the following (see also [4]).

Lemma 3.1

There exists a increasing sequence $\{\lambda_k\}_{k\in\mathbb{N}}$, $\lambda_k \longrightarrow +\infty$ as $k \to \infty$, such that the eigenvalues of A are given by $\{-\lambda_k\}_{k\in\mathbb{N}}$, and the corresponding eigenfunctions $\{\omega_k\}_{k\in\mathbb{N}}$ form a complete orthonormal system in $L^2(-1,1)$.

In this note we obtain the following result

Lemma 3.2

Let $v \in C^\infty([-1,1]), v > 0$ on $[-1,1]$, let $\alpha_*(x) = -\frac{(a(x)v_x(x))_x}{v(x)}$, $x \in (-1,1)$. Let A be the operator defined in (2.15) with $\alpha = \alpha_*$

$$\begin{cases} D(A) = H^2_a(-1,1) \\ \\ A = A_0 + \alpha_* I\ , \end{cases} \quad (3.18)$$

and let $\{\lambda_k\}, \{\omega_k\}$ be the eigenvalues and eigenfunctions of A, respectively, given by Lemma 3.1. Then

$$\lambda_1 = 0 \quad \text{and} \quad |\omega_1| = \frac{v}{\|v\|}.$$

Moreover, $\frac{v}{\|v\|}$ and $-\frac{v}{\|v\|}$ are the only eigenfunctions of A with norm 1 that do not change sign in $(-1,1)$.

Remark Problem (3.18) is equivalent to the following differential problem

$$\begin{cases} (a(x)\omega_x)_x + \alpha_*(x)\omega + \lambda\,\omega = 0 \qquad \text{in} \qquad (-1,1) \\ \\ a(x)\omega_x(x)|_{x=\pm 1} = 0 \quad . \end{cases} \quad (3.19)$$

□

Proof: (of Lemma 3.2)
STEP.1 We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad \text{and} \qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions of the operator (3.18) (see Lemma 3.1). Therefore,

$$\langle \omega_k, \omega_h \rangle_{L^2(-1,1)} = \int_{-1}^{1} \omega_k(x)\omega_h(x)dx = 0, \qquad \text{if } h \neq k\,. \tag{3.20}$$

We can see, by easy calculations, that an eigenfunction of the operator defined in (3.18) is the function

$$\frac{v(x)}{\|v\|}\,,$$

associated with the eigenvalue $\lambda = 0$. Taking into account the above and considering that $v(x) > 0$, $\forall x \in (-1,1)$

$$\exists\, k_* \in \mathbb{N}\ : \omega_{k_*}(x) = \frac{v(x)}{\|v\|} > 0 \ \text{ or } \ \omega_{k_*}(x) = -\frac{v(x)}{\|v\|} < 0,\ \forall x \in (-1,1)\,. \tag{3.21}$$

Writing (3.20) with $k = k_*$ we obtain

$$\langle \omega_{k_*}, \omega_h \rangle_{L^2(-1,1)} = \int_{-1}^{1} \omega_{k_*}(x)\omega_h(x)dx = 0, \qquad \forall h \neq k_*\,. \tag{3.22}$$

Therefore, considering (3.22) and keeping in mind that $\omega_{k_*} > 0$ or $\omega_{k_*} < 0$ in $(-1,1)$, we observe that ω_{k_*} is the only eigenfunction of the operator defined in (3.18) that doesn't change sign in $(-1,1)$.

STEP.2 Let us now prove that

$$k_* = 1\,, \tag{3.23}$$

that is, $\lambda_1 = 0$. By a well-known variational characterization of the first eigenvalue, we have

$$\lambda_1 = \inf_{u\in H^1_a(-1,1)} \frac{\int_{-1}^{1}\left(a\,u_x^2\ - \alpha_*\,u^2\right)\,dx}{\int_{-1}^{1} u^2\,dx}\,.$$

By Lemma 3.1, since $\lambda_{k_*} = 0$, it is sufficient to prove that $\lambda_1 \geq 0$, or

$$\int_{-1}^{1} \alpha_*\,u^2\,dx \leq \int_{-1}^{1} a\,u_x^2\,dx, \qquad \forall\, u \in H^1_a(-1,1) \tag{3.24}$$

Integrating by parts, we have

$$\int_{-1}^{1} \alpha_*\,u^2\,dx = -\int_{-1}^{1} \frac{(a\,v_x)_x}{v}\,u^2\,dx = \int_{-1}^{1} a\,v_x\left(\frac{u^2}{v}\right)_x dx =$$

$$= \int_{-1}^{1} a\, v_x \frac{2uu_x}{v}\, dx - \int_{-1}^{1} a\, v_x^2 \left(\frac{u^2}{v^2}\right)\, dx =$$

$$= 2\int_{-1}^{1} \sqrt{a}\, \frac{v_x}{v} u \sqrt{a} u_x\ dx - \int_{-1}^{1} a\, v_x^2 \left(\frac{u^2}{v^2}\right)\ dx \leq$$

$$\leq \int_{-1}^{1} a\ \left(\frac{v_x u}{v}\right)^2 dx + \int_{-1}^{1} au_x^2\, dx - \int_{-1}^{1} a\, v_x^2 \left(\frac{u^2}{v^2}\right)\, dx = \int_{-1}^{1} au_x^2\ dx\,,$$

from which (3.24). □

For the proof of Theorem 1.3 the following Lemma is necessary.

Lemma 3.4

Let $T > 0$, $\alpha \in L^\infty(Q_T)$, let $v_0 \in L^2(-1,1)$, $v_0(x) \geq 0$ a.e. $x \in (-1,1)$ and let $v \in \mathcal{B}(0,T)$ be the solution to the linear system

$$\begin{cases} v_t - (a(x)v_x)_x = \alpha(t,x)v & \textit{in} \quad Q_T = (0,T)\times(-1,1) \\ a(x)v_x(t,x)|_{x=\pm1} = 0 & t \in (0,T) \\ v(0,x) = v_0(x) & x \in (-1,1)\ . \end{cases}$$

Then

$$v(t,x) \geq 0, \quad \forall (t,x) \in Q_T\,.$$

Proof: Let $v \in \mathcal{B}(0,T)$ be the solution to the system (1.2), and we consider the positive-part and the negative-part. It is sufficient to prove that

$$v^-(t,x) \equiv 0 \qquad \text{in } Q_T\,.$$

Multiplying both members equation of the problem (1.2) by v^- and integrating it on $(-1,1)$ we obtain

$$\int_{-1}^{1} \left[v_t v^- - (a(x)v_x)_x v^- - \alpha v v^-\right] dx = 0. \tag{3.25}$$

Recalling the definition v^+ and v^-, we obtain

$$\int_{-1}^{1} v_t v^- dx = \int_{-1}^{1} (v^+ - v^-)_t v^- dx = -\int_{-1}^{1} (v^-)_t v^- dx = -\frac{1}{2}\frac{d}{dt}\int (v^-)^2 dx\,.$$

Integrating by parts and applying Theorem 2.1, we obtain $v^- \in H^1_a(-1,1)$ and the following equality

$$\int_{-1}^{1} (a(x)v_x)_x v^-\, dx = [a(x)v_x v^-]_{-1}^{1} - \int_{-1}^{1} a(x)v_x(-v)_x\, dx = \int_{-1}^{1} a(x)v_x^2\, dx\,.$$

We also have

$$\int_{-1}^{1} \alpha v v^- dx = -\int_{-1}^{1} \alpha (v^-)^2 dx$$

and therefore (3.25) becomes

$$-\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(v^-)^2 dx + \int_{-1}^{1} \alpha (v^-)^2 dx = \int_{-1}^{1} a(x) v_x^2 \geq 0,$$

from which

$$\frac{d}{dt}\int_{-1}^{1}(v^-)^2 dx \leq 2\int_{-1}^{1} \alpha (v^-)^2 dx \leq 2\|\alpha\|_\infty \int_{-1}^{1}(v^-)^2 dx.$$

From the above inequality, applying Gronwall's lemma we obtain

$$\int_{-1}^{1}(v^-(t,x))^2 dx \leq e^{2t\|\alpha\|_\infty} \int_{-1}^{1}(v^-(0,x))^2 dx\,.$$

Since

$$v(0,x) = v_0(x) \geq 0\,,$$

we have

$$v^-(0,x) = 0.$$

Therefore,

$$v^-(t,x) = 0, \qquad\qquad \forall (t,x) \in Q_T.$$

From this, as we mentioned initially, it follows that

$$v(t,x) = v^+(t,x) \geq 0 \qquad \forall (t,x) \in Q_T.$$

□

We are now ready to prove our main result.

Proof: (of Theorem 1.3)

STEP.1 To prove Theorem 1.3 it is sufficient to consider the set of target states

$$v_d \in C^\infty([-1,1]), \qquad v_d > 0 \text{ on } [-1,1]. \tag{3.26}$$

Indeed, regularizing by convolution, every function $v_d \in L^2(-1,1), v_d \geq 0$ can be approximated by a sequence of strictly positive $C^\infty([-1,1])-$ functions.

STEP.2 Taking any nonzero, nonnegative initial state $v_0 \in L^2(-1,1)$ and any target state v_d as described in (3.26) in STEP.1, let us set

$$\alpha_*(x) = -\frac{(a(x)v_{dx}(x))_x}{v_d(x)}, \qquad x \in (-1,1). \tag{3.27}$$

Then, by (3.26),

$$\alpha_*(x) \in L^\infty(-1,1)\,.$$

We denote by

$$\{-\lambda_k\}_{k\in\mathbb{N}} \qquad\qquad \text{and} \qquad\qquad \{\omega_k\}_{k\in\mathbb{N}},$$

respectively, the eigenvalues and orthonormal eigenfunctions[2] of the spectral problem $A\omega + \lambda\omega = 0$, with $A = A_0 + \alpha_* I$ (see Lemma 3.1).

We can see, by Lemma 3.2, that

$$\lambda_1 = 0 \qquad \text{and} \qquad \omega_1(x) = \frac{v_d(x)}{\|v_d\|} > 0,\ \forall x \in (-1,1)\,. \tag{3.28}$$

<u>STEP.3</u> Let us now choose the following static bilinear control

$$\alpha(x) = \alpha_*(x) + \beta,\ \forall x \in (-1,1), \quad \text{with } \beta \in \mathbb{R}\ (\beta \text{ to be determined below}).$$

The corresponding solution of (1.2), for this particular bilinear coefficient α, has the following Fourier series representation ([3])

$$v(t,x) = \sum_{k=1}^{\infty} e^{(-\lambda_k+\beta)t}\left(\int_{-1}^{1} v_0(s)\omega_k(s)ds\right)\omega_k(x) =$$

$$= e^{\beta t}\left(\int_{-1}^{1} v_0(s)\omega_1(s)ds\right)\omega_1(x) + \sum_{k>1} e^{(-\lambda_k+\beta)t}\left(\int_{-1}^{1} v_0(s)\omega_k(s)ds\right)\omega_k(x)$$

Let

$$r(t,x) = \sum_{k>1} e^{(-\lambda_k+\beta)t}\left(\int_{-1}^{1} v_0(s)\omega_k(s)ds\right)\omega_k(x)$$

where, recalling that $\lambda_k < \lambda_{k+1}$, we obtain

$$-\lambda_k < -\lambda_1 = 0 \quad \text{ for ever } k \in \mathbb{N},\ k > 1\,.$$

Owing to (3.28),

$$\|v(t,\cdot) - v_d\| \le \left\| e^{\beta t}\left(\int_{-1}^{1} v_0(s)\omega_1(s)ds\right)\omega_1 - \|v_d\|\omega_1 \right\| + \|r(t,x)\| =$$

$$= \left| e^{\beta t}\left(\int_{-1}^{1} v_0(x)\omega_1(x)dx\right) - \|v_d\| \right| + \|r(t,x)\|$$

[2]As first eigenfunction we take the one which is positive in $(-1,1)$.

[3]Observe that adding $\beta \in \mathbb{R}$ in the coefficient α_* there is a shift of the eigenvalues corresponding to α_* from $\{-\lambda_k\}_{k\in\mathbb{N}}$ to $\{-\lambda_k + \beta\}_{k\in\mathbb{N}}$, but the eigenfunctions remain the same for α_* and $\alpha_* + \beta$.

Since $v_0 \in L^2(-1,1)$, $v_0 \geq 0$ and $v_0 \not\equiv 0$ in $(-1,1)$ and by (3.28), we obtain

$$\int_{-1}^{1} v_0(x)\omega_1(x)dx > 0. \tag{3.29}$$

Then, it is possible choose β and $T > 0$ so that

$$e^{\beta T}\int_{-1}^{1} v_0\omega_1 dx = \|v_d\|,$$

that is,

$$\beta = \frac{1}{T}\ln\left(\frac{\|v_d\|}{\int_{-1}^{1} v_0\omega_1 dx}\right). \tag{3.30}$$

Since $-\lambda_k < -\lambda_2$, $\forall k > 2$, applying Parseval's equality we have

$$\|r(t,x)\|^2 \leq e^{2(-\lambda_2+\beta)t}\sum_{k>1}\left|\int_{-1}^{1} v_0\omega_k ds\right|^2 \|\omega_k(x)\|^2 =$$

$$= e^{2(-\lambda_2+\beta)t}\sum_{k>1}\langle v_0, \omega_k\rangle^2 = e^{2(-\lambda_2+\beta)t}\|v_0\|^2.$$

So, by (3.30), the last inequality, and the above estimate for $\|v(T,\cdot) - v_d(\cdot)\|$ we conclude that

$$\|v(T,\cdot) - v_d(\cdot)\| \leq e^{(-\lambda_2+\beta)T}\|v_0\| = e^{-\lambda_2 T}\frac{\|v_d\|}{\int_{-1}^{1} v_0\omega_1 dx}\|v_0\| \stackrel{T\longrightarrow\infty}{\longrightarrow} 0\,.$$

From which we have the conclusion. $\square$

Proof: (of Theorem 1.4)
The proof of Theorem 1.3 can be adapted to Theorem 1.4, keeping in mind that, in STEP.3, inequality (3.29) continues to hold in this new setting. In fact we have

$$\int_{-1}^{1} v_0(x)\omega_1(x)dx = \int_{-1}^{1} v_0(x)\frac{v_d(x)}{\|v_d\|}dx =$$

$$= \frac{1}{\|v_d\|}\int_{-1}^{1} v_0 v_d dx > 0, \text{ by assumptions (1.3).}$$

From this point on, one can proceed as in the proof of Theorem 1.3. $\square$

Acknowledgments

This research has been performed in the framework of the GDRE CONEDP. The authors wish to thank Institut Henri Poincaré (Paris, France) for providing a very stimulating environment during the "Control of Partial and Differential Equations and Applications" program in the Fall 2010.

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