On 2+1-dimensional magneto-hydrodynamics with a constant divergence constraint

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Abstract

Non-steady, two-dimensional magnetohydrodynamics is investigated under a constant divergence geometric constraint. An appropriate geometric parametrisation is introduced and a novel class of 2+1-dimensional motions thereby isolated.

1 Introduction

The intrinsic nonlinearity of the governing equations of magneto-hydrodynamics is a major impediment to construction of exact solutions. Lie group techniques may be employed systematically to construct similarity solutions [1–3]. In [4], a novel method was adopted wherein the nonlinear acceleration term in the governing Lundquist equations was assumed to be conservative.

Here, we proceed with a geometric approach. This is motivated by previous work in which a vanishing divergence constraint has been imposed in steady, spatial gasdynamics and magneto-hydrostatics and shown to lead, remarkably, to the nonlinear Schrödinger (NLS) equation of modern soliton theory [5–9]. For the 2+1-dimensional magneto-hydrodynamic equations under present investigation, a constant divergence constraint is also shown to lead to an integrable connection, namely to the modified Korteweg-de Vries (mkdV) hierarchy. The procedure allows the construction of a new class of exact solutions to the nonlinear magnetohydrodynamic system.

2 The magnetohydrodynamic system

The governing equations of non-dissipative magnetohydrodynamics are:

$$\operatorname{div}\mathbf{q} = 0 \, , \tag{2.1}$$

$$\rho\,[\,\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q}\,] - \mu\,\mathrm{curl}\,\mathbf{H}\times\mathbf{H} + \nabla p = \mathbf{0}\ , \tag{2.2}$$

$$\mathrm{div}\,\mathbf{H} = 0\ , \tag{2.3}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{curl}\,(\,\mathbf{q}\times\mathbf{H}\,)\ , \tag{2.4}$$

where $\mathbf{H}, \mathbf{q}, p, \rho$ denote, in turn, the magnetic field, velocity, pressure and constant density, while μ is the magnetic permeability. In the two-dimensional reduction, the magnetic induction equation (2.3) shows that

$$\mathbf{H} = \nabla A \times \mathbf{k} \tag{2.5}$$

where A is the magnetic flux. The Faraday Law (2.4) now yields

$$(\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla)A = 0\ , \tag{2.6}$$

so that A is convected with the conducting fluid.

Here, a geometric parametrisation recently introduced in [11] in the context of the kinematics of fibre-reinforced fluids is applied to construct a new class of solutions of the magnetohydrodynamic system (2.1)-(2.4) for which

$$\nabla\times[\,\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q}\,] = \mathbf{0}\ . \tag{2.7}$$

The latter constraint was introduced in a recent study of 2+1-dimensional magnetohydrodynamics in [5] in which the conducting motions so derived were termed 'accelerated'. In [5], an elastic gas law $p = c_s^2\,\rho$ was adopted (where c_s is constant sound speed) whereas here, incompressibility is assumed.

In view of (2.7), a potential Π exists such that

$$\rho\,[\,\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q}\,] = -\nabla\Pi \tag{2.8}$$

whence, the magnetohydrodynamic momentum equation (2.2) becomes

$$\nabla(p-\Pi) + \mu(\nabla^2 A)\nabla A = \mathbf{0} \tag{2.9}$$

with compatibility condition

$$\frac{\partial(\nabla^2 A, A)}{\partial(x,y)} = 0\ , \tag{2.10}$$

so that

$$\nabla^2 A = \Phi(A)\ . \tag{2.11}$$

Accordingly, 2+1-dimensional solutions of the pseudo-hydrodynamic system

$$\mathrm{div}\,\mathbf{q} = 0\ , \tag{2.12}$$

$$\rho\,[\,\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q}\,] + \nabla\Pi = 0 \tag{2.13}$$

will be sought. Associated magnetohydrodynamic flows are then determined via magnetic potentials A such that the convection condition (2.6) and the nonlinear constraint (2.11) are simultaneously satisfied.

3 A parametrisation

Here, attention is restricted to the 2+1-dimensional version of the magnetohydrodynamic system (2.1)-(2.4). A geometric parametrisation is introduced in terms of a 'fibre' direction $\mathbf{t}$ subject to the constraint [11]

$$\frac{\partial \mathbf{t}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{t} = (\mathbf{t} \cdot \nabla)\mathbf{q} \ . \tag{3.1}$$

The kinematic conditions (2.12)-(2.13) which attend the motion of fibre-reinforced fluids have been investigated in [10–12]. In the following, we seek to construct a class of 'accelerated' motions of the 2+1-dimensional system (2.12)-(2.13) wherein

$$\mathbf{q} = v\mathbf{t} + w\mathbf{n} \tag{3.2}$$

and $\mathbf{t}$ is subject to the condition (3.1), while $\mathbf{n}$ is the principal unit normal to these 'fibre' lines.

The Serret-Frenet relations for the $\mathbf{t}$-lines and the $\mathbf{n}$-lines are, respectively

$$\frac{\delta}{\delta s}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} , \tag{3.3}$$

and

$$\frac{\delta}{\delta n}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} , \tag{3.4}$$

where $\delta/\delta s = \mathbf{t} \cdot \nabla$ and $\delta/\delta n = \mathbf{n} \cdot \nabla$ denote, in turn, the directional derivatives in the tangential and principal normal directions to the $\mathbf{t}$-lines. The quantities κ and $-\theta = -\operatorname{div} \mathbf{t}$ are the curvatures of the $\mathbf{t}$-lines and $\mathbf{n}$-lines respectively.

The compatibility of the pair (3.3), (3.4) requires that

$$\frac{\delta \kappa}{\delta n} - \frac{\delta \theta}{\delta s} = \kappa^2 + \theta^2 \ , \tag{3.5}$$

on use of the commutator relation [9]

$$\left[\frac{\delta}{\delta n}, \frac{\delta}{\delta s} \right] = \frac{\delta^2}{\delta n \delta s} - \frac{\delta^2}{\delta s \delta n} = \kappa \frac{\delta}{\delta s} + \theta \frac{\delta}{\delta n} \ . \tag{3.6}$$

The general solution of (3.5) may parametrised in terms of an angle φ where

$$\kappa = \frac{\delta \varphi}{\delta s}, \qquad \theta = \frac{\delta \varphi}{\delta n} \tag{3.7}$$

and

$$\mathbf{t} = \cos\varphi \ \mathbf{i} + \sin\varphi \ \mathbf{j}, \qquad \mathbf{n} = -\sin\varphi \ \mathbf{i} + \cos\varphi \ \mathbf{j} \ , \tag{3.8}$$

with $\mathbf{i}$ and $\mathbf{j}$ the usual unit vectors in the direction of the Cartesian x- and y-axes. The parametrisation (3.8) implies that

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \bar{\mu} \\ -\bar{\mu} & 0 \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix} , \tag{3.9}$$

where

$$\bar{\mu} = \frac{\partial \varphi}{\partial t} \ . \tag{3.10}$$

The relations (3.3), (3.4) and (3.9) now together show that

$$\left[\frac{\partial}{\partial t}, \frac{\delta}{\delta s} \right] = \frac{\partial}{\partial t}\left(\frac{\delta}{\delta s}\right) - \frac{\delta}{\delta s}\left(\frac{\partial}{\partial t}\right) = \bar{\mu}\frac{\delta}{\delta n} \ , \tag{3.11}$$

$$\left[\frac{\partial}{\partial t}, \frac{\delta}{\delta n} \right] = \frac{\partial}{\partial t}\left(\frac{\delta}{\delta n}\right) - \frac{\delta}{\delta n}\left(\frac{\partial}{\partial t}\right) = -\bar{\mu}\frac{\delta}{\delta s} \ . \tag{3.12}$$

Curvilinear coordinates of the form

$$s = s(x, y, t), \ n = n(x, y, t), \ \tau = t \tag{3.13}$$

are now introduced with

$$\frac{\delta}{\delta s} = \frac{1}{\phi}\frac{\partial}{\partial s} \ , \tag{3.14}$$

$$\frac{\delta}{\delta n} = \frac{1}{\psi}\frac{\partial}{\partial n} \ , \tag{3.15}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \bar{\rho}\frac{\partial}{\partial s} + \bar{\sigma}\frac{\partial}{\partial n} \tag{3.16}$$

where s and n parametrise the **t**-lines and **n**-lines respectively. The quantities $\bar{\rho}(s,n,\tau)$, $\bar{\sigma}(s,n,\tau)$ and $\phi(s,n,\tau)$, $\psi(s,n,\tau)$ are determined by the requirement that the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial t$ commute. Thus, the commutator relation (3.6), on use of (3.14), (3.15) leads to the conditions

$$\psi_s = \varphi_n \phi, \tag{3.17}$$

$$\phi_n = -\varphi_s \psi \tag{3.18}$$

while the commutator relations (3.11), (3.12) acting, in turn, on ϕ and ψ, on use of (3.14), (3.15) yield

$$\phi_\tau + \bar{\rho}\phi_s + \bar{\sigma}\phi_n + \phi\bar{\rho}_s = 0 \ , \tag{3.19}$$

$$\bar{\mu}\phi + \psi\bar{\sigma}_s = 0 \ , \tag{3.20}$$

$$\psi_\tau + \bar{\rho}\psi_s + \bar{\sigma}\psi_n + \psi\bar{\sigma}_n = 0 \ , \tag{3.21}$$

$$\phi\bar{\rho}_n - \bar{\mu}\psi = 0 \ . \tag{3.22}$$

The relations (3.17)-(3.22) provide six equations in the six unknowns $\phi, \psi, \varphi, \bar{\rho}, \bar{\sigma}$ and $\bar{\mu}$.

As observed by Spencer [13], the continuity equation (2.1) together with the constraint (3.1) imply the condition

$$[\ \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\ ]\ \mathrm{div}\,\mathbf{t} = 0 \tag{3.23}$$

so that the quantity $\theta = \mathrm{div}\,\mathbf{t}$ is convected along the particle lines. This constraint holds, in particular, in the privileged case when the divergence is everywhere constant. It is with this class of motions that we shall be concerned in the sequel.

4 The constraint $\operatorname{div} \mathbf{t} = 1$

If $\operatorname{div} \mathbf{t}$ is everywhere constant, then, without loss of generality, we may set

$$\operatorname{div} \mathbf{t} = 1 \; . \tag{4.1}$$

Let $\mathbf{r}$ denote the position vector to a generic $\mathbf{t}$-line, so that $\delta \mathbf{r}/\delta n = \mathbf{n}$. Then, the constraint (4.1) shows, on use of (3.4) that $\delta(\mathbf{r} - \mathbf{t})/\delta n = \dfrac{1}{\psi}\partial(\mathbf{r} - \mathbf{t})/\partial n = \mathbf{0}$ whence, on integration

$$\mathbf{r}(s, n, \tau) = \mathbf{t}(s, n, \tau) + \mathbf{R}(s, \tau) \; . \tag{4.2}$$

Conversely, (4.2) shows that

$$\frac{\delta \mathbf{r}}{\delta s} = \mathbf{t} = \kappa \mathbf{n} + \frac{\delta \mathbf{R}}{\delta s} \tag{4.3}$$

and

$$\operatorname{div} \mathbf{r} = \operatorname{div} \mathbf{t} + \mathbf{t} \cdot \frac{\delta \mathbf{R}}{\delta s} = \operatorname{div} \, \mathbf{t} + \mathbf{t} \cdot \frac{\delta \mathbf{r}}{\delta s} \tag{4.4}$$

so that $\operatorname{div} \mathbf{t} = 1$. The relation (4.2) shows that the fibre lines are generalised tractrices associated with the base curve $\Gamma_0 : \mathbf{R} = \mathbf{R}(s, \tau)$(Eisenhart [14]). Moreover, (4.2) imples that $|\mathbf{r} - \mathbf{R}| = 1$ so that the fibre line distributions are confined to a strip bounded by the two curves $\Gamma_\pm$ which are parallel to and at unit distance from Γ_0.

If s is taken as arclength along the base curve Γ_0 and $\mathbf{T}, \mathbf{N}$ are its unit tangent and principal normal then

$$\mathbf{R}_s = \mathbf{T}, \quad \mathbf{R}_\tau = g\mathbf{T} + h\mathbf{N} \tag{4.5}$$

where $g = g(s, \tau)$ and $h = h(s, \tau)$ remain to be determined. Compatibility of these relations and of the Serret-Frenet and time evolution equations, namely

$$\begin{aligned} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_s &= \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix} , \\ \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_\tau &= \begin{pmatrix} 0 & \ell \\ -\ell & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix} , \end{aligned} \tag{4.6}$$

where $f(s, \tau)$ is the curvature of Γ_0, yields

$$\begin{aligned} f_\tau = \ell_s, \quad g_s = hf \; , \\ \ell = h_s + gf \; . \end{aligned} \tag{4.7}$$

Elimination of g in this system leads to

$$f_\tau = \mathcal{R}h, \tag{4.8}$$

where $\mathcal{R} = \partial_s^2 + f^2 + f_s\partial_s^{-1} f$ is the recursion operator for the mkdV hierarchy of solitonic equations [15]. Underlying geometry of the latter associated with planar motion of an inextensible curve is discussed in [9]. In particular, if $h_s = f$ then (4.8) reduces to the integrable mkdV equation

$$f_\tau = f_{sss} + (3/2)f^2 f_s \ . \tag{4.9}$$

If $\mathbf{t}$ and $\mathbf{n}$ are now decomposed in terms of $\mathbf{T}$ and $\mathbf{N}$ according to

$$\mathbf{t} = \cos\omega\ \mathbf{T} + \sin\omega\ \mathbf{N},\ \ \mathbf{n} = -\sin\omega\ \mathbf{T} + \cos\omega\ \mathbf{N} \tag{4.10}$$

then the requirement $\mathbf{r}_s = \phi\ \mathbf{t}$ applied to (4.2) implies, since

$$\frac{\delta \mathbf{r}}{\delta s} = \frac{\delta \mathbf{t}}{\delta s} + \frac{\delta \mathbf{R}}{\delta s}\ ,$$

the relation

$$(\cos\omega\ \mathbf{T} + \sin\omega\ \mathbf{N})\phi = -\omega_s \sin\omega\ \mathbf{T} + f\cos\omega\ \mathbf{N} + \omega_s\cos\omega\ \mathbf{N} - f\sin\omega\ \mathbf{T} + \mathbf{T}\ ,$$

whence

$$(\phi - \cos\omega)(\cos\omega\ \mathbf{T} + \sin\omega\ \mathbf{N}) = (\omega_s + f - \sin\omega)(-\sin\omega\ \mathbf{T} + \cos\omega\ \mathbf{N})\ .$$

so that

$$\phi = \cos\omega, \tag{4.11}$$

$$\omega_s = \sin\omega - f. \tag{4.12}$$

If $\mathbf{T}$ and $\mathbf{N}$ are, in turn, parametrised according to

$$\mathbf{T} = \cos\Delta\ \mathbf{i} + \sin\Delta\ \mathbf{j}, \quad \mathbf{N} = -\sin\Delta\ \mathbf{i} + \cos\Delta\ \mathbf{j} \tag{4.13}$$

then (4.2) translate into the pair of relations

$$\Delta_s = f, \tag{4.14}$$

$$\Delta_\tau = \ell\ =\ h_s + gf \tag{4.15}$$

compatible modulo $(4.7)_1$. Moreover,

$$\mathbf{t} = \cos\,(\omega + \triangle)\ \mathbf{i} + \sin(\omega + \triangle)\ \mathbf{j} \tag{4.16}$$

so that

$$\varphi = \omega + \Delta\ . \tag{4.17}$$

It is readily shown that the solution of the system (3.17)-(3.22) is given by

$$\phi = \cos\omega, \quad \psi = \omega_n\ ,$$

$$\bar{\rho} = -(h \sin\omega + g\cos\omega)/\phi, \;\; \bar{\sigma} = -(\omega_\tau + h_s + h\cos\omega + g(f - \sin\omega))/\psi \; , \\ \bar{\mu} = -h/\cos\omega \; . \tag{4.18}$$

augmented by the relation (4.18).

In addition, the following result may be established [12]:

Theorem I

The velocity

$$\mathbf{q} = v\mathbf{t} + w\mathbf{n} \tag{4.19}$$

where

$$v = -w_n/\omega_n \; , \quad \omega_n \neq 0 \tag{4.20}$$

$$w_s = w\cos\omega - h \tag{4.21}$$

and h, ω are the quantities in the decompositions (4.5) and (4.10), satisfies the kinematic conditions

$$\operatorname{div}\mathbf{q} = 0 \; , \\ \frac{\partial \mathbf{t}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{t} = \mathbf{t}\cdot\nabla\mathbf{q} \; . \tag{4.22}$$

□

In the following the above result is exploited to construct a new class of exact solutions of the 2+1-dimensional reduction of the magnetohydrodynamic system (2.1)-(2.4).

5 2+1-dimensional magnetohydrodynamics with $\operatorname{div}\mathbf{t} = 1$. A class of exact solutions

Here, we start with the pseudo-hydrodynamic system (2.12)-(2.13) and seek solutions subject to the constraint (3.1) where $\mathbf{q}$ has the decomposition (4.19). Now, if

$$\Omega = |\operatorname{curl}\mathbf{q}| \tag{5.1}$$

denotes the vorticity magnitude, then the compatibility condition for the momentum equation (2.13) adopts the form

$$\frac{\partial \Omega}{\partial t} + \mathbf{q}\cdot\nabla\Omega = 0 \; . \tag{5.2}$$

Accordingly, it is required to construct simultaneous solutions of the continuity equation (2.12) and constraint (3.1) modulo the restriction (5.2). Here, we exploit the base solution of the kinematic conditions (2.12) and (3.1) as presented in [12] in the context of fibre-reinforced fluids and in which the base curve Γ_0 is a straight line undergoing an arbitrary rigid motion. In that case,

$$\mathbf{R} = s\mathbf{T} + \mathbf{P}, \quad \mathbf{T} = \begin{pmatrix} \cos\Delta \\ \sin\Delta \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{5.3}$$

where Δ, α and β depend on τ alone. Thus,

$$\mathbf{R}_\tau = s\Delta_\tau \mathbf{N} + \mathbf{P}_\tau, \quad \mathbf{N} = \begin{pmatrix} -\sin\Delta \\ \cos\Delta \end{pmatrix}, \tag{5.4}$$

so that

$$g = \mathbf{P}_\tau \cdot \mathbf{T}, \quad h = s\Delta_\tau + \mathbf{P}_\tau \cdot \mathbf{N} \tag{5.5}$$

in the decomposition $(4.5)_2$. Since $h_s = \Delta_\tau$, it is seen that $f = 0$ whence (4.12) has the general solution

$$\omega = 2\tan^{-1}[\, c(n,\tau)e^s \,] \tag{5.6}$$

while (4.19)-(4.21) admit the associated velocity

$$\begin{aligned} \mathbf{q} &= (q_0\cos\omega + h\sin\omega)\,\mathbf{t} + (-q_0\sin\omega + h\cos\omega + \Delta_\tau)\,\mathbf{n}\,, \\ &= q_0\mathbf{T} + h\mathbf{N} + \Delta_\tau\mathbf{n} \end{aligned} \tag{5.7}$$

where $q_0 = q_0(\tau)$. Thus

$$\begin{aligned} \mathbf{\Omega} &= \operatorname{curl}\mathbf{q} = \operatorname{curl}\,(h\mathbf{N}) + \Delta_\tau \operatorname{curl}\mathbf{n} \\ &= \Delta h \times \mathbf{N} + \Delta_\tau\mathbf{b} = 2\Delta_\tau\mathbf{b}\,. \end{aligned} \tag{5.8}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla &= \frac{\partial}{\partial\tau} - \frac{1}{\phi}[\,h\sin\,\omega + g\cos\omega\,]\frac{\partial}{\partial s} - \frac{1}{\psi}[\,\omega_\tau + h_s + h\cos\omega - g\sin\omega\,]\frac{\partial}{\partial n} \\ &\quad + \frac{1}{\phi}[\,q_0\cos\omega + h\sin\omega\,]\frac{\partial}{\partial s} + \frac{1}{\psi}[\,-q_0\sin\omega + h\cos\omega + \Delta_\tau\,]\frac{\partial}{\partial n} \\ &= \frac{\partial}{\partial\tau} - \frac{\omega_\tau}{\omega_n}\frac{\partial}{\partial n} + (\,q_0 - g\,)\frac{\partial}{\partial s} + (\,q_0 - g\,)\frac{\partial}{\partial n}\,. \end{aligned} \tag{5.9}$$

Accordingly, the convective condition (5.2) requires that

$$\Delta_{\tau\tau} = 0\,. \tag{5.10}$$

whence we obtain the constant vorticity condition

$$\Omega = const\,, \tag{5.11}$$

Constant vorticity flows are *a fortiori* universal in the sense of Marris [16] in that $\operatorname{curl}\mathbf{\Omega} = \mathbf{0}$. (c.f. Fosdick and Truesdell [17]).

In order to construct associated 'accelerated' magnetohydrodynamic motions, it is required to satisfy simultaneously the two conditions

$$\frac{\partial A}{\partial t} + \mathbf{q} \cdot \nabla A = 0 \ , \tag{5.12}$$

and

$$\nabla^2 A = \Phi(A) \tag{5.13}$$

on the magnetic flux A.

Here, we illustrate the procedure by setting $g = q_0\,(\tau)$ consistent with $f = 0$ (c.f. $(4.7)_2$) whence, in view of (5.9), the convective condition (5.12) requires that $A = A(\omega, s)$. If we proceed with $A = A(\omega)$, so that it is required that

$$\frac{\partial \omega}{\partial t} + \mathbf{q} \cdot \nabla\, \omega = 0 \tag{5.14}$$

then

$$\begin{aligned}
\nabla^2 A &= \operatorname{div}(\operatorname{grad} A) = \operatorname{div}\left(\frac{\delta A}{\delta s}\,\mathbf{t} + \frac{\delta A}{\delta n}\,\mathbf{n}\right) \\
&= \operatorname{div}\left(\frac{A'\omega_s}{\phi}\mathbf{t} + \frac{A'\omega_n}{\psi}\mathbf{n}\right) = \operatorname{div}\left(A' \tan\omega\, \mathbf{t} + A'\mathbf{n}\right) \\
&= \frac{\delta}{\delta s}(A' \tan\omega) + A' \tan\omega\ \operatorname{div}\mathbf{t} + \frac{\delta}{\delta n}A' + A'\ \operatorname{div}\mathbf{n}\ .
\end{aligned}$$

Hence,

$$\nabla^2 A = \frac{1}{\phi}(A''\omega_s \tan\omega + A'\sec^2\omega\,\omega_s) + A'\tan\omega + \frac{1}{\psi}A''\omega_n - A'\kappa\ ,$$

where $\kappa = \dfrac{1}{\phi}\varphi_s = \dfrac{\omega_s}{\phi} = \tan\omega$ so that

$$\nabla^2 A = A''\sec^2\omega + A'\tan\omega\ \sec^2\omega \equiv \Phi(A) \tag{5.15}$$

and the requirement (5.13) is met. Moreover,

$$\begin{aligned}
\mathbf{H} &= \left[\ \frac{\delta A}{\delta s}\ \mathbf{t} + \frac{\delta A}{\delta n}\ \mathbf{n}\ \right] \times \mathbf{b} = \frac{A'(\omega)}{\cos\omega}[\ -\sin\omega\ \mathbf{n} + \cos\omega\ \mathbf{t}\ ] \\
&= \frac{A'(\omega)}{\cos\omega}\mathbf{T} = \frac{A'(\omega)}{\cos\omega}[\ \cos\Delta\,\mathbf{i} + \sin\Delta\,\mathbf{j}\ ]\ ,
\end{aligned} \tag{5.16}$$

while

$$\begin{aligned}
\mathbf{q} &= (\ q_0 - \Delta_\tau \sin\omega\ )\,\mathbf{T} + (\ h + \Delta_\tau \cos\omega\ )\mathbf{N} \\
&= [\ q_0 \cos\Delta - h\sin\Delta - \Delta_\tau \sin(\Delta+\omega)\ ]\ \mathbf{i} + [\ q_0 \sin\Delta + h\cos\Delta + \Delta_\tau \cos(\Delta+\omega)\ ]\ \mathbf{j}
\end{aligned} \tag{5.17}$$

where h is given by $(5.5)_2$ with $\Delta_\tau = \Delta_t = \text{const.}$

Since

$$\frac{1}{\cos\omega}\frac{\partial}{\partial s} = \cos(\omega+\Delta)\frac{\partial}{\partial x} + \sin(\omega+\Delta)\frac{\partial}{\partial y}\ ,$$
$$\frac{1}{\omega_n}\frac{\partial}{\partial n} = -\sin(\omega+\Delta)\frac{\partial}{\partial x} + \cos(\omega+\Delta)\frac{\partial}{\partial y} \tag{5.18}$$

in particular,

$$\frac{1}{\cos\omega} = \cos(\omega+\Delta)\, s_x + \sin(\omega+\Delta)\, s_y\ ,$$
$$0 = -\sin(\omega+\Delta)\, s_x + \cos(\omega+\Delta)\, s_y$$

whence

$$s_x = \frac{\cos(\omega+\Delta)}{\cos\omega}\ , \quad s_y = \frac{\sin(\omega+\Delta)}{\cos\omega}\ . \tag{5.19}$$

Moreover, (4.12) with $f=0$ yields

$$\omega_x \cos\omega\cos(\omega+\Delta) + \omega_y\cos\omega\sin(\omega+\Delta) = \sin\omega$$

so that

$$\omega = \sin^{-1}[\, y\cos\Delta - x\sin\Delta + T_1(t)\,] \tag{5.20}$$

where $T_1(t)$ is arbitrary. It is readily verified that the required convective condition (5.14) holds.

Equivalently, ω is given by (5.6), whence it may be shown that

$$\frac{\delta}{\delta n}[\, \ln c(n,\tau)\,] = \operatorname{cosec}\omega \tag{5.21}$$

whence

$$(-\sin(\omega+\Delta)\frac{\partial}{\partial x} + \cos(\omega+\Delta)\frac{\partial}{\partial y})\ln c = \operatorname{cosec}\omega$$

with general solution

$$c = T_2(t)[\, \operatorname{cosec}\omega + \cot\omega\,] \tag{5.22}$$

where $T_2(t)$ is arbitrary.

Accordingly, the relation (5.6) shows that the solution of the system (5.19) is given by

$$s = 2\ln\left(\tan\frac{\omega}{2}\right) - \ln T_2(t)\ . \tag{5.23}$$

In addition, the class of magnetohydrodynamic solutions have the property

$$\mathbf{H}\cdot\nabla\,\mathbf{H} = \Psi\Psi'(\,\cos\Delta\,\omega_x + \sin\Delta\,\omega_y\,)(\,\cos\Delta\,\mathbf{i} + \sin\Delta\,\mathbf{j}\,) = \mathbf{0} \tag{5.24}$$

on use of (5.20), so that the momentum equation reduces to (c.f. (2.13))

$$\rho\left[\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q}\right] + \nabla\Pi = 0\ , \tag{5.25}$$

where $\Pi = p + \dfrac{1}{2}\,\mu\, H^2$ is the total magnetic pressure. It is readily shown that $\Pi_{xy} = 0$.

The preceding may be summarised in :

Theorem II

The 2+1-dimensional magnetohydrodynamic system (2.1)-(2.5) admits the class of exact solutions with

$$\mathbf{q} = [\, q_0 \cos\Delta - h \sin\Delta - \Delta_t \sin(\Delta+\omega)\,]\,\mathbf{i} + \,[\, q_0 \sin\Delta + h\cos\Delta + \Delta_t \cos(\Delta+\omega)\,]\,\mathbf{j}\,,$$

$$\mathbf{H} = \Psi(\omega)(\,\cos\Delta\,\mathbf{i} + \sin\Delta\,\mathbf{j}\,)\,,$$

where

$$\omega = \sin^{-1}[\, y\cos\Delta - x\sin\Delta + T_1(t)\,]\,,$$

$$h = \Delta_t[\,2\ln\left(\tan\frac{\omega}{2}\right) - T_2(t)\,] - \alpha_t \sin\Delta + \beta_t\cos\Delta$$

and

$$\Delta_t = const.$$

while $\alpha(t),\ \beta(t)$ are the entries in $\mathbf{P}$. $\square$

It is noted that the superposition

$$\mathbf{H}^* = \mathbf{H} + \Omega(\omega)\mathbf{k}\,, \quad \mathbf{q}^* = \mathbf{q} + \Lambda(\omega)\mathbf{k} \tag{5.26}$$

extends the above class of base exact solutions to 3+1-dimensions.

Appendix

It may be shown that the system consisting of the kinematic conditions

$$\mathrm{div}\mathbf{q} = 0\,,$$

$$\frac{\partial\mathbf{t}}{\partial t} + (\mathbf{q}\cdot\nabla)\,\mathbf{t} = (\mathbf{t}\cdot\nabla)\mathbf{q}\,,$$

subject to the constraint

$$\mathrm{div}\,\mathbf{t} = 1$$

can be reduced to consideration of a single complex Liouville equation. The procedure is as follows.

The relation $\mathrm{div}\,\mathbf{t} = \delta\varphi/\delta n$, shows that

$$\varphi_n = \psi$$

whence, (3.17) implies that

$$\phi = \frac{\psi_s}{\psi} = \cos\omega\ .$$

Let $\rho_s = \dfrac{\psi_s}{\psi}$ so that

$$\psi = N(n,\tau)e^{\rho} \ .$$

Now,

$$\omega_{sn} = \cos\omega \ \omega_n, \quad \rho_{sn} = -\sin\omega \ \omega_n$$

and the latter relation shows that

$$\left(\frac{\rho_{sn}}{\psi}\right)_n = -\cos\omega\,\omega_n$$

whence

$$\left(\frac{\rho_{sn}}{\psi}\right)_n + \rho_s\psi = 0 \ ,$$

that is

$$\left(\frac{\rho_{sn}}{Ne^{p}}\right)_n + \rho_s N e^{\rho} = 0 \ .$$

The re-parametrisation $nN(n,\tau) \to n$ now produces a 3rd order nonlinear equation in ρ, namely

$$\left(\frac{\rho_{sn}}{e^{\rho}}\right)_n + \rho_s e^{\rho} = 0 \ .$$

Moreover,

$$\begin{aligned}(\rho + i\omega)_{sn} &= (\rho_s + i\omega_s)_n = (\,\cos\omega + i[\,\sin\omega - f(s,\tau\,)\,]\,)_n \\ &= \omega_n[\,-\sin\omega + i\cos\omega\,] = iN(n,\tau)e^{\rho+i\omega} \ .\end{aligned}$$

and the re-scaling $nN(n,\tau) \to n$ results in the complex Liouville equation

$$\gamma_{sn} = ie^{\gamma}$$

where $\gamma = \rho + i\omega$. It is noted that the latter admits the auto-Bäcklund transformation

$$\left.\begin{aligned}\partial_s(\gamma' - \gamma) &= \beta\, i \exp\left(\frac{\gamma + \gamma'}{2}\right) \ , \\ \partial_n(\gamma' + \gamma) &= \frac{4}{\beta}\sinh\left(\frac{\gamma' - \gamma}{2}\right) \ ,\end{aligned}\right\} \mathbb{B}_{\beta}$$

with associated permutability theorem

$$\gamma_{12} = \gamma_0 + 2\ln\left[\frac{\beta_1\, e^{\gamma_1/2} + \beta_2\, e^{\gamma_2/2}}{\beta_2\, e^{\gamma_1/2} - \beta_1\, e^{\gamma_2/2}}\right] \ ,$$

where β is a Bäcklund parameter, γ_0 is a seed solution and $\gamma_1 = \mathbb{B}_{\beta_1}\gamma_0$, $\gamma_2 = \mathbb{B}_{\beta_2}\gamma_0$, $\gamma_{12} = \mathbb{B}_{\beta_1}\mathbb{B}_{\beta_2}\gamma_0 = \mathbb{B}_{\beta_2}\mathbb{B}_{\beta_1}\gamma_0$.

The permutability theorem may be used iteratively to generate solutions of the original kinematic conditions via a seed solution γ_0. In particular, the base solution obtained in Section 5, for which

$$\begin{aligned}\gamma = \rho + i\,\omega &= \frac{\omega_{sn}}{\omega_n} + i\,\omega = \cos\omega + i\,\omega \\ &= \frac{1-c^2e^{2s}}{1+c^2e^{2s}} + 2i\tan^{-1}[ce^s]\end{aligned}$$

may be used in this connection.

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